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ON PLURILINEARITIES AMONG PROJECTIVE SPACES

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The goal of the present paper is to give a suitable and extensive definition of graphic plurilinearities, i. e. of those pluricorrespondences among projective spaces which generalize the concept of homographies, collineations, etc. between two linear spaces, with the inclusion of the « singular » cases.

For the history of special cases already studied by other Authors and by myself, [1] and [5] can be consulted, while [2] is systematically used here as a set of preliminary results.

Besides the definition (Sect. 2), the main results of this exposition are: sets of necessary and sufficient conditions (Sect. 6), some properties of plurilinearities (Sect. 7), a sufficient condition (Sect. 8), and the classification of the plurilinearities among three projective lines (Sect. 5).

A more detailed study of plurilinearities among special spaces, such as linear, may be object of a future paper.

0. **Notations**: In this section only those notations which may somehow differ from the ordinary usage are listed.

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0.1. The symbol $\{\ldots \mid :::\}$ denotes the set consisting of all the elements ... for which ::: .

$A \subseteq B$ means: A is a subset of B.

$A \subset B$ means: A is a proper subset of B.

$\varnothing$ denotes the empty (or void) set.

$\times$, also in the form $\times_{j \in J}$, denotes the Cartesian product of a finite number of disjoint sets; it is commutative and associa tive by convention.

$|J|$ denotes the order of the set J.

Throughout this paper, the letter I shall indicate a prescribed finite set, which shall be taken, for simplicity, to be the set $\{1, 2, \ldots, t\}$ of the first t natural numbers ($t \geqslant 1$).

0.2. Whenever a finite collection of sets is given, all denoted by the same main letter S (e. g. S^j, or S^{*j}, with $j \in J$), then the symbol $\boldsymbol{S}\langle J\rangle$ (respectively $\boldsymbol{S}^*\langle J\rangle$) shall denote their Cartesian product, otherwise indicated by a symbol like $\times_{j \in J} S^j$.

Similar conventions shall be adopted for x^j ($x^j \in S^j$; $j \in J$) and $\boldsymbol{x}\langle J\rangle$ ($\boldsymbol{x}\langle J\rangle \in \boldsymbol{S}\langle J\rangle$), for y^j, z^j, etc.

0.2.1. If some of the sets S^j ($j \in J$) had elements in common, they would be previously replaced by disjoint copies.

0.3. The definitions of *projective space* (equivalently: *graphic irreducible space*) and related concepts are assumed (cfr. [2], [3], [4]). However, the following facts have to be noted.

0.3.1. A projective line is any set of order not less than 3; if it is though a subspace of dimension 1 of a projective space of larger dimension, then it may possess some additional structure. In any case a line shall be systematically denoted by the main letter R (such as R', R^*, R^i).

0.3.2. A *pencil* of $[k]$'s in a projective space $[n]$ is defined as the set of all the $[k]$'s satisfying the condition $C \subset [k] \subset D$, where C is a given $[k-1]$, D is a given $[k+1]$, $C \subset D$, $0 \leqslant k < n$. The subspace C shall be called the ***axis*** (or ***centre***) of the pencil, D the ***carrier***.

0.3.3. The symbol $[H]$ ($H \subseteq S$; S = projective space) denotes the minimum subspace of S containing H.

1. Pluricorrespondences.

1.1. Definition: A *pluricorrespondence* (short: plc.) *among the sets* S^j ($j \in J$; $|J| = u$ = natural number), or *plc. on* $S\langle J\rangle$, is a subset T of their Cartesian product: $T \subseteq S\langle J\rangle$.

1.1.1. In particular, $\varnothing$ and $S\langle J\rangle$ are plc.'s on $S\langle J\rangle$. For $u = 2$, 3, the terms *bicorrespondence* (or *correspondence*) and *tricorrespondence* are used. For $u = 1$, a plc. is evidently the same as a subset of S^1.

1.2. Definition: For every plc. T on $S\langle J\rangle$ and for every subset $J^\square$ ($J^\square \subseteq J$; $J^\square \neq \varnothing$), the *projection from* T *into* $S\langle J^\square\rangle$ is the mapping $\pi\langle J^\square; T\rangle$: $T \to S\langle J^\square\rangle$, which associates with every $x\langle J\rangle$ ($x\langle J\rangle \in T$) the element $x\langle J^\square\rangle$; in other words: $(\pi\langle J^\square; T\rangle)(\times_{j\in J} x^j) = \times_{j\in J^\square} x^j$.

The plc. $P\langle J^\square; T\rangle$, defined on $S\langle J^\square\rangle$ by

$$P\langle J^\square; T\rangle = (\pi\langle J^\square; T\rangle)T$$

is called the *projection of* T *on* $S\langle J^\square\rangle$.

1.3. Definition: For every plc. T on $S\langle J\rangle$, for every subset $J^\square$ ($J^\square \subseteq J$), and for every choice of the subsets S^{*j} ($S^{*j} \subseteq S^j$; $j \in J$), the *restriction of* T *to the* S^* *'s* is the plc.

$$T \cap (S^*\langle J^\square\rangle \times S\langle J - J^\square\rangle) .$$

It shall be denoted by one of the symbols: $R\langle S^*\langle J^\square\rangle; T\rangle$; $R\langle S^{*j}(j \in J^\square); T\rangle$; $R\langle S^{*j_1}, \ldots, S^{*j_v}; T\rangle$, assuming, in the last instance, that $J^\square = \{j_1, \ldots, j_v\}$.

1.4. Definition: For every plc. T on $S\langle J\rangle$, for every subset $J^\square$ ($J^\square \subset J$), and for every choice of the subsets S^{*j} ($S^{*j} \subseteq S^j$; $j \in J^\square$), the plc.

$$P\langle J - J^\square; R\langle S^*\langle J^\square\rangle; T\rangle\rangle$$

shall be denoted by $T(S^*\langle J^\square\rangle)$, or $T(S^{*j}(j \in J^\square))$, or $T(S^{*j_1}, \ldots, S^{*j_v})$ (assuming, in the last instance, that $J^\square = \{j_1, \ldots, j_v\}$) and shall be called the *image of* $S^*\langle J^\square\rangle$ *under* T.

1.4.1. In particular, $\boldsymbol{T}(\{x^{i_1}\} \times \dots \times \{x^{i_v}\})$ can be denoted also by $\boldsymbol{T}(\boldsymbol{x}\langle J^{\square}\rangle)$, or similarly.

1.5. The operations $\boldsymbol{P}\langle\dots\rangle$, $\boldsymbol{R}\langle\dots\rangle$, $\boldsymbol{T}(\dots)$ enjoy several properties, all of immediate proof, such as the following.

1.5.1. Suppose $\boldsymbol{T}\subseteq\boldsymbol{S}\langle J\rangle$, $\varnothing\subset J^{\triangle}\subseteq J^{\square}\subseteq J$; then

$$\boldsymbol{P}\langle J^{\triangle};\, \boldsymbol{P}\langle J^{\square};\, \boldsymbol{T}\rangle\rangle = \boldsymbol{P}\langle J^{\triangle};\, \boldsymbol{T}\rangle\ .$$

1.5.2. Suppose $\boldsymbol{T}\subseteq\boldsymbol{S}\langle J\rangle$, $J^{\triangle}\subseteq J^{\square}\subseteq J$, $S^{*j}\subseteq S^j$ $(j\in J^{\square})$.

(i) If $S^{**j}\subseteq S^{*j}$ $(j\in J^{\triangle})$, then

$$\boldsymbol{R}\langle \boldsymbol{S}^{**}\langle J^{\triangle}\rangle;\, \boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{T}\rangle\rangle = \\ = \boldsymbol{R}\langle \boldsymbol{S}^{**}\langle J^{\triangle}\rangle \times \boldsymbol{S}^{*}\langle J^{\square}-J^{\triangle}\rangle;\, \boldsymbol{T}\rangle\ .$$

(ii) If $S^{*j} = S^j$ $(j\in J^{\square}-J^{\triangle})$, then

$$\boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{T}\rangle = \boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\triangle}\rangle;\, \boldsymbol{T}\rangle\ .$$

1.5.3. Suppose $\boldsymbol{T}^{*}\subseteq\boldsymbol{T}\subseteq\boldsymbol{S}\langle J\rangle$, $J^{\square}\subseteq J$, $S^{*j}\subseteq S^j$ $(j\in J^{\square})$.

(i) $\boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{T}^{*}\rangle\subseteq\boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{T}\rangle$;

(ii) if $J^{\square}\neq\varnothing$, then $\boldsymbol{P}\langle J^{\square};\, \boldsymbol{T}^{*}\rangle\subseteq\boldsymbol{P}\langle J^{\square};\, \boldsymbol{T}\rangle$;

(iii) if $J^{\square}\subset J$, then $\boldsymbol{T}^{*}(\boldsymbol{S}^{*}\langle J^{\square}\rangle)\subseteq\boldsymbol{T}(\boldsymbol{S}^{*}\langle J^{\square}\rangle)$.

1.5.4. If $\boldsymbol{T}\subseteq\boldsymbol{S}\langle J\rangle$, $\varnothing\subset J^{\square}\subseteq J$, and $S^{*j}\subseteq S^j$ $(j\in J^{\square})$, then

$$\boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{P}\langle J^{\square};\, \boldsymbol{T}\rangle\rangle = \boldsymbol{P}\langle J^{\square};\, \boldsymbol{R}\langle \boldsymbol{S}^{*}\langle J^{\square}\rangle;\, \boldsymbol{T}\rangle\rangle\ .$$

2. Graphic Plurilinearities.

2.0. From now on let S^i be a projective space of dimension s^i $(i\in I;\ |I| = t \geqslant 1;\ s^i \geqslant 0)$ and set $s = \Sigma_{i\in I}\, s^i$.

2.1. DEFINITION: A *graphic plurilinearity* (g. pll.) on $\boldsymbol{S}\langle I\rangle$, or *plurilinearity among projective spaces* S^i, is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$ which satisfies the following properties.

(P.1) For every choice of k, $J^{\triangle}$, and $\boldsymbol{y}^{\triangle}$ ($k \in I$, $J^{\triangle} \subseteq I - \{k\}$, $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$), the set $\boldsymbol{P}\langle\{k\}; \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle$ is a subspace of S^k.

If $t \geqslant 2$, then for every choice of h, k, $J^{\triangle}$, and $\boldsymbol{y}^{\triangle}$ ($h \in I$, $k \in I$, $h \neq k$, $J^{\triangle} \subseteq I - \{h, k\}$, $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$) set

$$(\bigstar) \qquad \boldsymbol{F} = \boldsymbol{P}\langle\{h, k\}; \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle .$$

(P.2) If $t \geqslant 2$ and if there is a k ($k \in I$) such that $s^k \geqslant 1$, then for every such k, for every $\boldsymbol{F}$ defined as above, and for every line R^k ($R^k \subseteq S^k$), one of the following two cases takes place:

(i) for every y^k ($y^k \in R^k$) the subspace $\boldsymbol{F}(y^k)$ is non-empty, and the set F given by

$$F = \{\boldsymbol{F}(y^k) \mid y^k \in R^k\}$$

is a pencil is S^h; or

(ii) there exists a point y^{*k} on R^k such that $\boldsymbol{F}(y^k) \subseteq \boldsymbol{F}(y^{*k})$ for every y^k ($y^k \in R^k$).

2.2. Remark: When $t = 2$, then Def. 2.1 yields the *graphic linearities*, which coincide with the homonymous correspondences between two spaces, as defined in Sect. 1.1 of [2]. Consequently, for $t \geqslant 2$, conditions (P.1), (P.2) can be replaced by the following:

(P′) if $t \geqslant 2$, then for every choice of h, k, $J^{\triangle}$, and $\boldsymbol{y}^{\triangle}$ ($h \in I$, $k \in I$, $h \neq k$, $J^{\triangle} \subseteq I - \{h, k\}$, $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$) the plc. $\boldsymbol{F}$ given by 2.1($\bigstar$) is a graphic linearity (Sect. 1.1 of [2]) between S^h and S^k.

For $t = 1$, the only requirement is

(P°) $\boldsymbol{T}$ is a subspace of S^1.

2.3. Evidently: *$\boldsymbol{S}\langle I\rangle$ and $\varnothing$ are g. pll.'s on $\boldsymbol{S}\langle I\rangle$.*

3. Special Pluricorrespondences.

3.0. In order to investigate properties of the g. pll.'s and, in particular, to find sets of necessary and sufficient conditions which characterize them among the plc.'s among projective spaces, some special types of plc.'s are introduced in the present section.

Notations are as in 2.0.

3.1. DEFINITION: A T^A *plc.* is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$, which satisfies the following properties:

(A.1) for every i $(i \in I)$ the set $P\langle\{i\}; \boldsymbol{T}\rangle$ is a subspace of S^i;

(A.2) if $t \geqslant 2$ and if $s^i > 0$ for some i $(i \in I)$, then for every such i and for every hyperplane $S^{☆i}$ of S^i, the plc. $\boldsymbol{R}\langle S^{☆i}; \boldsymbol{T}\rangle$ is a T^A plc.;

(A.3) if $t \geqslant 2$ and if, for some i $(i \in I)$, $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ is a point, say y^i, then $\boldsymbol{T}(y^i)$ is a T^A plc.

3.1.1. REMARKS: Def. 3.1. proceeds evidently by induction on the two indices t, s $(t = 1, 2, ...; s = 0, 1, ...)$.

3.1.2. When $t = 1$, then condition (A.1) implies that $\boldsymbol{T}$ is a subspace of S^1. Conversely, still when $t = 1$, every subspace $\boldsymbol{T}$ of S^1 satisfies trivially (A.1), while (A.2) and (A.3) are inapplicable.

When $s = 0$ and therefore $s^i = 0$ for every i $(i \in I)$, then the only subsets of $\boldsymbol{S}\langle I\rangle$ are $\varnothing$ and $\boldsymbol{S}\langle I\rangle$. Each of them satisfies trivially (A.1) and (A.3); (A.2) is inapplicable. Thus the following statement holds.

3.1.3. *When* $t = 1$, *the* T^A *plc.'s are the same as the subspaces of* S^1. *When* $s = 0$, *all the plc.'s (i. e.* $\varnothing$ *and* $\boldsymbol{S}\langle I\rangle$*) are* T^A.

3.2. If $s^i = 1$ for every i $(i \in I)$, then conditions (A.2) and (A.3) can be combined, and the following definition arises.

DEFINITION: A T^R *plc.* is a plc. $\boldsymbol{T}$ among projective lines R^i $(i \in I)$, which satisfies the following properties:

(R.1) for every i $(i \in I)$ the set $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ is a subspace of R^i (thence void, or a point, or R^i);

(R.2) if $t \geqslant 2$, then for every i $(i \in I)$ and for every x^i $(x^i \in R^i)$ the plc. $\boldsymbol{T}(x^i)$ is T^R.

3.2.1. Evidently: *The* T^R *plc.'s are the same as the* T^A *plc.'s among projective lines.*

3.3. DEFINITION: A T^B *plc.* is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$ which satisfies the following properties:

(B.1) if $s^i \leqslant 1$ for every i $(i \in I)$, then $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ is a subspace of S^i $(i \in I)$;

(B.2) if $s > 0$, then for every choice of the subspaces S^{*i} $(i \in I;$ $\varnothing \subset S^{*i} \subseteq S^i$, $\Sigma_{i \in I} \dim S^{*i} < s)$ the plc. $\boldsymbol{R}\langle \boldsymbol{S}^*\langle I\rangle; \boldsymbol{T}\rangle$ is T^B.

3.4. DEFINITION: A T^C *plc.* is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$, which satisfies the following properties:

(C.1) if $s^i > 0$ for every i $(i \in I)$, then for every choice of the lines R^i $(R^i \subseteq S^i;\ i \in I)$ the plc. $\boldsymbol{R}\langle R^i\ (i \in I);\ \boldsymbol{T}\rangle$ is T^R;

(C.2) if $t \geqslant 2$ and if $s^i = 0$ for some i $(i \in I)$, then for every such i the plc. $\boldsymbol{T}(S^i)$ is T^C.

3.5. DEFINITION: A T^D *plc.* is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$, which satisfies the following properties:

(D.1) if $s^i = 1$ for every i $(i \in I)$, then $\boldsymbol{P}\langle\{i\};\ \boldsymbol{T}\rangle$ is a subspace of S^i $(i \in I)$;

(D.2) if $s^i > 1$ for some i $(i \in I)$, then for every such i and for every line R^i of S^i, the plc. $\boldsymbol{R}\langle R^i;\ \boldsymbol{T}\rangle$ is T^D;

(D.3) if $t \geqslant 2$ and if $s^i \leqslant 1$ for every i $(i \in I)$, then for every i and for every z^i $(i \in I;\ z^i \in S^i)$ the plc. $\boldsymbol{T}(z^i)$ is T^D.

3.6. REMARK: As already noted for T^A plc.'s, the definitions of T^A, T^B, T^C, T^D, and T^R plc.'s proceed by induction on the indices t, s. In each case a statement analogous to 3.1.3 holds, namely:

3.6.1. LEMMA: (i) *When* $t = 1$, *then g. pll.'s,* T^A *plc.'s,* T^B *plc.'s,* T^C *plc.'s,* T^D *plc.'s, and (if* $s^1 = 1$*)* T^R *plc.'s are all the same as the subspaces of* S^1. (ii) *When* $s = 0$, *then all the plc.'s (i. e.* $\varnothing$ *and* $\boldsymbol{S}\langle I\rangle$*) are g.pll.'s and* T^A, T^B, T^C, *and* T^D *plc.'s.*

Proof. (i) is implied immediately by (P.1); (A.1); (B.1) and (B.2); (C.1) and (R.1); (D.1) and (D.2); (R.1), in the respective cases. In cases T^B, T^C, T^D the result is obtained by proving that, if $R^1 \subseteq S^1$ and if R^1 and $\boldsymbol{T}$ have at least two points in common, then $R^1 \subseteq \boldsymbol{T}$. (ii) follows from a direct verification.

3.7. REMARK: Most of the proofs given in the sequel are conducted by induction on the two indices t and s $(t = 1, 2, ...;$ $s = 0, 1, ...)$, on the ground that, by 3.6.1, the statement in dispute is true when $t = 1$ and when $s = 0$.

4. T^R Bicorrespondences.

The classification of the T^R plc.'s between two projective lines R^1, R^2 is immediately derived.

4.1. Either

$T_{(2)}$.I $$\boldsymbol{T} = \varnothing\ ,$$

or, if $\boldsymbol{T} \neq \varnothing$, both $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ $(i = 1, 2)$ are non-empty. Three possibilities arise: (α) $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle = \{x^{*i}\}$ $(x^{*i} \in R^i;\ i = 1, 2)$; (β) $\boldsymbol{P}\langle\{j\}; \boldsymbol{T}\rangle = \{x^{*j}\}$, $\boldsymbol{P}\langle\{h\}; \boldsymbol{T}\rangle = R^h$ $(\{j, h\} = \{1, 2\});\ x^{*j} \in R^j)$; (γ) $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle = R^i$ $(i = 1, 2)$. Cases (α) and (β) yield respectively

$T_{(2)}$.II $$\boldsymbol{T} = \{x^{*1} \times x^{*2}\}$$
$$(x^{*i} = \text{fixed point on } R^i;\ i = 1, 2);$$

$T_{(2)}$.III $$\boldsymbol{T} = \{x^{*j}\} \times R^h$$
$$(\{j, h\} = \{1, 2\};\ x^{*j} = \text{fixed point on } R^j).$$

Case (γ) implies that $\boldsymbol{T}(x^i) \neq \varnothing$ for every i and for every x^i $(i = 1, 2;\ x^i \in R^i)$. Therefore:

$(\gamma\alpha)$ $\boldsymbol{T}(x^i)$ has always dimension 0. Let $\tau\colon R^1 \to R^2$ be defined by the position $\tau x^1 = \boldsymbol{T}(x^1)$ $(x^1 \in R^1)$. Consequently τ is a mapping and, since $\tau^{-1}x^2 = \boldsymbol{T}(x^2)$ $(x^2 \in R^2)$, τ is bijective. This gives

$T_{(2)}$.IV $$\boldsymbol{T} = \{x^1 \times \tau x^1 \mid x^1 \in R^1\}\ ,$$

where $\tau\colon R^1 \to R^2$ is a bijective mapping.

$(\gamma\beta)$ $\boldsymbol{T}(x^{*j}) = R^h$ for at least one j and for only one point x^{*j} of R^j. Set $\{h\} = I - \{j\}$. If $y^j \in R^j - \{x^{*j}\}$, then $\boldsymbol{T}(y^j) = \{x^{*h}\}$, with $x^{*h} \in R^h$; $\boldsymbol{T}(x^{*h}) \supseteq \{x^{*j}, y^j\}$; therefore $\boldsymbol{T}(x^{*h}) = R^j$. Thus $\boldsymbol{T} \subseteq \boldsymbol{T}'$, with $\boldsymbol{T}' = (\{x^{*1}\} \times R^2) \cup (\{x^{*2}\} \times R^1)$.

Suppose $y^1 \times y^2 \in \boldsymbol{T} - \boldsymbol{T}'$; then $\{x^{*1} \times x^{*2}, x^{*1} \times y^2, y^1 \times x^{*2}, y^1 \times y^2\} \subseteq \boldsymbol{T}$, and each one of the sets $\boldsymbol{T}(x^{*1})$, $\boldsymbol{T}(x^{*2})$, $\boldsymbol{T}(y^1)$, $\boldsymbol{T}(y^2)$ is a line; this is in contradiction to assumption $(\gamma\beta)$, so $\boldsymbol{T} = \boldsymbol{T}'$ and

$T_{(2)}$.V $$\boldsymbol{T} = (\{x^{*1}\} \times R^2) \cup (R^1 \times \{x^{*2}\})$$
$$(x^{*i} = \text{fixed point on } R^i;\ i = 1, 2)$$

($\gamma\gamma$) $\boldsymbol{T}(x^j) = R^h$ for at least one j and for at least two points, say y^j and z^j, of R^j ($j = 1, 2$). Then for every x^h ($x^h \in R^h$) the set $\boldsymbol{T}(x^h)$ contains y^j and z^j; therefore $\boldsymbol{T}(x^h) = R^j$ and

$$\mathrm{T}_{(2)}.\mathrm{VI} \qquad \boldsymbol{T} = R^1 \times R^2 .$$

4.2. The previous analysis proves that every T^R correspondence between two lines is necessarily of one of the types above. It is immediately verified that each of them represents actually a T^R plc. between two lines, for every choice of the arbitrary elements appearing in each type. Therefore

THEOREM: The T^R correspondences between two projective lines are those described in Sect. 4.1, formulae $\mathrm{T}_{(2)}$.I-VI.

5. T^R Tricorrespondences.

Although not necessary for the sequel, the classification of the T^R plc.'s among three projective lines is given in this section.

5.1. Let $\boldsymbol{T}$ be a T^R plc. on $R^1 \times R^2 \times R^3$, the R^i 's being projective lines. In the following, the letters j, h, k shall denote a permutation of 1, 2, 3; i. e. $\{j, h, k\} = \{1, 2, 3\} = I$.

The first obvious case is

$$\mathrm{T}_{(3)}.\mathrm{I} \qquad \boldsymbol{T} = \varnothing .$$

Another trivial case is $\boldsymbol{T} = R^1 \times R^2 \times R^3$, listed below as case XIX. In the remainder of this section $\boldsymbol{T}$ is supposed to verify

$$(\ast\ast) \qquad \varnothing \subset \boldsymbol{T} \subset R^1 \times R^2 \times R^3 ;$$

therefore $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ is, for each i ($i \in I$), either a fixed point x^{*i} of R^i or the line R^i itself.

The following conventions shall be adopted:

(□) ^{*i} denotes a fixed point on R^i;

(□□) τ: $R^j \rightarrow R^h$ an/dor φ: $R^h \rightarrow R^k$ are bijective mappings.

(α) If $P\langle\{k\}; T\rangle = \{x^{*k}\}$, then $T = \{x^{*k}\} \times T(x^{*k})$, with $T(x^{*k}) \neq \varnothing$. By (R.2) and 4.1 the following cases arise for T:

T$_{(3)}$.II $\quad T = \{x^{*1} \times x^{*2} \times x^{*3}\}$ (cfr. ($\square$));

T$_{(3)}$.III $\quad T = \{x^{*j} \times x^{*k}\} \times R^h$ (cfr. ($\square$));

T$_{(3)}$.IV $\quad T = \{x^j \times \tau x^j \times x^{*k} \mid x^j \in R^j\}$ (cfr. ($\square$), ($\square\square$));

T$_{(3)}$.V $\quad T = (\{x^{*j} \times x^{*k}\} \times R^h) \cup (\{x^{*h} \times x^{*k}\} \times R^j)$ (cfr. ($\square$));

T$_{(3)}$.VI $\quad T = R^j \times R^h \times \{x^{*k}\}$ (cfr. ($\square$)).

(β) Suppose now $P\langle\{i\}; T\rangle = R^i$ for every i ($i \in I$). Set $T^i = P\langle I - \{i\}; T\rangle$. By 1.5.4, 3.2, and 4, T^i is a T^R plc. Since $P\langle\{i\}; T^{i'}\rangle = P\langle\{i\}; T\rangle = R^i$ for every i' ($i' \in I - \{i\}$), it follows that T^i is of one of the types T$_{(2)}$.IV-VI. Accordingly, the following cases arise: at least one T^i is of type T$_{(2)}$.IV; no T^i is of type T$_{(2)}$.IV, at least two are of type T$_{(2)}$.V; no T^i is of type T$_{(2)}$.IV, exactly one is of type T$_{(2)}$.V; all T^i are of type T$_{(2)}$.VI.

($\beta\alpha$) T^j is of type T$_{(2)}$.IV. Therefore $T^j = \{x^h \times \varphi x^h \mid x^h \in R^h\}$ (cfr. ($\square\square$)) and $T = \{x^j \times x^h \times \varphi x^h \mid x^j \times x^h \in T^k\}$; the classification of T depends only on T^k. The following cases are thus obtained:

T$_{(3)}$.VII $\quad T = \{x^j \times \tau x^j \times \varphi\tau x^j \mid x^j \in R^j\}$ (cfr. ($\square\square$));

T$_{(3)}$.VIII $\quad T = (R^j \times \{x^{*h} \times \varphi x^{*h}\}) \cup \{x^{*j} \times x^h \times \varphi x^h \mid x^h \in R^h\}$ (cfr. ($\square$), ($\square\square$));

T$_{(3)}$.IX $\quad T = R^j \times \{x^h \times \varphi x^h \mid x^h \in R^h\}$ (cfr. ($\square\square$)).

($\beta\beta$) T^j and T^k are of type T$_{(2)}$.V, T^h is of type T$_{(2)}$.V or VI. Set $T^j = (\{x^{*h}\} \times R^k) \cup (\{x^{*k}\} \times R^h)$, $T^k = (\{x^{*j}\} \times R^h) \cup (\{y^{*h}\} \times R^j)$.

($\beta\beta\alpha$) If $y^{*h} \neq x^{*h}$, then $T^k(x^{*h}) = \{x^{*j}\}$ and, for every x^k ($x^k \in R^k$), $x^{*h} \times x^k \in T^j$, whence $T(x^{*h} \times x^k) \neq \varnothing$ and $T(x^{*h}) = \{x^{*j}\} \times R^k$. Similarly $T(y^{*h}) = \{x^{*k}\} \times R^j$. Suppose $z^h \in R' =$

$= R^h - \{x^{*h}, y^{*h}\}$; then evidently $T(z^h) = \{x^{*j} \times x^{*k}\}$ and

$$(\bigstar\bigstar\bigstar) \qquad T = \bigcup_{x^h \in R^h} \{x^h\} \times T(x^h) =$$
$$= (\{x^{*h}\} \times T(x^{*h})) \cup (\{y^{*h}\} \times T(y^{*h})) \cup (\bigcup_{z \in R'} \{z^h\} \times T(z^h)) .$$

The fact that $x^{*h} \times x^{*j} \times x^{*k}$ and $y^{*h} \times x^{*j} \times x^{*k}$ belong, respectively, to the first two terms of $(\bigstar\bigstar\bigstar)$ implies

$$T_{(3)}.X \qquad T = (\{x^{*h} \times x^{*j}\} \times R^k) \cup (\{y^{*h} \times x^{*k}\} \times R^j) \cup$$
$$\cup (\{x^{*j} \times x^{*k}\} \times R^h) \qquad \text{(cfr. } (\square)).$$

$(\beta\beta\beta)$ If $y^{*h} = x^{*h}$, then $z^h \in R''$ $(R'' = R^h - \{x^{*h}\})$ implies necessarily $T(z^h) = \{x^{*j} \times x^{*k}\}$. Therefore $T(x^{*j} \times x^{*k}) \supseteq R''$, whence $T(x^{*j} \times x^{*k}) = R^h$ and $x^{*j} \times x^{*h} \times x^{*k} \in T$. Therefore

$$T = \bigcup_{x^h \in R^h} \{x^h) \times T(x^h) = (\{x^{*h}\} \times T(x^{*h})) \cup$$
$$\cup (\bigcup_{z^h \in R''} \{z^h\} \times T(z^h)) = (\{x^{*h}\} \times T(x^{*h})) \cup (\{x^{*j} \times x^{*k}\} \times R^h)$$

and

$$T^h = \bigcup_{x^h \in R^h} T(x^h) = T(x^{*h}) \cup (\bigcup_{z^h \in R''} T(z^h)) =$$
$$= T(x^{*h}) \cup \{x^{*j} \times x^{*k}\} = T(x^{*h}) ,$$

whence

$$T = (\{x^{*h}\} \times T^h) \cup (\{x^{*j} \times x^{*k}\} \times R^h) .$$

This reduces the study of T to the study of T^h. If $T^h = (\{y^{*j}\} \times \times R^k) \cup (\{y^{*k}\} \times R^j)$ with $y^{*j} \neq x^{*j}$ or $y^{*k} \neq x^{*k}$, then case $(\beta\beta\alpha)$ arises again. Therefore either $y^{*j} = x^{*j}$ and $y^{*k} = x^{*k}$, or $T^h = R^j \times R^k$. Correspondingly:

$$T_{(3)}.XI \qquad T = (\{x^{*h} \times x^{*j}\} \times R^k) \cup (\{x^{*h} \times x^{*k}\} \times R^j) \cup$$
$$(\{x^{*j} \times x^{*}\} \times R^h) \qquad \text{(cfr. } (\square));$$

$$T_{(3)}.XII \qquad T = (\{x^{*h}\} \times R^j \times R^k) \cup (\{x^{*j} \times x^{*k}) \times R^h) \quad \text{(cfr. } (\square)).$$

($\beta\gamma$) $T^j = (\{x^{*h}\} \times R^k) \cup (\{x^{*k}\} \times R^h)$, $T^h = R^j \times R^k$, $T^k = R^j \times R^h$. Then $T(x^j \times z^k) \neq \varnothing$ and $\{z^k\} \times T(x^j \times z^k) \in T^j$ for every choice of x^j and z^k ($x^j \in R^j$; $z^k \in R''' = R^k - \{x^{*k}\}$). Therefore $T(x^j \times z^k) = \{x^{*h}\}$, $T(x^j \times x^{*h}) \supseteq R'''$, and $T(x^j \times x^{*h}) = R^k$. Similarly, $T(x^j \times z^h) = \{x^{*k}\}$ and $T(x^j \times x^{*k}) = R^h$ for every x^j as above and every z^h ($z^h \in R'' = R^h - \{x^{*h}\}$). Consequently $T = \bigcup_{x^j \times x^h \in R^j \times R^h} \{x^j \times x^h\} \times T(x^j \times x^h) = \bigcup_{x^j \in R^j} (\{x^j \times z^h \times x^{*k} \mid z^h \in R''\} \cup (\{x^j \times x^{*h}\} \times R^k))$; in other words

$\mathrm{T}_{(3)}$.XIII $T = (\{x^{*h}\} \times R^j \times R^k) \cup (\{x^{*k}\} \times R^j \times R^h)$ (cfr. (□)).

($\beta\delta$) All T^i 's are of type $\mathrm{T}_{(2)}$.VI. Then, under hypotheses (✱✱) and ($\beta\delta$), the following statements are rather evident.

5.1.1. $P\langle\{j\}; T^h\rangle = R^j$;

5.1.2. *There is at most one point x^{*j} ($x^{*j} \in R^j$) such that $T(x^{*j}) = R^h \times R^k$.*

In fact, if also $T(x^{**j}) = R^h \times R^k$, then $T(x^h \times x^k) \supseteq \{x^{*j}, x^{**j}\}$ for every x^h, x^k ($x^h \in R^h$; $x^k \in R^k$), and $T = R^1 \times R^2 \times R^3$, in contradiction to (✱✱).

5.1.3. *For every choice of i and x^i ($i \in I$; $x^i \in R^i$) the plc. $T(x^i)$ is of type $\mathrm{T}_{(2)}$.IV or $\mathrm{T}_{(2)}$.V, with the exception of no more than one point, for which it can be of type $\mathrm{T}_{(2)}$.VI.*

On the ground of these statements, there are the following possibilities.

($\beta\delta\alpha$) On R^k there is a point x^{*k} such that $T(x^{*k}) = R^j \times R^h$. For every choice of y^k, z^k, x^j, x^h ($\{y^k, z^k\} \subseteq R''' = R^k - \{x^{*k}\}$; $x^j \times x^h \in T(y^k)$) the following formulae hold: $T(x^j \times x^h) \supseteq \{y^k, x^{*k}\}$, $T(x^j \times x^h) = R^k$, $x^j \times x^h \in T(z^k)$; therefore $T(y^k) \subseteq T(z^k)$ and, by symmetry, $T(y^k) = T(z^k) = U$, say. It follows that $T = \bigcup_{x^k \in R^k} \{x^k\} \times T(x^k) = (\{x^{*k}\} \times R^j \times R^h) \cup (U \times R^k)$. If U is of type $\mathrm{T}_{(2)}$.IV or $\mathrm{T}_{(2)}$.V, the following cases arise respectively:

$\mathrm{T}_{(3)}$.XIV $T = (\{x^{*k}\} \times R^j \times R^h) \cup (\{x^j \times \tau x^j \mid x^j \in R^j\} \times R^k)$ (cfr. (□), (□□));

$\mathrm{T}_{(3)}$.XV $T = (\{x^{*1}\} \times R^2 \times R^3) \cup (\{x^{*2}\} \times R^3 \times R^1) \cup (\{x^{*3}\} \times R^1 \times R^2)$ (cfr. (□)).

($\beta\delta\beta$) For every choice of i and x^i ($i \in I$; $x^i \in R^i$) the plc. $T(x^i)$ is of type $T_{(2)}$.IV or V. If (✱✱) and ($\beta\delta\beta$) hold, then the following lemmas (5.1.4 to 5.1.8) are valid.

5.1.4. Lemma: $T(y^j) \nsubseteq T(z^j)$ *for every choice of* j, y^j, *and* z^j ($j \in I$; $y^j \in R^j$; $z^j \in R^j$; $y^j \neq z^j$).

Proof. Suppose $T(y^j) \subseteq T(z^j)$; then for every $x^h \times x^k$ ($x^h \times x^k \in T(y^j)$) the set $T(x^h \times x^k)$ contains y^j and z^j, therefore it coincides with R^j. This implies $T(y^j) \subseteq T(x^j)$ for every x^j ($x^j \in R^j$); because of the structure of the plc. 's of types $T_{(2)}$.IV and V, this implies $T(y^j) = T(x^j)$, therefore $T^j = T(y^j)$, in contradiction to hypothesis ($\beta\delta$).

5.1.5. Lemma: *Suppose* $y^j \in R^j$, $z^j \in R^j$, $y^j \neq z^j$, *and that* $T(y^j)$ *and* $T(z^j)$ *are of type* $T_{(2)}$.V. *Set* $T(y^j) = (\{y^{*h}\} \times R^k) \cup (\{y^{*k}\} \times R^h)$, $T(z^j) = (\{z^{*h}\} \times R^k) \cup (\{z^{*k}\} \times R^h)$. *Then*: (i) $y^{*h} \neq z^{*h}$; (ii) $y^{*k} \neq z^{*k}$; (iii) $T(y^{*h} \times y^{*k}) = \{y^j\}$; (iv) $T(z^{*h} \times z^{*k}) = \{z^j\}$.

Proof. (i) If $y^{*h} = z^{*h}$, then $\{y^j, z^j\} \subseteq T(y^{*h} \times x^k)$ for every x^k ($x^k \in R^k$); therefore $R^j = T(y^{*h} \times x^k)$ and $R^j \times R^k = T(y^{*h})$, in contradiction to hypothesis ($\beta\delta\beta$). (ii) follows by symmetry.

(iii) Evidently $y^j \in T(y^{*h} \times y^{*k})$. If the statement were not true, then the following implications would follow: $T(y^{*h} \times y^{*k}) = R^j$, $y^{*h} \times y^{*k} \in T(z^j)$, and $y^{*h} = z^{*h}$ or $y^{*k} = z^{*k}$, in contradiction to (i) or (ii) respectively. (iv) follows by symmetry.

5.1.6. Lemma: *For every* j ($j \in I$) *the plc.* $T(x^j)$ *is of type* $T_{(2)}$.IV *for all points* x^j *of* R^j *with the exception of no more than two of them.*

Proof. Suppose y^j and z^j are two distinct points of R^j such that $T(y^j)$ and $T(z^j)$ are of type $T_{(2)}$.V. Let x'^j be any point of $R^j - \{y^j, z^j\}$. By 5.1.5 (iii) (iv) the pairs $y^{*h} \times y^{*k}$ and $z^{*h} \times z^{*k}$ are not in $T(x'^j)$. On the other hand, $T(y^{*h} \times z^{*k}) \supseteq \{y^j, z^j\}$, therefore $T(y^{*h} \times z^{*k})$ (and similarly $T(z^{*h} \times y^{*k})$) coincides with R^j. Therefore $T(x'^j)$ contains $y^{*h} \times z^{*k}$ and $z^{*h} \times y^{*k}$, but not $y^{*h} \times y^{*k}$ or $z^{*h} \times z^k$; thus it cannot be of type $T_{(2)}$.V.

5.1.7. Lemma: *If* $x^{*h} \times x^{*k} \in T(y^j) \cap T(z^j)$ ($y^j \neq z^j$), *then* $\{x^{*h} \times x^{*k}\} \times R^j \subseteq T$; *thence* $T(x^{*h})$ *and* $T(x^{*k})$ *are of type* $T_{(3)}$.V.

5.1.8. Lemma: *For every choice of* i, y^i *and* z^i ($i \in I$; $y^i \in R^i$; $z^i \in R^i$; $y^i \neq z^i$) *the plc.'s* $T(y^i)$ *and* $T(z^i)$ *have no more than two pairs in common.*

Proof. Immediate consequence of 5.1.5, 5.1.6, and 5.1.7.

Still in case $(\beta\delta\beta)$, the following possibilities have to be considered.

$(\beta\delta\beta\alpha)$ On R^j there are two distinct points y^{*j}, z^{*j} such that $\boldsymbol{T}(y^{*j})$ and $\boldsymbol{T}(z^{*j})$ are of type $T_{(2)}$.V. Set $\boldsymbol{T}(y^{*j}) = (\{z^{*h}\} \times R^k) \cup \cup (\{z^k\} \times R^h)$, $\boldsymbol{T}(z^{*j}) = (\{y^{*h}\} \times R^k) \cup (\{y^{*k}\} \times R^h)$. Therefore: $y^{*j} \times z^{*h} \times y^{*k} \in \boldsymbol{T}$, $y^{*j} \times z^{*h} \times z^{*k} \in \boldsymbol{T}$, $y^{*j} \times y^{*h} \times z^{*k} \in \in \boldsymbol{T}$, $z^{*j} \times y^{*h} \times z^{*k} \in \boldsymbol{T}$, $z^{*j} \times y^{*h} \times y^{*k} \in \boldsymbol{T}$, $z^{*j} \times z^{*h} \times \times y^{*k} \in \boldsymbol{T}$. By 5.1.5, $y^{*h} \neq z^{*h}$, $y^{*k} \neq z^{*k}$; this and 5.1.8 imply $\boldsymbol{T}(y^{*h}) = (\{z^{*j}\} \times R^k) \cup (\{z^{*k}\} \times R^j)$, $\boldsymbol{T}(z^{*h}) = (\{y^{*j}\} \times R^k) \cup \cup (\{y^{*k}\} \times R^j)$, and similarly for $\boldsymbol{T}(y^{*k})$ and $\boldsymbol{T}(z^{*k})$.

Therefore for every i $(i \in I)$ there are exactly two points x^i $(x^i \in R^i)$ for which $\boldsymbol{T}(x^i)$ is of type $\mathrm{T}_{(2)}$.V. For the moment let this case be described as the:

$\mathrm{T}_{(3)}$.XVI Hyperbolic Case.

$(\beta\delta\beta\beta)$ On R^j there is exactly one point x^{*j} for which $\boldsymbol{T}(x^{*j})$ is of type $\mathrm{T}_{(2)}$.V. If $\boldsymbol{T}(x^{*j}) = (\{x^{*h}\} \times R^k) \cup (\{x^{*k}\} \times R^h)$, then $\{x^{*j} \times x^{*h}\} \times R^k \subseteq \boldsymbol{T}$ and $\{x^{*j} \times x^{*k}\} \times R^h \subseteq \boldsymbol{T}$, which imply that $\boldsymbol{T}(x^{*h})$ and $\boldsymbol{T}(x^{*k})$ are also of type $\mathrm{T}_{(2)}$.V. Thus the conclusion of $(\beta\delta\beta\alpha)$ implies that for every i $(i \in I)$ there is only one point, x^{*i}, for which $\boldsymbol{T}(x^{*i})$ is of type $\mathrm{T}_{(2)}$.V. Let this case be described as the

$\mathrm{T}_{(3)}$.XVII Parabolic Case.

$(\beta\delta\beta\gamma)$ For every x^j of R^j the plc. $\boldsymbol{T}(x^j)$ is of type $\mathrm{T}_{(2)}$.IV. The conclusions of $(\beta\delta\beta\alpha)$ and $(\beta\delta\beta\beta)$ imply that the same fact is valid for every i and every x^i $(i \in I;\ x^i \in R^i)$. Let this case be described as the

$\mathrm{T}_{(3)}$.XVIII Elliptic Case.

$(\beta\delta\gamma)$ As already noted before formula (✦✦), the following case has to be added:

$\mathrm{T}_{(3)}$.XIX $\boldsymbol{T} = R^1 \times R^2 \times R^3$.

5.2. As for the existence of each of the found types and the possibility of a subclassification, the following facts have to be considered.

5.2.1. Types $T_{(3)}$.I-XV and XIX certainly exist for every prescribed triad of lines, with the only restriction, in cases IV, VII, VIII, and XIV, that whenever a bijective mapping between two lines appears (cfr. (□□)), the lines be of the same cardinality. In each case the construction of the T^R plc.'s depends on the choice of one fixed element on some of the lines, possibly of one other element on one of the lines (case X), and/or the choice of one or two mappings (□□). Evidently these choices can be made in essentially one way, if the lines do not posses any additional structure.

5.2.2. As for types $T_{(3)}$.XVI-XVIII, the following constructions answer the question of existence and uniqueness.

(*a*) Since bijective mappings do appear between any two of the lines, all three R^i 's must have the same cardinality.

(*b*) If the lines contain exactly 3 or 4 points (in case XVI) or exactly 3 points (in case XVII), then extremely simple possibilities arise, for which the conclusions (although not the proofs) of this section are valid.

(*c*) With the exclusion of cases (*b*), let $R^{\star i}$ be the set (projective line) obtained from R^i by deleting the points y^{*i}, z^{*i} in case XVI, and the point x^{*i} in case XVII (cfr. 5.1 ($\beta\delta\beta\alpha$); $i \in I$). Let $T^{\star}$ be the set obtained from T by deleting those elements $x^1 \times x^2 \times x^3$ of T for which $x^i = y^{*i}$ or $x^i = z^{*i}$, or, respectively, $x^i = x^{*i}$ for at least one i of I. It is immediately verified that $T^{\star}$ is a plc. of type $T_{(3)}$.XVIII on $R^{\star 1} \times R^{\star 2} \times R^{\star 3}$.

(*d*) Conversely, if $T^{\star}$ is a plc. of type $T_{(3)}$.XVIII on $R^{\star 1} \times$ $\times R^{\star 2} \times R^{\star 3}$, the inverse construction gives a T^R plc. of type XVI or, respectively, XVII on $R^1 \times R^2 \times R^3$.

(*e*) The structure of a T^R plc. of type XVIII, say T, can be described as follows. For every x^3 ($x^3 \in R^3$), identify the element x^3 with the bijective mapping $T(x^3)$, interpreted as a mapping x^3: $R^1 \rightarrow R^2$. A set R^3 of bijective mappings $R^1 \rightarrow R^2$ is then obtained, such that (cfr. 5.1 ($\beta\delta\beta\gamma$))

($\triangle$) for every $x^1 \times x^2$ of $R^1 \times R^2$ there is exactly one mapping x^3 of R^3 such that $x^3(x^1) = x^2$.

(*f*) Conversely, if any two sets R^1, R^2 of the same cardinality are given, it is always possible to construct a set R^3 of bijective mappings satisfying condition ($\triangle$), and the plc. $\boldsymbol{T}$, defined by the position $\boldsymbol{T} = \{x^1 \times x^2 \times x^3 \mid x^1 \in R^1; x^2 \in R^2; x^3 = \textit{the}$ mapping of R^3 such that $x^3(x^1) = x^2\}$, is immediately verified to be T^R and of type $T_{(3)}$.XVIII.

(*g*) Parts (*d*), (*f*), with remark (*b*), prove the existence of the desired types of plc. for any prescribed lines. The possibilities within each case depend on the choice of the set R^3 as described in part (*f*).

(*h*) Any additional structure on the lines might give rise to a subclassification of the T^R plc. 's.

6. Characteristic Properties of Graphic Plurilinearities.

6.0 THEOREM: *Let S^i $(i \in I)$ be projective spaces, let $\boldsymbol{T}$ be a plc. on $\boldsymbol{S}\langle I\rangle$. Then $\boldsymbol{T}$ is a g. pll. if and only if it is of either type T^A, T^B, T^C, or T^D.*

In other words each of the sets of conditions: (P.1-2), (A.1-3), (B.1-2), (C.1-2), (D.1-3) is equivalent to each other.

Proof. The statement is equivalent to the propositions that each one of the sets given above implies the next one, and that (D.1-3) implies (P.1-2) or (P^0, P'). These propositions, together with some auxiliary lemmas, are proved simultaneously, by induction on t, s (cfr. Remark 3.7), in the following subsections 6.1-6.6.

6.1. Let $\boldsymbol{T}$ be a g. pll. on $\boldsymbol{S}\langle I\rangle$.

6.1.1. LEMMA: *If $i \in I$, S^{*i} is a hyperplane of S^i, and $\boldsymbol{T}^* = \boldsymbol{R}\langle S^{*i}; \boldsymbol{T}\rangle$, then $\boldsymbol{T}^*$ is a g. pll.*

Proof. If $t = 1$, then the property is trivial. If $t \geqslant 2$, let notations be as in 2.2, with the additional condition $\boldsymbol{P}\langle\{i\}; \boldsymbol{y}^\triangle\rangle \subseteq S^{*i}$ if $i \in J^\triangle$. Set $\boldsymbol{F}^* = \boldsymbol{P}\langle\{h, k\}; \boldsymbol{T}^*(\boldsymbol{y}^\triangle)\rangle$.

If $i \neq h$ and $i \neq k$, then $\boldsymbol{F}^* = \boldsymbol{F}$; $\boldsymbol{F}^*$ is therefore a graphic linearity. If $i = h$ or $i = k$, say $i = h$, then $F^* = R\langle S^{*i}; F\rangle$ and F^* is again a graphic linearity by Thm. 2.1 of [2]. Therefore (P') is valid for T^*.

6.1.2. LEMMA: *If* $t \geqslant 2$, $i \in I$, $y^i \in S^i$, *and* $T^* = T(y^i)$, *then* T^* *is a g. pll.*

Proof. Notations as in 2.2. If $t = 2$, then the statement follows from (P.1) and 3.6.1. Suppose $t > 2$. For every choice of h, k, J^*, y^* ($\{h, k\} \subseteq I - \{i\}$; $h \neq k$; $J^* \subseteq I - \{i, h, k\}$; $y^* \in S\langle J^*\rangle$) set $y^\Delta = y^i \times y^*$. Then $T^*(y^*) = T(y^\Delta)$ and $P\langle\{h, k\}; T^*(y^*)\rangle$, being the same as F, is a graphic linearity by 2.2. Therefore T^* verifies (P').

6.1.3. PROPOSITION: *Every g. pll.* T *is a* T^A *plc.*

Proof. (*a*) Condition (P.1) (with $J^\Delta = \varnothing$) implies (A.1).

(*b*) Notations as in 3.1. By 6.1.2 [respectively 6.1.1] $R\langle S^{\star i}; T\rangle$ $[T(y^i)]$ is a g. pll. and, by induction on s [on t], a T^A plc.; this is precisely (A.2) [(A.3)].

6.2. PROPOSITION: *Every* T^A *plc.* T *is a* T^B *plc.*

Proof. Notations as in 3.2. (*a*) Condition (B.1) is satisfied as a particular case of (A.1).

(*b*) Suppose $S^{*j} \subset S^j$ for some j $(j \in I)$; therefore a hyperplane $S^{\star j}$ exists, such that $S^{*j} \subseteq S^{\star j} \subset S^j$. By (A.2), $R\langle S^{\star j}; T\rangle$ is T^A; by induction on s, it is T^B. Since $R\langle S^{*i}\,(i \in I); T\rangle = R\langle S^{*i}\,(i \in I); R\langle S^{\star j}; T\rangle\rangle$, the validity of (B.2) for $R\langle S^{\star i}; T\rangle$ implies its validity for T.

(*c*) (A.3) and induction on t imply immediately (B.3).

6.3. PROPOSITION: *Every* T^B *plc.* T *is a* T^C *plc.*

Proof. (*a*) Suppose $s^i > 0$ for every i $(i \in I)$. (*aa*) If $s^j > 1$ for some j $(j \in I)$, then for every choice of the lines R^i ($R^i \subseteq S^i$; $i \in I$) the plc. $R\langle R^i (i \in I); T\rangle$ is T^B by (B.2), thence T^C by induction on s; thus (C.1) holds. (*ab*) Suppose $s^i = 1$ for every i $(i \in I)$. Then for every i and for every x^i ($i \in I$; $x^i \in S^i$) the plc, $T(x^i)$ is T^B by (B.2), thence T^R by induction on t. In other words. T satisfies (R.2); since (B.1) implies (R.1), T is T^R and satisfies therefore (C.1).

(*b*) Suppose $t \geqslant 2$ and $s^i = 0$ for some i $(i \in I)$. Then, for every such i, $T(S^i)$ satisfies evidently (B.1) and (B.2), thence it is T^B. By induction on t, it is T^C, thus (C.2) is valid for T.

6.4.1. LEMMA: *If* $\boldsymbol{T}^*$ *is a* T^D *plc. on* $\boldsymbol{S}\langle I - \{1\}\rangle$, *if* S^1 *is a point, and if* $\boldsymbol{T} = S^1 \times \boldsymbol{T}^*$, *then* $\boldsymbol{T}$ *is a* T^D *plc. on* $\boldsymbol{S}\langle I\rangle$.

Proof. (*a*) Since $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}\rangle$ is the same as $\boldsymbol{P}\langle\{i\}; \boldsymbol{T}^*\rangle$ for $i \neq 1$, and since it coincides with S^1 if $i = 1$, then property (D.1) is valid for $\boldsymbol{T}$.

(*b*) Property (D.2) follows, by induction on s, from the identity $\boldsymbol{R}\langle R^i; \boldsymbol{T}\rangle = \boldsymbol{R}\langle R^i; S^1 \times \boldsymbol{T}^*\rangle = S^1 \times \boldsymbol{R}\langle R^i; \boldsymbol{T}^*\rangle$ and from (D.2) applied to $\boldsymbol{T}^*$.

(*c*) When i is taken equal to 1, then property (D.3) is valid by construction. When $i \neq 1$, then $\boldsymbol{T}(z^i) = (S^1 \times \boldsymbol{T}^*)(z^i) = S^1 \times \boldsymbol{T}^*(z^i)$; property (D.3) for $\boldsymbol{T}$ follows from (D.3) applied to $\boldsymbol{T}^*$ and from the hypothesis of induction on t (or trivially if $t = 2$).

6.4.2. PROPOSITION: *Every* T^C *plc.* $\boldsymbol{T}$ *is a* T^D *plc.*

Proof. (*a*) Suppose $s^i = 1$ for every i ($i \in I$); then, by (C.1), $\boldsymbol{T}$ is T^R, and (R.1) implies (D.1).

(*b*) Suppose $s^i > 1$ for some i ($i \in I$), let R^i be a line of S^i, and set $\boldsymbol{T}^* = \boldsymbol{R}\langle R^i; \boldsymbol{T}\rangle$. (C.1) is obviously true for $\boldsymbol{T}^*$. As for (C.2): suppose $s^j = 0$ for some j ($j \in I - \{i\}$); then $\boldsymbol{T}(S^j)$ is T^C by (C.2), thence a g. pll. by induction. Since $\boldsymbol{T}^*(S^j) = \boldsymbol{R}\langle R^i; \boldsymbol{T}(S^j)\rangle$, $\boldsymbol{T}^*(S^j)$ is a g. pll. by property (D.2) applied to $\boldsymbol{T}(S^j)$, therefore it is T^D, and (D.2) is valid for $\boldsymbol{T}$.

(*c*) Suppose $t \geqslant 2$, $s^j \leqslant 1$ for every j ($j \in I$), $i \in I$, $z^i \in S^i$. If $s^j = 1$ for every j ($j \in I$), then $\boldsymbol{T}$ is T^R by (C.1), so is $\boldsymbol{T}(z^i)$ by (R.2); by induction on t, $\boldsymbol{T}(z^i)$ is a g. pll., thence T^D. If $s^i = 0$ for some i ($i \in I$), then $\boldsymbol{T}(S^i)$ is T^C by (C.2), thence T^D by induction, and $\boldsymbol{T}$ is T^D by 6.4.1. (D.3) is therefore valid for $\boldsymbol{T}$.

6.5. Let $\boldsymbol{T}$ be a T^D plc. on $\boldsymbol{S}\langle I\rangle$.

6.5.1. LEMMA: *Suppose*: $J^{\square} = \{i \mid i \in I; s^i \geqslant 1\}$; $J^{\star} \subseteq J^{\square}$; *$R^j$ is a line of S^j for every j ($j \in J^{\star}$); $y^{\Delta j} \in S^j$ for every j ($j \in I - J^{\star}$). Then* $\boldsymbol{R}\langle R^j\ (j \in J^{\star}); \boldsymbol{T}\rangle$ *and* $\boldsymbol{R}\langle R^j\ (j \in J^{\star}); \boldsymbol{T}(y^{\Delta}\langle I - J^{\star}\rangle)\rangle$ *are* T^D.

Proof. The first part is a repeated application of (D.2) (or trivial if $s^i \leqslant 1$). As for the second part, it is a repeated application of (D.3) if $s^i \leqslant 1$ for all i's; otherwise it is reduced to the case $s^i \leqslant 1$ by choosing the line R^j such that $y^{\Delta j} \in R^j \subseteq S^j$ ($j \in J^{\square} - J^{\star}$), by considering $\boldsymbol{R}\langle R^j\ (j \in J^{\square}); \boldsymbol{T}\rangle$, which is T^D, and by applying the identity $\boldsymbol{R}\langle R^j\ (j \in J^{\square});\ \boldsymbol{T}(\boldsymbol{y}^{\triangle}\langle I - J^{\star}\rangle)\rangle = (\boldsymbol{R}\langle R^j\ (j \in J^{\square});\ \boldsymbol{T}\rangle)(\boldsymbol{y}^{\triangle}\langle I - J^{\star}\rangle)$.

6.5.2. Lemma: *If $k \in I$, if $J^{\triangle} \subseteq I - \{k\}$, and if $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$, then $\boldsymbol{P}\langle\{k\};\ \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle$ is a subspace of S^k.*

Proof. The statement is equivalent to the fact that, if y^k and z^k are distinct points of $\boldsymbol{P}\langle\{k\};\ \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle$, then $[y^k, z^k] \subseteq \boldsymbol{P}\langle\{k\};\ \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle$. In fact there are in $\boldsymbol{T}$ two elements $\boldsymbol{y}$ and $\boldsymbol{z}$, the projections of which are y^k and z^k on S^k, and $\boldsymbol{y}^{\triangle}$ (for both) on $\boldsymbol{S}\langle J^{\triangle}\rangle$. By setting $S^{*i} = [y^i, z^i]$ for every i $(i \in I)$, the plc. $\boldsymbol{T}^*$, given by $\boldsymbol{T}^* = \boldsymbol{R}\langle \boldsymbol{S}^*\langle I\rangle;\ \boldsymbol{T}\rangle$, is T^D by Lemma 6.5.1. Then $\boldsymbol{T}^*(\boldsymbol{y}^{\triangle})$ is T^D by (D.3). The application of (D.1) to $\boldsymbol{T}^*(\boldsymbol{y}^{\triangle})$ gives $S^{*k} = \boldsymbol{P}\langle\{k\};\ \boldsymbol{T}^*(\boldsymbol{y}^{\triangle})\rangle \subseteq \boldsymbol{P}\langle\{k\};\ \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle$; the statement follows.

6.5.3. Lemma: *If $J^{\triangle} \subset I$, if $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$, and if $\boldsymbol{T}^{\triangle} = \boldsymbol{T}(\boldsymbol{y}^{\triangle})$, then $\boldsymbol{T}^{\triangle}$ is T^D.*

Proof. Notations as in 3.5. There is nothing to prove if $J^{\triangle} = \varnothing$. Suppose $J^{\triangle} \neq \varnothing$. (*a*) (D.1) is true for $\boldsymbol{T}^{\triangle}$ as a particular case of Lemma 6.5.2.

(*b*) Suppose $s^i > 1$ for some i $(i \in I - J^{\triangle})$. Then $\boldsymbol{R}\langle R^i;\ \boldsymbol{T}^{\triangle}\rangle = (\boldsymbol{R}\langle R^i;\ \boldsymbol{T}\rangle)(\boldsymbol{y}^{\triangle})$, which is T^D by (D.2) and by induction on s. Therefore (D.2) is true for $\boldsymbol{T}^{\triangle}$.

(*c*) If $t \geqslant 2$ and $s^i \leqslant 1$ for every i $(i \in I - J^{\triangle})$, then $\boldsymbol{T}^{\triangle}(z^i)$ is T^D by Lemma 6.5.1. Therefore (D.3) is valid for $\boldsymbol{T}^{\triangle}$.

6.5.4. Lemma: *If $\varnothing \subset J^{\square} \subseteq I$ and if $\boldsymbol{T}^{\square} = \boldsymbol{P}\langle J^{\square};\ \boldsymbol{T}\rangle$, then $\boldsymbol{T}^{\square}$ is T^D.*

Proof. Notations as in 3.5. (*a*) (D.1) is valid for $\boldsymbol{T}^{\square}$ as a particular case of Lemma 6.5.2 and as a consequence of the identity $\boldsymbol{P}\langle\{k\};\ \boldsymbol{T}^{\square}\rangle = \boldsymbol{P}\langle\{k\};\ \boldsymbol{T}\rangle$ $(k \in J^{\square})$.

(*b*) Suppose $s^i > 1$ for some i $(i \in J^{\square})$; then $\boldsymbol{R}\langle R^i;\ \boldsymbol{T}^{\square}\rangle = \boldsymbol{P}\langle J^{\square};\ \boldsymbol{R}\langle R^i;\ \boldsymbol{T}\rangle\rangle$ and (D.2) is true for $\boldsymbol{T}^{\square}$ by induction on s and as a consequence of the application of (D.2) to $\boldsymbol{T}$.

(*c*) Suppose $|J^{\square}| \geqslant 2$ and $s^j \leqslant 1$ for every j $(j \in J^{\square})$; then $\boldsymbol{T}^{\square}(z^i) = \boldsymbol{P}\langle J^{\square} - \{i\};\ \boldsymbol{T}(z^i)\rangle$. By Lemma 6.5.3, $\boldsymbol{T}(z^i)$ is T^D and so is $\boldsymbol{T}^{\square}(z^i)$ by induction on t. Thence (D.3) is valid for $\boldsymbol{T}^{\square}$.

6.5.5. Lemma: *Suppose: $J^{\star} \subseteq J^{\square} \subseteq I$; $J^{\triangle} \subseteq I - J^{\square}$; $J^{\square} \neq \varnothing$; $\boldsymbol{y}^{\triangle} \in \boldsymbol{S}\langle J^{\triangle}\rangle$; R^j is a line of S^j for every j $(j \in J^{\star})$. Then $\boldsymbol{R}\langle R^j\ (j \in J^{\star});\ \boldsymbol{P}\langle J^{\square};\ \boldsymbol{T}(\boldsymbol{y}^{\triangle})\rangle\rangle$ is T^D.*

Proof. Corollary of Lemmas 6.5.3, 6.5.4, and 6.5.1.

6.6.1. LEMMA: *A bicorrespondence between two lines is* T^D *if and only if it is* T^R.

Proof. In the present instance, conditions (R.1), (R.2) coincide respectively with (D.1), (D.3); (D.2) is inapplicable.

6.6.2. PROPOSITION: *Every* T^D *plc.* $\boldsymbol{T}$ *is a g. pll.*

Proof. Notations as in 2.1. Suppose $t \geqslant 2$ and define $\boldsymbol{F}$ as in 2.1(✦). All amounts to proving that (P') holds, i. e. that $\boldsymbol{F}$ is a graphic linearity.

In every case $\boldsymbol{P}\langle\{h\}; \boldsymbol{F}\rangle$ [respectively, $\boldsymbol{P}\langle\{k\}; \boldsymbol{F}\rangle$] is a subspace of S^h [S^k] by 6.5.5, 6.5.2, and by Thm. 10.2 of [2]. This proves the statement completely if $s^h s^k = 0$. Suppose now $s^h s^k \geqslant 1$ and let R^h, R^k be lines in S^h, S^k respectively. By 6.5.5, $\boldsymbol{R}\langle R^h, R^k; \boldsymbol{F}\rangle$ is T^D; by 6.6.1, it is T^R, thence of one of the types $T_{(2)}$.I-VI of Sect. 4. These types coincide with 10.1(*a*)-(*f*) of [2]; thus the hypotheses of Thm. 10.2 of [2] are satisfied and $\boldsymbol{F}$ is a graphic linearity.

7. Properties of Graphic Plurilinearities.

Notations as in Sect. 6. The following properties of g. pll.'s follow from Thm. 6.0.

7.1. THEOREM: *If*: $\boldsymbol{T}$ *is a g. pll. on* $\boldsymbol{S}\langle I\rangle$; $J^{\triangle} \subset I$; *for every* j $(j \in J^{\triangle})$, S^{*j} *is a subspace of* S^j; $\varnothing \subset J^{\square} \subseteq I - J^{\triangle}$; $J^{\text{☆}} \subseteq J^{\square}$; *for every* j $(j \in J^{\text{☆}})$, $S^{\text{☆}j}$ *is a subspace of* S^j; *then* $\boldsymbol{R}\langle S^{\text{☆}j}\ (j \in J^{\text{☆}}); \boldsymbol{P}\langle J^{\square}; \boldsymbol{T}(\boldsymbol{S}^*\langle J^{\triangle}\rangle)\rangle\rangle$ *is a g. pll.*

Proof. By (B.2), $\boldsymbol{R}\langle S^{*j}\ (j \in J^{\triangle}); \boldsymbol{T}\rangle$ is a g. pll. Since $\boldsymbol{T}(\boldsymbol{S}^*\langle J^{\triangle}\rangle) = \boldsymbol{P}\langle I - J^{\triangle}; \boldsymbol{R}\langle S^{*j}\ (j \in J^{\triangle}); \boldsymbol{T}\rangle\rangle$, $\boldsymbol{T}(\boldsymbol{S}^*\langle J^{\triangle}\rangle)$ is a g. pll. by 6.5.4. Then 6.5.4 and (B.2) imply the statement.

7.2. THEOREM: *If* $\boldsymbol{T}$ *is a g. pll. on* $\boldsymbol{S}\langle I\rangle$, *and if* $k \in I$, *then* $\boldsymbol{P}\langle\{k\}; \boldsymbol{T}\rangle$ *is a subspace of* S^k.

Proof. (P.1).

7.3. THEOREM: *If, for every* i $(i \in I)$, S^{*i} *is a subspace of* S^i, *and if* $\boldsymbol{T}$ *is a g. pll. on* $\boldsymbol{S}^*\langle I\rangle$, *then* $\boldsymbol{T}$ *is a g. pll. on* $\boldsymbol{S}\langle I\rangle$.

Proof. By induction (cfr. Remark 3.7); notations as in 3.4.

Properties (D.1), (D.3) are evidently satisfied.

As for (D.2), if $R^i \subseteq S^{*i}$ or if $R^i \cap S^{*i} = \varnothing$, then the property is trivial. Suppose $R^i \cap S^{*i} = \{y^i\}$ ($y^i \in R^i$; $i \in I$). Then $T(y^i)$ is a g. pll. on $S\langle I - \{i\}\rangle$ by 7.1 and by induction on t. By 6.4.1 and by the identity $R\langle R^i; T\rangle = \{y^i\} \times T(y^i)$, $R\langle R^i; T\rangle$ is a g. pll. on $\{y^i\} \times S\langle I - \{i\}\rangle$, thence, by induction on s, it is a g. pll. on $R^i \times S\langle I - \{i\}\rangle$. (D.2) is therefore valid.

7.4. THEOREM: *If*: $I = I' \cup I''$; $I' \cap I'' = \varnothing$; $T^{\star}$ *and* T^* [$T^{\star\star}$ *and* T^{**}] *are g. pll.'s on* $S\langle I'\rangle$ [*on* $S\langle I''\rangle$]; $T^* \subseteq T^{\star}$; $T^{**} \subseteq T^{\star\star}$; $T = (T^* \times T^{\star\star}) \cup (T^{\star} \times T^{**})$; *then* T *is a g. pll. on* $S\langle I\rangle$.

Proof. Notations as in 2.1. For $t = 1$ the statement is trivial; suppose $t \geqslant 2$ and define F as in 2.1(✦). All amounts to proving that (P') is satisfied, i. e. that F is a graphic linearity.

(a) Suppose first $J^{\Delta} = \varnothing$. If $\{h, k\} \subseteq I'$ (and similarly if $\{h, k\} \subseteq I''$), then $F = P\langle\{h, k\}; T^{\star}\rangle$, which is a graphic linearity as a consequence of the application of (P') to $T^{\star}$. If $h \in I'$ and $k \in I''$, then F is a graphic linearity because of the following facts: (i) $P\langle\{h\}; T^{\star}\rangle$, $P\langle\{h\}; T^*\rangle$, $P\langle\{k\}; T^{\star\star}\rangle$, and $P\langle\{k\}; T^{**}\rangle$ are subspaces of S^h, S^k respectively; (let them be temporarily denoted by W, X, Y, Z); (ii) $F = (X \times Y) \cup (W \times Z)$; (iii) $W \supseteq X$, $Y \supseteq Z$; (iv) Thm. 8.9 of [2].

(b) If $J^{\Delta} \neq \varnothing$, set $y^{\Delta} = y' \times y''$, with $y' \in S\langle I' \cap J^{\Delta}\rangle$, $y'' \in S\langle I'' \cap J^{\Delta}\rangle$. Then the statement follows from the application of part (a) to the images of y', y'', and y^{Δ} under the plc.'s involved.

7.4.1. COROLLARY: $T^{\star} \times T^{\star\star}$ *is a g. pll.*

Proof. From 7.4, when $T^* = T^{\star}$.

7.4.2. COROLLARY: *If*: T *is a g. pll. on* $S\langle I\rangle$; $I = I' \cup I''$; $I' \cap I'' = \varnothing$; $y = y' \times y'' \in T$; $y' \in S\langle I'\rangle$; $y'' \in S\langle I''\rangle$; *then* $(\{y'\} \times T(y')) \times (\{y''\} \times T(y''))$ *is a g. pll.*

Proof. From 7.4, when $T^{\star} = T(y'')$, $T^* = \{y'\}$, $T^{\star\star} = T(y')$, $T^{**} = \{y''\}$.

7.5. THEOREM: *Notations as in* 2.1. *For the set* F *given by* $F = \{F(x^k) \mid x^k \in R^k\}$, *there are the following three possibilities*:

(a) $F(x^k)$ *is constant for all the points* x^k *of* R^k;

(b) *there exists exactly one point* x^{*k} *on* R^k *such that*

(i) $\boldsymbol{F}(x^k)$ *is constant for all the points* x^k *of* $R^k - \{x^{*k}\}$, *and* (ii) $\boldsymbol{F}(x^k) \subset \subset \boldsymbol{F}(x^{*k})$;

(*c*) F *is a pencil and the mapping* $f\colon R^k \to F$ *defined by* $fx^k = \boldsymbol{F}(x^k)$, *is bijective.*

Proof. (P') and Theorem 3.2 of [2].

7.6. An immediate consequence of Thm. 10.2 of [2] is the following.

THEOREM: *Notations as in. Sect.* 2. *If* $\boldsymbol{T} \subseteq \boldsymbol{S}\langle I \rangle$, *then* $\boldsymbol{T}$ *is a g. pll. if and only if one of the following conditions is satisfied*:

(P^0) $t = 1$ *and* $\boldsymbol{T}$ *is a subspace of* S^1;

(P'') $t > 1$ *and for every choice of* h, k, J^{Δ}, *and* $\boldsymbol{y}^{\Delta}$ *as in* 2.1:

(i) *either* $s^h s^k = 0$ *and* $\boldsymbol{P}\langle\{h\}; \boldsymbol{T}(\boldsymbol{y}^{\Delta})\rangle$ *and* $\boldsymbol{P}\langle\{k\}; \boldsymbol{T}(\boldsymbol{y}^{\Delta})\rangle$ *are subspaces of* S^h, S^k *respectively*;

(ii) *or for every choice of the lines* R^h, R^k *in* S^h, S^k *respectively, the plc.* $\boldsymbol{G}$ *given by*

$$\boldsymbol{G} = \boldsymbol{R}\langle R^h, R^k; \boldsymbol{P}\langle\{h, k\}; \boldsymbol{T}(\boldsymbol{y}^{\Delta})\rangle\rangle$$

is of one of the types $T_{(2)}$.I-VI *of Sect.* 4.

7.7. THEOREM: *The classification of the g. pll.'s between two or three projective lines is given in Sects.* 4 *and* 5 *respectively.*

Proof. 4, 5, and 6.0.

8. T^J Pluricorrespondences.

In this section another special type of plc.'s among projective spaces S^i $(i \in I)$ is considered, as well as its relationships to g. pll.'s.

For every i $(i \in I)$ let $S^{\star i}$ be a point or a projective line $\boldsymbol{R}^i$; let L be defined by $L = \{i \mid i \in I;\ \dim S^{\star i} = 0\}$.

8.1.1. Suppose: $I = J^0 \cup K^0$, $J^0 \cap K^0 = \varnothing$, $K^0 \supseteq L$, $\boldsymbol{z}^0 \in \in \boldsymbol{S}\langle K^0 \rangle$. Furthermore, if $J^0 \neq \varnothing$, a collection of bijective mappings $\tau^0_{jk}\colon R^j \to R^k$ is given, such that $\tau^0_{jk} \circ \tau^0_{ij} = \tau^0_{ik}$ $(i \in J^0)$, $j \in J^0$, $k \in J^0$). This is equivalent to fixing j^0 $(j^0 \in J^0)$, prescribing arbitrary bijective mappings $\sigma^0_j\colon R^{j_0} \to R^j$ $(j \in J^0;\ \sigma^0_{j_0} = 1)$, and setting $\tau^0_{jk} = \sigma^0_k \circ (\sigma^0_j)^{-1}$.

DEFINITION: With notations as above, a T^I *plc.* is a plc. U^0, on $S\langle I\rangle$, given by,

$$U^0 = \{\{z^0\} \times (\times_{j\in J_\bullet}\sigma_j^0 x^{j^\bullet}) \mid x^{j^\bullet} \in R^{j^\bullet}\} .$$

In other words, U^0 consists of the elements obtained by choosing fixed points on some of the $S^{\star i}$'s and, on the remaining ones, variable points which correspond to each other by means of prescribed coherent bijective mappings.

8.1.2. A T^J plc. can now be defined as follows.

Let the following elements be given: a partition $I = K \cup \cup (\bigcup_{\alpha\in A} J^\alpha)$ $(K \supseteq L; K \cap J^\alpha = \varnothing;\ J^\alpha \neq \varnothing;\ J^\alpha \cap J^\beta = \varnothing$ for $\alpha \in A$, $\beta \in A$, $\alpha \neq \beta$; $0 \leqslant |A| =$ integer); two elements a $(= a\langle I\rangle)$ and b $(= b\langle I\rangle)$ in $S^{\star}\langle I\rangle$, such that $a^i = b^i$ whenever $i \in K$ and $a^i \neq b^i$ whenever $i \in I - K$. Furthermore, if $K \subset I$ (i. e. if $A \neq \varnothing$): a proper order is given in A, say $\prec$ (for instance by setting $A = \{1^*, 2^*, ..., w^*\}$, with $1^* \prec 2^* \prec ... \prec w^*$); for each α $(\alpha \in A)$ let a collection of coherent mappings τ^α_{jh} be given, with the same properties as in 8.1.1, provided $\square^0$ is everywhere replaced by $\square^\alpha$, and with the additional condition $\tau^\alpha_{jh} a^j = a^h$, $\tau^\alpha_{jh} b^j = b^h$ $(j \in J^\alpha;\ h \in J^\alpha)$.

For every α $(\alpha \in A)$ set: $K^\alpha = I - J^\alpha$, $z^\alpha = a\langle K \cup J^{1^*} \cup ... \cup \cup J^{(\alpha-1)^*}\rangle \times b\langle J^{(\alpha+1)^*} \cup ... \cup J^{W^*}\rangle$, and let U^α be the T^I plc. obtained from Def. 8.1.1 by substituting $^\alpha$ for ^{0}.

DEFINITION: With notations as above, a T^J *plc.* is a plc. U on $S^{\star}\langle I\rangle$, given by $U = \{a\}$ if $K = I$ and by $U = \bigcup_{\alpha\in A} U^\alpha$ if $K \subset I$.

8.1.3. Evidently

LEMMA: *If $J^\triangle \subset I$, if U is T^J, if $P\langle\{j\}; U\rangle = \{y^j\} \subseteq S^{\star j}$ for every j $(j \in J^\triangle)$, then $U(y\langle J^\triangle\rangle)$ is T^J.*

8.2.1. Let the S^i*'s* $(i \in I)$ be projective spaces and suppose $a = a\langle I\rangle \in S\langle I\rangle$, $b = b\langle I\rangle \in S\langle I\rangle$.

DEFINITION: A *join* of a and b (in $S\langle I\rangle$) is any one of the T^J plc.'s obtained by applying Def. 8.1.2 to the entities a, b, $S^{\star i}$, with $S^{\star i} = [a^i, b^i]$ $(i \in I)$. The set K is necessarily given by $K = \{i \mid i \in I;\ a^i = b^i\}$; the other elements which appear in Def. 8.1.2 can be chosen arbitrarily, within the allowable possibilities.

8.2.2. DEFINITION: If, in addition to the data of Def. 8.2.1, a plc. $\boldsymbol{T}$ is given on $\boldsymbol{S}\langle I\rangle$, and if $\boldsymbol{a}$ and $\boldsymbol{b}$ lie in $\boldsymbol{T}$, then a *join of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in* $\boldsymbol{T}$ is (whenever existent) a join $\boldsymbol{U}$ of $\boldsymbol{a}$ and $\boldsymbol{b}$, satisfying the condition $\boldsymbol{U}\subseteq\boldsymbol{T}$.

Phrases like « $\boldsymbol{U}$ joins $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{T}$ » shall be used.

8.3. DEFINITION: A T^E *plc.* is a plc. $\boldsymbol{T}$ on $\boldsymbol{S}\langle I\rangle$, which satisfies the following condition:

(E) if $\boldsymbol{a}\in\boldsymbol{T}$ and $\boldsymbol{b}\in\boldsymbol{T}$, then there exists a T^J plc. which joins $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{T}$.

8.4. PROPOSITION: *Every* T^E *plc. is a g. pll.*

Proof. By Thm. 6.0 the statement is equivalent to the fact that if $\boldsymbol{T}$ is T^E, then it is T^B. In order to prove this, choose notations as in 3.3 and proceed by induction on s, t (cfr. Remark 3.7); the fact that T^E and T^B plc.'s are the same when $t = 1$ and when $s = 0$ is trivial.

(*a*) Suppose $s^i \leqslant 1$ for every i $(i\in I)$; set $X = \boldsymbol{P}\langle\{i\};\boldsymbol{T}\rangle$. If $X = \varnothing$ or if X is a point, then (B.1) is satisfied; otherwise, suppose $a^i\in X$, $b^i\in X$, $a^i\neq b^i$; then $\boldsymbol{a}$ and $\boldsymbol{b}$ can be found in $\boldsymbol{T}$, such that their projections be a^i and b^i. Let $\boldsymbol{U}$ join $\boldsymbol{a}$ and $\boldsymbol{b}$ in T, which implies $S^{\star i} = [a^i, b^i]\subseteq S^i$. Because of Def. 8.2, $S^{\star i}\subseteq X$, therefore X is a subspace of S^i and (B.1) holds.

(*b*) Suppose $s > 0$, $\{\boldsymbol{a}, \boldsymbol{b}\}\subseteq\boldsymbol{T}^* = \boldsymbol{R}\langle\boldsymbol{S}^*\langle I\rangle;\boldsymbol{T}\rangle$. If $\boldsymbol{U}$ joins $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{T}$, then the inclusions $[a^i, b^i]\subseteq S^{*i}$ imply that $\boldsymbol{U}\subseteq\boldsymbol{T}^*$. Therefore $\boldsymbol{T}^*$ is T^E and, by induction on s, it is a g. pll. Thence (B.2).

(*c*) If $t\geqslant 2$, $i\in I$, $\boldsymbol{P}\langle\{i\};\boldsymbol{T}\rangle = \{y^i\}$, $y^i\in S^i$, then $a^i = b^i = y^i$ for any two elements $\boldsymbol{a}\langle I\rangle$, $\boldsymbol{b}\langle I\rangle$ of $\boldsymbol{T}$. If $\boldsymbol{U}$ joins these elements in $\boldsymbol{T}$, then $\boldsymbol{U}(y^i)$ is T^J by 8.1.3 and it joins $\boldsymbol{a}\langle I - \{i\}\rangle$ and $\boldsymbol{b}\langle I - \{i\}\rangle$ in $\boldsymbol{T}(y^i)$. By induction on t, property (B.3) follows.

8.5. REMARK: Property 8.4 cannot be reversed, namely not every g. pll. is a T^E plc. A counterexample is the following. Suppose $R^1 = \{a, b, c, d\}$ and consider it as a projective line; let R^2, R^3 be two copies of R^1; for brevity let pqr denote the element $p\times q\times r$, with $p\in R^1$, $q\in R^2$, $r\in R^3$. Let $\boldsymbol{T}$ be given by $\boldsymbol{T} = \{aaa, bba, cca, dda, add, bad, cbd, dcd, acc, bdc, cac, dbc, abb,$

bcb, *cdb*, *dab*$\}$. $\boldsymbol{T}$ is a g. pll. of type $T_{(3)}$.XVIII, however the two elements *bba*, *cac* cannot be joined, in $\boldsymbol{T}$, by any T^J plc.

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