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Some multiplication formulas

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SOME MULTIPLICATION FORMULAS

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Some years ago Rainville [5] proved the formula

$$(1) \qquad \left(\frac{\sin\beta}{\sin\alpha}\right)^n P_n(\cos\alpha) = \sum_{r=0}^{n}\binom{n}{r}\left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r P_{n-r}(\cos\beta)\,,$$

where $P_n(x)$ is the Legendre polynomial. The writer [1] showed that

$$(2) \left(\frac{\sin\beta}{\sin\alpha}\right)^n C_n^\lambda(\cos\alpha) = \sum_{r=0}^{n}\binom{2\lambda+n-1}{r}\left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r C_{n-r}^\lambda(\cos\beta)\,,$$

where $C_n^\lambda(x)$ is the ultraspherical polynomial defined by

$$(1-2xz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) z^n\,.$$

For $\lambda = 1/2$, (2) reduces to (1).

Chatterjea [2] has recently proved the following formulas:

$$(3) \qquad (\cot\alpha\sin\beta)^n \Phi_n(x\tan\alpha) =$$

$$= \sum_{r=0}^{n}\binom{n}{r}\left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r (\cos\beta)^{n-r}\Phi_{n-r}(x\cot\alpha\tan\beta)\,,$$

where $\Phi_n(x)$ is defined by

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$$e^t I_0(xt) = \sum_{n=o}^{\infty} \Phi_n(x) \frac{t^n}{n!},$$

and

$$(4) \qquad (\cot \alpha \sin \beta)^n L_n(\tan \alpha)$$
$$= \sum_{r=0}^{n} \binom{n}{r} \left(\frac{(\sin(\beta - \alpha)}{\sin \alpha} \right)^r (\cos \beta)^{n-r} L_{n-r}(\tan \beta),$$

where $L_n(x)$ is the Laguerre polynomial.

These results and in particular (3) suggest the consideration of the functional equation

$$(5) \qquad (\cot \alpha \sin \beta)^n f_n(x \tan \alpha)$$
$$= \sum_{r=0}^{n} \binom{n}{r} \left(\frac{\sin(\beta - \alpha)}{\sin \alpha} \right)^r (\cos \beta)^{n-r} f_n(x \tan \beta).$$

If we take $\beta = \pi/4$ it is easily verified that (5) reduces to

$$(\cot \alpha)^n f_n(x \tan \alpha) = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{\cos \alpha - \sin \alpha}{\sin \alpha} \right)^r f_n(x).$$

For $\lambda = \tan \alpha$ this becomes

$$(6) \qquad f_n(\lambda x) = \sum_{r=0}^{n} \binom{n}{r} (1 - \lambda)^r \lambda^{n-r} f_{n-r}(x).$$

For $x = 1$, (6) becomes

$$(7) \qquad f_n(\lambda) = \sum_{r=0}^{n} \binom{n}{r} (1 - \lambda)^r \lambda^{n-r} c_{n-r},$$

where

$$(8) \qquad c_r = f_r(1) \qquad (r = 0, 1, 2, \ldots).$$

Conversely if $f_n(\lambda)$ is defined by (7), where the c_r are arbitrary constants, it is easily verified that (6) holds. Indeed we have

$$f_n(\lambda x) = \sum_{r=0}^{n} \binom{n}{r} [1 - \lambda + \lambda(1 - x)]^r (\lambda x)^{n-r} c_{n-r}$$

$$= \sum_{r=0}^{n} \binom{n}{r} (\lambda x)^{n-r} \sum_{s=0}^{r} \binom{r}{s} (1-\lambda)^s \lambda^{r-s} (1-x)^{r-s} c_{n-r}$$

$$= \sum_{s=0}^{n} \binom{n}{s} (1-\lambda)^s \lambda^{n-s} \sum_{r=s}^{n} \binom{n-s}{r-s} (1-x)^{r-s} x^{n-r} c_{n-r}$$

$$= \sum_{s=0}^{n} \binom{n}{s} (1-\lambda)^s \lambda^{n-s} f_{n-s}(x) .$$

We see therefore that (6) is equivalent to (7) and that the general solution of (6) is given by (7) with arbitrary c_r. Moreover it follows from (6) with x replaced by μx and λ replaced by λ/μ that

$$f_n(\lambda x) = \sum_{r=0}^{n} \binom{n}{r} \left(1 - \frac{\lambda}{\mu}\right)^r \left(\frac{\lambda}{\mu}\right)^{n-r} f_{n-r}(\mu x) ,$$

$$\mu^n f_n(\lambda x) = \sum_{r=0}^{n} \binom{n}{r} (\mu - \lambda)^r \lambda^{n-r} f_{n-r}(\mu x) .$$

If mow take $\lambda = \tan\alpha$, $\mu = \tan\beta$ we get (5). Thus (5) is equivalent to (7).

We recall that Feldheim [3] has proved that the Laguerre polynomials $L_n^{(\alpha)}(x)$ are the only orthogonal polynomials with a multiplacation theorem of the form

$$\lambda^n f_n(x/\lambda) = \sum_{r=0}^{n} A_{nr} (\lambda - 1)^{n-r} f_n(x) .$$

It may be of interest to note that (7) implies

$$\sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = e^{(1-x)z} \sum_{n=0}^{\infty} c_n \frac{(xz)^n}{n!}$$

or, if we prefer,

$$\sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = e^z \sum_{n=0}^{\infty} d_n \frac{(xz)^n}{n!} \tag{9}$$

with

$$d_n = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s} c_s = (c-1)^n , \tag{10}$$

where the last is symbolic. Rainville [4, p. 244] has noted that the polynomials $f_n(x)$ defined by (9) satisfy (6).

It follows from (9) that the polynomials

$$g_n(x) = x^n f_n\left(\frac{1}{x}\right) \tag{11}$$

constitute an Appell set, that is

$$g_n'(x) = ng_{n-1}(x) . \tag{12}$$

Also (6) becomes

$$g_n(\lambda x) = \sum_{r=0}^{n} \binom{n}{r} (\lambda - 1)^r x^r q_{n-r}(x) . \tag{13}$$

Conversely if the Appell set $\{g_n(x)\}$ is assigned then the polynomials $\{f_n(x)\}$ defined by (11) satisfy (5).

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