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NOTES ON A FORMULA OF CARLITZ

Nota () di S. K. CHATTERJEA (a Calcutta)*

1. - In a previous note ¹⁾, Carlitz has proved the formula

$$(1.1) \quad \left(\frac{\sin \beta}{\sin \alpha} \right)^n P_n^{\lambda+1/2}(\cos \alpha) = \\ = \sum_{r=0}^n \binom{2\lambda + n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r P_{n-r}^{\lambda+1/2}(\cos \beta)$$

where $P_n^\lambda(x)$ denotes the ultraspherical polynomial of degree n . For $\lambda = 0$, we get the formula of Rainville ²⁾;

$$(1.2) \quad \left(\frac{\sin \beta}{\sin \alpha} \right)^n P_n(\cos \alpha) = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r P_{n-r}(\cos \beta)$$

where $P_n(x)$ denotes the Legendre polynomial of degree n .

Following the method of Carlitz, one can obtain

$$(1.3) \quad (\cot \alpha \sin \beta)^n \Phi_n(\tan \alpha) = \\ = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta \Phi_{n-r}(\tan \beta)$$

(*) Pervenuta in redazione il 13 ottobre 1960.

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¹⁾ CARLITZ, L., Bull. Cal. Math. Soc., 51 (1959), pp. 132-133.

²⁾ RAINVILLE, E. D., Bull. Amer. Math. Soc., 51 (1945), pp. 268-271.

and

$$(1.4) \quad (\cot \alpha \sin \beta)^n L_n(\tan \alpha) = \\ = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta L_{n-r}(\tan \beta)$$

$$\alpha \neq (2n + 1)\pi/2, \quad \beta \neq (2n + 1)\pi/2; \quad n = 0, \pm 1, \pm 2, \dots$$

where $\Phi_n(x)$ denotes the polynomial of degree n , encountered by Karle (2, p. 269) in the study of the contribution to electron scattering of a freely rotating group within molecules comprising a jet of gas, and $L_n(x)$ denotes the Laguerre polynomial of degree n and of order zero.

The generating function

$$(1.5) \quad e^t I_0(tx) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!}$$

for the polynomials $\Phi_n(x)$ is obtained by Rainville (2, p 269).

From (1.5) we have

$$(1.6) \quad e^{t \cos \beta} I_0(tx \cos \beta) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!} \cos^n \beta$$

$\therefore$

$$(1.7) \quad \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!} \cos^n \beta = \\ = e^{t \sin \beta \cos \alpha / \sin \alpha} I_0(tx \cos \beta)$$

Writing $\beta = \alpha$ and $t = t \sin \beta / \sin \alpha$ in (1.6) we obtain

$$(1.8) \quad \sum_{n=0}^{\infty} \Phi_n(x) \left(\frac{\sin \beta}{\sin \alpha} \right)^n \frac{t^n}{n!} \cos^n \alpha = e^{t \sin \beta \cos \alpha / \sin \alpha} I_0(tx \cot \alpha \sin \beta)$$

Again using $x = x \cot \alpha \tan \beta$ in (1.7) we get

$$(1.9) \quad \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \sum_{n=0}^{\infty} \Phi_n(x \cot \alpha \tan \beta) \frac{t^n}{n!} \cos^n \beta = \\ = e^{t \sin \beta \cos \alpha / \sin \alpha} I_0(tx \cot \alpha \sin \beta)$$

Comparing (1.8) and (1.9) we have on equating coefficients,

$$(1.10) \quad (\cot \alpha \sin \beta)^n \Phi_n(r) = \\ = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta \Phi_{n-r}(x \cot \alpha \tan \beta)$$

Thus we get finally

$$(\cot \alpha \sin \beta)^n \Phi_n(\tan \alpha) = \\ = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta \Phi_{n-r}(\tan \beta)$$

$$\alpha \neq (2n+1)\pi/2, \quad \beta \neq (2n+1)\pi/2; \quad n=0, \pm 1, \pm 2, \dots$$

Secondly, starting with the generating function ³⁾

$$e^t J_0[2(tx)^{1/2}] = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}$$

we can prove, in precisely the same way, that

$$(\cot \alpha \sin \beta)^n L_n(\tan \alpha) = \\ = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta L_{n-r}(\tan \beta)$$

$$\alpha \neq (2n+1)\pi/2, \quad \beta \neq (2n+1)\pi/2; \quad n=0, \pm 1, \pm 2, \dots$$

2. - Special cases of the formulae just proved:

Putting $x = i \tan \alpha$ in (1.10) and observing that (2, p 269)

$$\Phi_n(x) = (1 - x^2)^{n/2} P_n[(1 - x^2)^{-1/2}],$$

³⁾ RAINVILLE, E. D., Bull. Amer. Math. Soc., 51 (1945), pp. 266-267.

we get

$$\left(\frac{\sin \beta}{\sin \alpha}\right)^n P_n(\cos \alpha) = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta - \alpha)}{\sin \alpha}\right]^r P_{n-r}(\cos \beta),$$

$$\therefore \Phi_n(i \tan \alpha) = \sec^n \alpha P_n(\cos \alpha).$$

which in (1.2).

Next using $\beta = 2\alpha$ we derive from (1.3)

$$(2.1) \quad (2 \cos^2 \alpha)^n \Phi_n(\tan \alpha) = \sum_{r=0}^n \binom{n}{r} \cos^{n-r} 2\alpha \Phi_{n-r}(\tan 2\alpha)$$

Further using $\cos 2\alpha = x$, (2.1) can be put in the form:

$$(2.2) \quad (1+x)^n \Phi_n\left(\sqrt{\frac{1-x}{1+x}}\right) = \sum_{r=0}^n \binom{n}{r} x^{n-r} \Phi_{n-r}(\sqrt{1-x^2}/x)$$

In like manner, we get from (1.4)

$$(2.3) \quad (1+x)^n L_n\left(\sqrt{\frac{1-x}{1+x}}\right) = \sum_{r=0}^n \binom{n}{r} x^{n-r} L_{n-r}(\sqrt{1-x^2}/x)$$

The polynomials $\Phi_n(x)$ are defined by (2, p 269)

$$\Phi_n(x) = \frac{1}{\pi} \int_0^\pi (1+x \cos \theta)^n d\theta.$$

$$(2.4) \quad \therefore (1+x)^n \Phi_n\left(\sqrt{\frac{1-x}{1+x}}\right) =$$

$$= \sum_{k=0}^n \binom{n}{k} x^k \cdot \frac{1}{\pi} \int_0^\pi \left(1 + \frac{\sqrt{1-x^2}}{x} \cos \theta\right)^k d\theta.$$

$$= \frac{1}{\pi} \int_0^\pi \left\{ \sum_{k=0}^n \binom{n}{k} (x + \sqrt{1-x^2} \cos \theta)^k \right\} d\theta.$$

$$= \frac{1}{\pi} \int_0^\pi [(1+x) + \sqrt{1-x^2} \cos \theta]^n d\theta,$$

which is analogous to the result ⁴⁾ for the Legendre polynomials.

3. - Some Definite Integrals:

when $\beta = 2\alpha$ and $\cos 2\alpha = x$, (1.1) can be put in the form:

$$(3.1) \quad 2^{n/2}(1+x)^{n/2}P_n^{\lambda+1/2}\left(\sqrt{\frac{1+x}{2}}\right) = \sum_{k=0}^n \binom{2\lambda+n}{n-k} P_k^{\lambda+1/2}(x)$$

From (3.1) and from the orthogonal property of ultraspherical polynomials we easily obtain

$$\begin{aligned} (3.2) \quad & \int_{-1}^1 (1+x)^{n/2}(1-x^2)^{\lambda} P_n^{\lambda+1/2}\left(\sqrt{\frac{1+x}{2}}\right) P_r^{\lambda+1/2}(x) dx \\ &= 0, \quad r > n \\ &= 2^{-(n+2\lambda)/2} \binom{2\lambda+n}{n-r} \pi \cdot \frac{\Gamma(2\lambda+r+1)}{\left(\lambda+r+\frac{1}{2}\right) \Gamma(r+1) \left[\Gamma\left(\lambda+\frac{1}{2}\right)\right]^2}, \\ & \quad 0 \leq r \leq n \quad (\lambda > -1) \end{aligned}$$

For $\lambda = 0$, we obtain

$$\begin{aligned} (3.3) \quad & \int_{-1}^1 (1+x)^{n/2} P_n\left(\sqrt{\frac{1+x}{2}}\right) P_r(x) dx \\ &= 0, \quad r > n \\ &= 2^{-(n-2)/2} \binom{n}{r} / (2r+1), \quad 0 \leq r \leq n. \end{aligned}$$

which was established by Bhonsle (4, p 9).

⁴⁾ BHONSLE, B. R., Ganita., 8 (1957), pp. 9-16.

4. - Recently Rainville (3, p 267) has obtained the following expression for the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ as a series of products of Laguerre polynomials of orders α and β and of different arguments:

$$(4.1) \quad P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n (-)^k \frac{\Gamma(n + \beta + 1) \Gamma(n + \alpha + 1)}{\Gamma(k + \beta + 1) \Gamma(n - k + \alpha + 1)} \cdot L_k^{(\beta)}\left(\frac{1+x}{2}\right) L_{n-k}^{(\alpha)}\left(\frac{1-x}{2}\right)$$

We at once obtain from (4.1)

$$(4.2) \quad P_n^{\lambda}(\cos 2\theta) \equiv P_n^{(\lambda-1/2, \lambda-1/2)}(\cos 2\theta) \cdot g_n \\ = \sum_{k=0}^n (-)^k \frac{\left[\Gamma\left(n + \lambda + \frac{1}{2}\right) \right]^2 \cdot g_n}{\Gamma\left(k + \lambda + \frac{1}{2}\right) \Gamma\left(n - k + \lambda + \frac{1}{2}\right)} \cdot L_k^{(\lambda-1/2)}(\cos 2\theta) L_{n-k}^{(\lambda-1/2)}(\sin^2 \theta)$$

$$\text{where } g_n = \frac{(2\lambda)_n}{(\lambda + 1/2)_n} ; \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}.$$

Again from (1.1) we obtain

$$(4.3) \quad (2 \cos \alpha)^n P_n^{\lambda}(\cos \alpha) = \sum_{k=0}^n \binom{2\lambda + n - 1}{r} P_{n-k}^{\lambda}(\cos 2\alpha) \\ \therefore P_n^{\lambda}(\cos 2\theta) = (2 \cos 2\theta)^{-n} \sum_{k=0}^n \binom{2\lambda + n - 1}{k} P_{n-k}^{\lambda}(\cos 4\theta).$$