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ON A TRACE PROBLEM

Nota (*) *di* JACQUES LOUIS LIONS (*a Nancy*) [1])

1. *Introduction*

In the plane $\{x_1, x_2\}$, let Ω be the open set $\{x_2 > 0\}$; let u_1 and u_2 be two functions given in Ω with the properties

$$(1.1) \qquad u_j \in L^2(\Omega), \quad \frac{\partial u_j}{\partial x_1}, \quad \frac{\partial u_j}{\partial x_2} \in L^2(\Omega)\ ^{2)}, \qquad j = 1, 2,$$

and

$$(1.2) \qquad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 .$$

Assuming that only (1.1) holds, one can define, as it is well known (Aronszajn [1], J. L. Lions [1], G. Prodi [1]), the « traces » $u_j(x_1, 0)$ of $u_j(x)$ on the boundary Γ of Ω, and $u_j(x_1, 0) \in H^{1/2}(\Gamma)$ i.e.

$$(1.3) \qquad \int (1 + | y_1 |) \, | \hat{u}_j(y_1, 0) |^2 dy_1 < \infty, \qquad j = 1, 2$$

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2) $L^2(\Omega)$, as usually, denotes the space of square integrable functions in Ω for the measure $dx = dx_1 dx_2$; the derivatives are taken in the sense of distributions in Ω cf. SCHWARTZ [1], SOBOLEV [1].

where $\hat{u}_j(y_1, 0)$ denotes the Fourier transform in x_1 of $u_j(x_1, 0)$.

If we assume now that *moreover* (1.2) *holds*, then $u_2(x_1, 0)$ satisfies *a stronger condition than* (1.3), as we shall see in sections 2 and 3 — where we shall define the best (i.e the smallest) spaces spanned by $u_1(x_1, 0)$ and $u_2(x_1, 0)$ —. The problem of characterizing the « traces spaces » spanned by $u_1(x_1, 0)$, $u_2(x_1, 0)$ arises in connection with the boundary value problems related to the Navier-Stokes equations.

Let us now replace in the above problem $L^2(\Omega)$ by $L^p(\Omega)$, $1 < p < \infty$, $p \neq 2$. Then, *without assuming* (1.2), the condition which plays the role of (1.3) has been found by E. Gagliardo [1] — Next, the Gagliardo's result was extended in Lions [3]. The combination of this last paper and the above remarks leads naturally to the following general problem.

Let E be a complex Banach space; if $e \in E$, $\| e \|$ will denote the norm of e in E. Let Λ be an unbounded operator in E satisfying:

$$(1.4) \begin{cases} \Lambda \text{ is the infinitesimal generator of a semi-group } G(t), \\ \text{strongly continuous in } E \text{ for } t \geqslant 0, \text{ and bounded}^{3)}. \end{cases}$$

We shall denote by $D(\Lambda)$ the domain of Λ; provided with the norm

$$\| e \|_{D(\Lambda)} = (\| e \|^2 + \| \Lambda e \|^2)^{1/2},$$

$D(\Lambda)$ becomes a Banach Space.

Let p and α be given with the following properties:

$$(1.5) \qquad 1 < p \leqslant \infty, (1/p) + \alpha = \vartheta \in]0, 1[.$$

We denote by $W(p, \alpha, D(\Lambda), E)$ (cf. Lions [3]) the space of function u satisfying,

$$(1.6) \qquad t^\alpha u \in L^p(0, \infty, D(\Lambda))^{4)},$$

3) For the semi-group theory, the reader is referred to Hille-Phillips [1].

4) i.e. u is measurable with values in $D(\Lambda)$ and $\left(\int_0^\infty \| t^\alpha u(t) \|^p_{D(\Lambda)} dt \right)^{1/p} < \infty$. Standard modification when $p = \infty$.

$$t^{\alpha} u' \in L^p(0, \infty, E)\ ^{5)}; \tag{1.7}$$

provided with the norm

$$\| u \|_{W(p,\alpha,D(\wedge),E)} = \max \left[\left(\int_0^\infty \| t^{\alpha} u(t) \|^p_{D(\wedge)} \, dt \right)^{1/p}, \left(\int_0^\infty \| t^{\alpha} u'(t) \|^p \, dt \right)^{1/p} \right],$$

$W(p, d, D(\wedge), E)$ becomes a Banach space. If E is a Hilbert space and if $p = 2$, it is a Hilbert space.

Now let u_1 and u_2 be given in $W(p, \alpha, D(\wedge), E)$, with the property:

$$\wedge u_1 + u_2' = 0 \, . \tag{1.8}$$

We set the following problem:

Problem 1.1: To characterize the spaces spanned by $u_1(0)$ and $u_2(0)$ when u_1 and u_2 span the space $W(p, \alpha, D(\wedge), E)$, subject to condition (1.8).

We give in section 2 necessary conditions; we conjecture these condition to be sufficient — and we prove in section 3 that this is true in the Hilbert case.

2. *Necessary conditions.*

2.1 Let us set

$$u_1(0) = f_1 \, , \quad u_2(0) = f_2 \, .$$

It follows from Lions [3] that f_1 and f_2 satisfy

$$\int_0^\infty \| t^{(\alpha-1)}(G(t)f_j - f_j) \|^p \, dt < \infty, \qquad j = 1, 2. \tag{2.1}$$

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5) $u' = (du/dt)$ is the derivative of u considered as a distribution in the open set $]\,0, \infty\,[$ with values in $D(\wedge)$. (Cf. Schwartz [2]).

We are going to prove that f_2 satisfies a stronger condition.

2.2 Let us denote by $\tilde{u}_j$, $j = 1, 2$, the function which equals $u_j(t)$ for $t > 0$ and 0 for $t < 0$. One has (taking the derivative d/dt in the distribution's sense on the whole line):

$$\frac{d}{dt}\tilde{u}_2 - \Lambda\,\tilde{u}_2 = f_2 \otimes \delta + \left(\frac{du_2}{dt}\right)^{\sim} - \Lambda\,\tilde{u}_2$$

where δ = measure of mass 1 at the origin.

But by (1.8), $\left(\frac{du_2}{dt}\right)^{\sim} = -\Lambda\,\tilde{u}_1$ hence

$$\frac{d\tilde{u}_2}{dt} - \Lambda\,\tilde{u}_2 = f_2 \otimes \delta - \Lambda\,(\tilde{u}_1 + \tilde{u}_2)\,. \tag{2.2}$$

The solution of the Cauchy problem (2.2) is given by

$$\tilde{u}_2(t) = G(t)f_2 - G * \Lambda\,(\tilde{u}_1 + \tilde{u}_2)\,, \tag{2.3}$$

(where we extend $G(t)$ by 0 for $t < 0$).

But since

$$G * (d/dt - \Lambda) = \delta \otimes I$$

where I = identity mapping from $D(\Lambda)$ into itself, we have

$$-\,G * \Lambda\,(\tilde{u}_1 + \tilde{u}_2) = -\,D * G * (u_1 + u_2)^{\sim} + \tilde{u}_1 + \tilde{u}_2$$

and therefore (2.3) gives

$$G(t)f_2 = D * G * (u_1 + u_2)^{\sim} - \tilde{u}_1\,. \tag{2.4}$$

From this equality it follows that

$$\frac{1}{t}\int_0^t G(\sigma)f_2 d\sigma = \frac{1}{t}\int_0^t G(t-\sigma)(u_1(\sigma) + u_2(\sigma))d\sigma - \tag{2.5}$$
$$-\frac{1}{t}\int_0^t u_1(\sigma)d\sigma\,.$$

Applying here an inequality due to Hardy (cf. Hardy-Littlewood-Polya [1], p. 245 (9.9.8)) we obtain:

$$(2.6) \qquad \int_0^\infty \left\| t^{\alpha-1} \int_0^t G(\sigma) f_2 d\sigma \right\|_{D(\wedge)}^p dt < \infty .$$

Summing up:

Theorem 2.1.: *Let* u_1 *and* u_2 *be given in* $W(p, \alpha, D(\wedge), E)$ *subject to* (1.8). *Then* $u_2(0) = f_2$ *satisfies condition* (2.6).

2.3 Since

$$\wedge \left(\int_0^t G(\sigma) f_2 d\sigma \right) = G(t) f_2 - f_2 ,$$

condition (2.6) implies that

$$t^{(\alpha-1)}(G(t)f_2 - f_2) \in L^p(0, \infty; E) ,$$

i.e. condition (2.1) for $j = 2$.

Reciprocally, if f_2 is given with (2.1), then

$$\wedge \left(t^{\alpha-1} \int_0^t G(\sigma) f_2 d\sigma \right) \in L^p(0, \infty; E)$$

but in general, if $\wedge$ is not an isomorphism from $D(\wedge)$ onto E,

$$t^{\alpha-1} \int_0^t G(\sigma) f_2 d\sigma$$

does not belong to $L^p(0, \infty; E)$ — so that (2.6) is in general a *stronger condition* than (2.1), $j = 2$ —.

Example: We consider $E = L^p(R)$, $\wedge = d/dx$, $G(t)f(x) =$

$= f(x+t)$. Then f_2 satisfies:

$$f_2 \in L^p(R)\,, \tag{2.7}$$

$$t^{\alpha-1}\int_0^t f_2(x+\sigma)d\sigma \in L^p(R\times(0,\infty)) \tag{2.8}$$

and

$$t^{\alpha-1}(f_2(x+t)-f_2(x)) \in L^p(R\times(0,\infty))\,. \tag{2.9}$$

And in general (2.8) does not hold for a function satisfying (2.7) and (2.9).

When $p=2$, (2.8) is equivalent to

$$\int_{-\infty}^{+\infty} |y|^{-2\vartheta}\,|\hat{f}_2(y)|^2\,dy < \infty\,, \qquad \frac{1}{2}+\alpha=\vartheta\,,$$

$\hat{f}_2 =$ Fourier transform of f_2.
We can also notice that $f_2 \in D(\Lambda^\infty)$ [6]) does not imply (2.6) in general.

2.4 *Remark.*

The proof of Theorem 2.1 give also the inequality

$$\|f_1\| + \|f_2\| + \left(\int_0^\infty \|t^{\alpha-1}(G(t)f_1 - f_1)\|^p\,dt\right)^{1/p} +$$

$$+\left(\int_0^\infty \|t^{(\alpha-1)}\int_0^t G(\sigma)f_2 d\sigma\|^p_{D(\Lambda)}\,d\sigma\right)^{1/p} \leqslant$$

$$\leqslant C\left(\|u_1\|_{W(p,\alpha,D(\Lambda),E)} + \|u_2\|_{W(p,\alpha,D(\Lambda),E)}\right)$$

where c is a suitable constant [7]).

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[6]) i.e. $f_2 \in D(\Lambda)$, $\Lambda f_2 \in D(\Lambda)$, ...

[7]) We made no attempt for calculating the best constant c.

2.5 We conjecture that the result of Theorem 2.1 is the best possible, i.e. given f_1 with (2.1) and f_2 with (2.6), f_1, $f_2 \in E$, there exists u_1, $u_2 \in W(p, \alpha, D(\wedge), E)$, with the properties:

$$u_1(0) = f_1, \; u_2(0) = f_2 \quad \text{and} \quad \wedge u_1 + u_2' = 0 .$$

We have been unable to prove this result in general; we shall prove in section 3 that this is indeed correct when E is a Hilbert space, $p = 2$ and $\wedge$ (or $i \wedge$) self adjoint.

3. *Hilbertian case. Necessary and sufficient conditions.*

Let E be a separable Hilbert space and A be a self-adjoint operator in E. By diagonalization of A, we can always assume that

$$E = \int^{\oplus} h(\lambda) d\mu(\lambda) ,$$

$d\mu$ = positive measure on R,
$h(\lambda) = d\mu$ – measurable family of Hilbert spaces (cf. Dixmier [1]), and that for $f \in D(A)$ (domain of A),

$$Af(\lambda) = \lambda f(\lambda) , \quad d\mu - \text{a.e.} \tag{3.1}$$

If we take

$$\wedge = iA \tag{3.2}$$

then $\wedge$ is the infinitesimal generator of the (unitary) group given by

$$G(t)f(\lambda) = \exp(i\lambda t)f(\lambda) , \quad f \in E . \tag{3.3}$$

We apply Theorem 2.1 in this situation, with $p = 2$.

Condition (2.1) for f_1 becomes

$$\int_{-\infty}^{+\infty}\int_0^{\infty} t^{2(\alpha-1)} |1 - e^{i\lambda t}|^2 |f_1(\lambda)|^2_{h(\lambda)} d\mu(\lambda)dt < \infty$$ [8])

i.e.

$$(3.4) \qquad \int_{-\infty}^{+\infty} |\lambda|^{1-2\alpha} |f_1(\lambda)|^2_{h(\lambda)} d\mu(\lambda) < \infty$$

condition (2.6) becomes

$$\int_{-\infty}^{+\infty}\int_0^{\infty} t^{2(\alpha-1)} \left|\int_0^t e^{i\lambda\sigma}d\sigma\right|^2 (1 + |\lambda|)^2 |f_2(\lambda)|^2_{h(\lambda)} d\mu(\lambda)dt < \infty$$ [9])

i.e.

$$(3.5) \qquad \int_{-\infty}^{+\infty} |\lambda|^{-1-2\alpha}(1 + |\lambda|)^2 |f_2(\lambda)|^2_{h(\lambda)} d\mu(\lambda) < \infty .$$

We can now prove the

THEOREM 3.1.: *Let f_1 and f_2 be given in E, satisfying conditions* (3.4) *and* (3.5). *Then there exist u_1 and u_2 such that*

$$(3.6) \qquad t^{\alpha}u_j \in L^2(0, \infty;\ D(\wedge)), \qquad t^{\alpha}\frac{du_j}{dt} \in L^2(0, \infty;\ E), \quad j = 1, 2,$$

$$(3.7) \qquad \wedge u_1 + \frac{\partial u_2}{\partial t} = 0,$$

$$(3.8) \qquad u_1(\lambda, 0) = f_1(\lambda), \quad u_2(\lambda, 0) = f_2(\lambda) .$$

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[8]) $|f(\lambda)|_{h(\lambda)}$ denotes the norm in $h(\lambda)$; one has

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2_{h(\lambda)} d\mu(\lambda) .$$

[9]) $$\|f\|^2_{D(\wedge)} = \int_{-\infty}^{+\infty} (1 + |\lambda|)^2 |f(\lambda)|^2_{h(\lambda)} d\mu(\lambda) .$$

Proof.: Let M and N be two functions given on $t \geqslant 0$, real valued, twice continuously differentiable, with compact support, and satisfying

$$M(0) = 1, \quad M'(0) = 0, \quad N(0) = 1. \tag{3.9}$$

We introduce $u_2(\lambda, t)$ by

$$u_2(\lambda, t) = M(|\lambda| t) f_2(\lambda) - i\lambda t N(t(1 + |\lambda|)) f_1(\lambda). \tag{3.10}$$

The second condition (3.8) is fulfilled.

Let us check that (3.6) holds, for $j = 2$. We can check separately that

$$\int_{-\infty}^{+\infty} \int_0^{\infty} t^{2\alpha}(1 + |\lambda|)^2 M(|\lambda| t)^2 |f_2(\lambda)|^2_{h(\lambda)} \, d\mu(\lambda) dt < \infty,$$

and that

$$\int_{-\infty}^{+\infty} \int_0^{\infty} t^{2\alpha}(1 + |\lambda|)^2 \lambda^2 t^2 N(t(1 + |\lambda|))^2 |f_1(\lambda)|^2_{h(\lambda)} \, d\mu(\lambda) dt < \infty.$$

The first condition is equivalent to

$$\int_{-\infty}^{+\infty} |\lambda|^{-2\alpha-1}(1 + |\lambda|)^2 |f_2(\lambda)|^2_{h(\lambda)} d\mu(\lambda) < \infty$$

and this is (3.5); the second condition is equivalent to

$$\int_{-\infty}^{+\infty} \lambda^2(1 + |\lambda|)^{-2\alpha-1} |f_1(\lambda)|^2_{h(\lambda)} d\mu(\lambda) < \infty$$

and this follows from (3.4).

Next for $\frac{\partial u_2}{\partial t}(\lambda, t)$:

$$\frac{\partial u_2}{\partial t}(\lambda, t) = |\lambda| M'(|\lambda| t) f_2(\lambda) - i\lambda N(t(1 + |\lambda|)) f_1(\lambda) - \\ - i\lambda(1 + |\lambda|) t N'(t(1 + |\lambda|)) f_1(\lambda). \tag{3.11}$$

One has to check that

$$\int_{-\infty}^{+\infty}\int_{0}^{\infty} t^{2\alpha} \mid \lambda \mid^2 M'(\mid \lambda \mid t)^2 \mid f_2(\lambda) \mid_{h(\lambda)}^2 d\mu(\lambda)dt < \infty$$

and that

$$\int_{-\infty}^{+\infty}\int_{0}^{\infty} t^{2\alpha}\lambda P(t(1 + \mid \lambda \mid))^2 \mid f_1(\lambda) \mid_{h(\lambda)}^2 d\mu(\lambda)dt < \infty ,$$

where $P(s) = N(s)$ or $sN'(s)$.
The first condition is equivalent to

$$\int_{-\infty}^{+\infty} \mid \lambda \mid^{1-2\alpha} \mid f_2(\lambda) \mid_{h(\lambda)}^2 d\mu(\lambda) < \infty$$

and this follows from (3.5); the second condition is equivalent to

$$\int_{-\infty}^{+\infty} (1 + \mid \lambda \mid)^{-1-2\alpha} \mid \lambda \mid^2 \mid f_1(\lambda) \mid_{h(\lambda)}^2 d\mu(\lambda) < \infty$$

and this follows from (3.4).

Therefore, it is proved that (3.6) holds, $j = 2$.

We choose now u_1 in such a way that (3.7) is true, i.e. $i\lambda u_1 + (\partial u_2)/(\partial t) = 0$; by comparison with (3.11), it follows that

$$(3.12) \quad \begin{cases} u_1(\lambda, t) = N(t(1 + \mid \lambda \mid))f_1(\lambda) + t(1 + \mid \lambda \mid)N'(t(1 + \\ + \mid \lambda \mid))f_1(\lambda) + i\dfrac{\mid \lambda \mid}{\lambda} M'(t \mid \lambda \mid)f_2(\lambda) . \end{cases}$$

We notice that $u_1(\lambda, 0) = f_1(\lambda)$, so that it remains only to check (3.6) for $j = 1$.

The verifications, which follow the same lines than above, are left to the reader.

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