Rendiconti del Seminario Matematico della Università di Padova

Giovanni Zacher

On lattice dual-homomorphisms between finite groups

Rendiconti del Seminario Matematico della Università di Padova, tome 30 (1960), p. 65-75

<http://www.numdam.org/item?id=RSMUP_1960__30__65_0>

ON LATTICE DUAL-HOMOMORPHISMS BETWEEN FINITE GROUPS

Nota (*) *di* GIOVANNI ZACHER (**) (*a Padova*)

Given two groups G and G, we say that φ is a dual-homorphism between these two groups if the following conditions are satisfied:

1) Every subgroup $\bar{H}$ of $\bar{G}$ is the image by φ of at least one subgroup H of G; $\bar{H} = \varphi(H)$;

2) For any two subgroups H, K of G we have

$$\varphi(H \cup K) = \varphi(H) \cap \varphi(K)$$

$$\varphi(H \cap K) = \varphi(H) \cup \varphi(K).$$

The aim of this paper is to give necessary and sufficient conditions for a (finite) group G to be a dual-homomorphic image of a finite group G. We shall prove that $\bar{G}$ has the following structure: $\bar{G} = \bar{H}_1 \times \bar{H}_2 \dots \times \bar{H}_t$, with $t \geq 1$, where the order of $\bar{H}_i$ is relatively prime to that of H_j for $i \neq j$, and H_i belongs to one of the following types of groups:

1) A modular non-Hamiltonian p-group;

2) A non-abelian P-group;

3) A simple non-abelian group with dual.

It is still an open question if groups of type 3) exist.

The group $\bar{G}$ is hence dual-isomorphic to a group $\bar{H}$, and

---

(*) Pervenuta in Redazione il 23 settembre 1959.

Indirizzo dell'A.: Seminario matematico, Università, Padova.

(**) This paper has been written while the author had an appointment supported by the International Cooperation Administration under a program administered by the National Academy of Sciences at the University of Illinois, Urbana, Ill. U.S.A.

therefore applying known results for lattice-homomorphisms between groups we can also get the structure of G.

Our proofs will rely heavily on results contained in [1] [1]), and also our terminology will follow that used in [1].

1. - Notation: Capital latin letters like $G, H, K, \ldots$ stand for groups, meanwhile small latin letters, like $a, b, \ldots$ for elements of a group.

S_{p^α} = Sylow group of order p^α; $\{a\}$ = cyclic group generated by a; $\mathfrak{L}(G)$ = lattice of the subgroups of G; $\Phi(G)$ = Frattini subgroup of G; $F(G)$ = union of all minimal subgroups of G; $\mathfrak{N}_G(H)$ = normalizer of H in G; $C_G(H)$ = centralizer of H in G; $H \triangleleft G = H$ normal in G; $[G:H]$ = index of H in G; 1 = identity group; $H \subset K$ means that H is a proper subgroup of K. A Hall subgroup of a finite group G is a group which has order relatively prime to its index in G.

2. - In this section we shall be concerned with some properties of finite groups with duals.

PROP. I: If N is a characteristic element of $\mathfrak{L}(G)$, and if φ is a dual-isomorphism between G and $\bar{G}$, then $\bar{N} = \varphi(N)$ is a characteristic element in $\mathfrak{L}(\bar{G})$.

COROLLARY: If N is a characteristic element in $\mathfrak{L}(G)$, then N has a dual if G has one.

PROP. II: If N is a characteristic element in $\mathfrak{L}(G)$, and if H/N is such in $\mathfrak{L}(G/N)$, then H is a characteristic element in $\mathfrak{L}(G)$.

PROP. III: If $G = H \times K$, with H a simple non abelian group and if G is dual-isomorphic to $\bar{G}$, then $\bar{G} = \bar{H} \times \bar{K}$, where $\bar{H} = \varphi(H)$, $\bar{K} = \varphi(K)$ and $H, K, \bar{H}, \bar{K}$ are Hall subgroups respectively of G and $\bar{G}$ [2]).

From our assumptions it follows, applying known results

---

[1]) Number in square brackets refer to the bibliography listed at the end of this paper.

[2]) I am indebt to Prof. M. Suzuki for valuable suggestion in the proof of this theorem.

on direct products [3]), that $H \times K = H_1 \times K$ implies $H_1 = H$, and $H \times K = H \times K_1$ implies $K_1 = K$. We have moreover that

$$\bar{G} = \bar{H} \cup \bar{K}, \quad \bar{H} \cap \bar{K} = 1 \tag{1}$$

To prove that $\bar{H}$ is normal in $\bar{G}$ we have only to show thow that for $\bar{k} \in \bar{K}$, $\bar{H} = \bar{k}\bar{H}\bar{k}^{-1}$. We consider the lattice automorphism ψ of G defined by $\psi = \varphi^{-1}\bar{\boldsymbol{k}}\varphi$ with φ a dual-isomorphism between G and $\bar{G}$, and $\boldsymbol{k}$ the inner-automorphism of $\bar{G}$ defined by the element $\bar{k}$. We have then $\psi(K) = K$ and $\psi(H)$ normal [4]) in G. So from $H \times K = \psi(H) \times K$ it follows that $\psi(H) = H$ and therefore $kHk^{-1} = H$. With the same argument one proves that also $\bar{K}$ is normal in $\bar{G}$, and so from (1) we get that $\bar{G} = \bar{H} \times \bar{K}$.

Now let's assume that q is a prime divisor of $[H:1]$ and $[K:1]$. Then also [5]) $[\bar{H}:1]$, $[\bar{K}:1]$ must have a common prime divisor p. Consider in $\bar{G}$ a group $\bar{P}$ of order p such that

$$\bar{P} \cap \bar{H} = \bar{P} \cap \bar{K} = 1, \quad 1 \subset \bar{P} \cup \bar{H} \subset \bar{G}. \tag{2}$$

Applying the inverse lattice isomorphism φ^{-1} to P, $\bar{H}$, $\bar{K}$, 1, $\bar{G}$, (2) gives us

$$P \cup H = P \cup K = G; \quad 1 \subset P \cap H \subset H \tag{2'}$$

If we put $H = \{h \mid hk = u \in P\}$, $K_1 = \{k \mid hk = u \in P\}$, then

$$H \supseteq H_1, \quad K \supseteq K_1, \quad H_1 \times K_1 \supseteq P, \quad P \cap H \lhd H_1, \quad P \cap K \lhd K_1. \tag{3}$$

The group P is maximal in G, therefore by (3) we have either $P = H_1 \times K_1$, or $G = H_1 \times K_1$.

If $P = H_1 \times K_1$, we must have either $H_1 = H$ or $K_1 = K$, which is not possible by (2'). If $G = H_1 \times K_1$, then $H_1 = H$, $K_1 = K$. But then $P \cap H \lhd H$, $1 \subset P \cap H \subset H$ give a contradiction, recalling that H is simple. Hence $[H:1]$ must be relatively prime to $[K:1]$.

---

[3]) See Ch. III in [2].

[4]) See th. 14, II in [1].

[5]) See th. 4, I in [1].

COROLLARY: Let G be a group dual-isomorphic to a group $\bar{G}$. Let N be a minimal non abelian subgroup of G. Then the group N is a characheristic element in $\mathfrak{L}(G)$.

Let ψ be an automorphism of $\mathfrak{L}(G)$; then $\psi(N)$ is normal in G, is simple and has order equal to that of N [6]). Therefore if we consider the minimal characteristic element H of $\mathfrak{L}(G)$ which contains N, it is a direct product of simple non abelian groups all of the same order. But H has a dual (Corollary to prop. I), and so by prop. III, H must coincide with N.

We now prove the following

THEOREM I: *Let G be a finite group dual-isomorphic to a group $\bar{G}$. Then G is the direct product of groups with pairwise relatively prime orders where each factor is either a simple non-abelian group with dual, or a P-group, or a modular non-Hamiltonian p-group.*

If G is solvable the theorem has been proved by Suzuki [7]). We shall use induction on the order of G. Let N be a minimal normal subgroup of G and assume that N is simple non-abelian. The group G/N is dual-isomorphic to $\bar{N}$, and therefore by induction, G/N is a direct product of groups $\tilde{H}_1, \ldots, \tilde{H}_t$, belonging to the types mentioned above. The group $\tilde{H}_i = H_i/N$ is a characteristic element of $\mathfrak{L}(G/N)$, N is such in G (Corollary to prop. III), therefore H_i is a characteristic element of $\mathfrak{L}(G)$ (prop. II). It follows that H_i has a dual. We consider the centralizer $C(N)$ of N in H_i; N is simple and therefore $C(N) \cup N = C(N) \times N$. If $C(N) \times N = H_i$, we have only to apply prop. III to reach the conclusion. Assume now $1 \subset C(N) \subset C(N) \times G$. The group $H_i/C(N)$ has a dual and therefore by induction we have $H_i/C(N) = F_i \times N \subset (N)/C(N)$; but this implies $C(N) \times N = H_i$, against our assumption. The only case left to consider is that for which $C(N) = 1$. If we set $\Phi(H_i/N) = M_i/N$, the group M_i is a characteristic element in $\mathfrak{L}(H_i)$. If $M_i \supset N$, H_i/N is a modular p-group, and applying induction to M_i we conclude with the desired result. Hence let

---

[6]) See th. 14, II and th. 15, II in [1].

[7]) See th. 5, IV in [1].

$M_i = N$, so that H_i/N is or simple non-abelian or a P-group [8]). The group $\bar{H}_i = \varphi(H_i)$ has the following structure: $\varphi(N) = \bar{N} \lhd \bar{H}_i$, $\bar{H}_i/\bar{N}$ is simple non abelian, and N is or simple non abelian, or a P-group.

If $C(\bar{N}) \subseteq \bar{N}$, then it is easy to see that we can find two groups $\bar{Q}$, $\bar{Q}_1$ of H_i such that we have

$$(4) \qquad \bar{H}_i \supset \bar{U} = \bar{Q}\bar{N} = \bar{Q}_1\bar{N} \supset \bar{N}; \; \bar{Q} \cap \bar{Q}_1 = \bar{T} \subseteq \bar{N}.$$

If ψ is a dual-isomorphism between $\bar{H}_i$ and H_i, then applying ψ to (4), we get

$$(4') \qquad 1 \subset U = Q \cap N = Q_1 \cap N \subset N; \; Q \cup Q_1 = T \supseteq N \supset U.$$

Now N is normal in H_i, so U is normal in N, which is impossible because N is simple. Hence $C(N) \cup N = \bar{H}_i$. If $\bar{N}$ is not an abelian P-group, then $C(\bar{N}) \cup \bar{N} = C(N) \times \bar{N}$, and we may apply prop. III to reach the conclusion. If N is an elementary abielan group, then $\bar{H}_i = C(\bar{N}) \cup \dot{N} = C(N)$ and N is the center of $\bar{H}_i$. We show that $\bar{N}$ can't be a proper subgroup of a Sylow group $\bar{S}$ of $\bar{H}_i$. S can't be cyclic, because by a th.of Zassenhaus [9]) $H_i/\bar{N}$ would not be simple. But then $\exists$ a group $\bar{U} \subseteq S$ which covers N and two groups $\bar{Q}$, $\bar{Q}_1$ such that the following relations are satisfied

$$\bar{H}_i \supset \bar{U} = \bar{Q} \cup \bar{N} = \bar{Q}_1 \cup \bar{N} \supset \bar{N}; \; \bar{Q} \cap \bar{Q}_1 = \bar{T} \subseteq \bar{N} \subset \bar{U}.$$

and we reach the same contradiction as previously for (4).

To complete our proof there is left to consider the case that G does not contain a simple non abelian normal subgroup. With N we indicate the union of all normal subgroups of G. Then N must coincide with G. Otherwise G/N would be a direct product of simple non abelian groups $\tilde{H}_1, \dots, \tilde{H}_t$. If $H_i/N = \tilde{H}_i$, then as we saw before, H_i would have a dual $\bar{H}_i$ and $\bar{N}$ would be a simple non abelian normal subgroup of $\bar{H}_i$. But then $H_i = N \times \bar{T}$ where N and $\bar{T}$ have order

---

8) See for definition pag. 11 in [1].

9) See [3].

relatively prime, and H_i and therefore G would contain a normal simple non abelian subgroup, which is against our assumptions. Thus the theorem is proved.

3. - We pass now to the study of the groups which are dual-homomorphic images of finite groups.

Let G be a group and φ a fixed dual-homomorphism between the finite groups G and G. With G_0 we indicate the intersection of all subgroups H of G such that $\varphi(H)=1$, and with E the union of all subgroups K of G such that $\varphi(K)=\bar{G}$.

In order to determine the structure of the group $\bar{G}$, we prove the following propositions:

PROP. IV: Let φ be a dual-homomorphim between G and $\bar{G}$. If $E=1$, then φ induces a one to one correspondence between minimal and maximal subgroups respectively of G and G; it follows $\varphi(F(G))=\Phi(G)$. If $G=G_0$, then φ induces a one to one correspondence betwen the maximal and minimal subgroups respectively of G and $\bar{G}$; it follows $\varphi(\Phi(G))=F(\bar{G})$.

The proof is obvious.

PROP. V: Let ψ be a lattice homomorphism of G on a lattice L, and assume that the lower kernel E of ψ is 1. Then the restriction ψ_1 of ψ on $F(G)$ is a lattice isomorphism.

Obviously the lower kernel of $F(G)$ is 1, and $F(F(G))=F(G)$. Now suppose that ψ_1 is a proper lattice homorphism. Then there $\exists$ at least one Sylow subgroup S of $F(G)$ of order p^α with $\alpha>1$ on which ψ_1 induces a proper lattice homomorphism; therefore we have[10] $F(G)=S\cup N$ where N is a normal complement of S, and S is a cyclic or a generalized quaternion group. In the latter case, $S\cup N=S\times N$; but then $F(F(G))\subset F(G)$, which is impossible. Hence S must be cyclic. Now consider two minimal subgroups P and Q of $F(G)$ with $P\subset S$. The group $P\cup N$ is a proper normal subgroup of $F(G)$. If Q has order a divisor of $[N:1]$, then $Q\subseteq N\subset PN$. Otherwise $Q\subset S'$, where S' is a conjugate to P in PN and

10) See pp. 70-71 in [1].

therefore $Q \subset PN \subset F(G)$. But then $F(F(G)) \subseteq PN \subset F(G)$ which is impossible. Hence ψ_1 is a lattice isomorphism.

If ψ is a homomorphism of the lattice L onto the lattice $\bar{L}$, if $\bar{a}$ is an element of L, then with $\psi^{-1}(\bar{a})$ we indicate the union of all those elements of L for which $\psi(a) = \bar{a}$. We assume that ψ is a complete homomorphism of the lattice $\mathfrak{L}(G)$ of a group, finite or infinite, onto a lattice $\bar{L}$, cardinal product of sublattices $\bar{L}_1, \bar{L}_2, \dots, \bar{L}_n$. $\bar{L}$ has a maximal and a minimal element, and therefore also $L_i (i = 1, 2, \dots, n)$. If we set $G_i = \psi^{-1}(0_1 \dots I_i \dots 0_n)$ then we have the following.

PROP. VI: G is a torsion group, $G = G_1 \cup G_2 \dots \cup G_n$, $G_1 \cap \dots \cap G_{n-1} \cap G_n = E$, $G/E = G_1/E \times G_2/E \times \dots \times G_n/E$ and each element of G_i/E has order relatively prime to every element of G_j/E for $i \neq j$.

We give the proof in the case $n = 2$. The extension to the general case is obvious.

We have $\psi(G_1 \cap G_2) = \psi(G_1) \cap \psi(G_2) = (I_1, 0_2) \cap (0_1, I_2) = 0$, therefore $E = \psi^{-1}(0) \supseteq G_1 \cap G_2$. On the other hand, $0 < (I_1, 0_2)$, $0 < (0_1, I_2)$ and therefore $\psi^{-1}(0) \subseteq \psi^{-1}(I_1, 0_2)$, $\psi^{-1}(0) \subseteq \varphi(0_1, I_2)$ and so $E = \psi^{-1}(0) \subseteq G_1 \cap G_2$; but then $E = G_1 \cap G_2$ and $G_1 \cap G_2$ is a normal subgroup of G We want to prove now that G_1 is normal in G. Let be $g_1 \notin E$, $g_1 \in G_1$, $g_2 \notin E$, $g_2 \in G_2$. We consider the group $H = g_2\{g_1\}g_2^{-1}$ and we shall see that $H \subset G_1$ Let $\varphi(H) = [l_1, l_2]$. All what we have to show is that $l_2 = 0_2$. From $l_2 > 0_2$ follows $(l_1, l_2) \geq (0_1, l_2) > 0$ and $g_2\{g_1\}g_2^{-1}$ contains a subgroup $\{t\}$ such that $\varphi(\{t\}) = [0_1, l_2] > 0$, so $t \notin E$. t is then given by $t = g_2 g_1^m g_2^{-1}$ with m integer greater then 0, and $\varphi(\{t, g_2\}) = \varphi(\{t\}) \cup \varphi(\{g_2\}) \in L_2$, so that $\varphi(\{g_1^m\}) \in L_2$. In other words $g_1^m \in G_1 \cap G_2$; but $G_1 \cap G_2$ is normal in G, so t is also in E and therefore $\varphi(\{t\}) = 0$, against our assumption. We conclude that $l_2 = 0_2$, and so $g_2\{g_1\}g_2^{-1} \in G_1$. G_1 is therefore normal in $G_1 \cup G_2$; similarly one shows that also G_2 is normal in $G_1 \cup G_2$. Now we prove that all the elements of G have finite order. Assume that g is not periodic. Then $E = 1$[11]).

---

[11]) See th. 5, III in [1].

Now $\varphi(\{g\} \cap (G_1 \cup G_2)) = \varphi(\{g\}) \cap I = \varphi(\{g\}) \neq 0$, therefore $\exists\ m > 0$ such that $1 \subset \{g^m\} \in G_1 \cup G_2$. We put $g^m = t$, and t has infinite order too. We may assume that g^m does not belong either to G_1 or G_2, otherwise if for example $g^m \in G_1$, if $g_2 \neq 1$ is an element of G_2, then $g^m g_2 \notin G_1$, $g^m g_2 \notin G_2$ and $g^m g_2$ is aperiodic being $G_1 \cup G_2 = G_1 \times G_2$. Hence $\varphi(\{t\}) = [l_1, l_2]$ with $l_i > 0$ $(i = 1, 2)$. Therefore exists a subgroup $\{t_0\}$ of $\{t\}$ and two subgroups $\{t_1\}$, $\{t_2\}$ of $\{t_0\}$, different from 1 such that $\{t_1\} \cup \{t_2\} = \{t_0\}$, $\{t_1\} \cap \{t_2\} = 1$, which is impossible, because t is torsion free. Hence g has finite order and G is a torsion group. Now we want to prove that every element of G_1/E has order prime to each element of G_2/E. Let $a_i \in G_i$ $a_i \notin E \cdot a_i^p \in E$ with p a prime number. Then $a_1 a_2 \notin E$, thus $\varphi(\{a_1 a_2\}) = [l_1, l_2]$ with $l_i > 0_i$, which is impossible. Finally assume that there exists an element $g \notin G_1 \cup G_2$; then $g \notin E$, and therefore $\varphi(\{g\}) \neq 0$. We may assume that g is of prime power order; but then $\varphi(\{g\}) = [l_1, 0_2]$ or $[0_1, l_2]$, so that g belongs to G_1 or G_2, against our assumption. Hence $G = G_1 \cup G_2$. Our proposition is completely proved.

We call a group G a P_1-group if G has order $p^\alpha q^\beta (\alpha \geq 1, \beta \geq 1)$ with $p > q$ prime numbers in which S_{q^β} is cyclic, S_{p^α} is elementary abelian and if $\{b\} = S_{q^\beta}$, $a \in S_{p^\alpha}$, then $bab^{-1} = a^r$ with $r \not\equiv 1$ mod. p and independent of a.

Prop. VII: If G is a P_1-group dual-homomorphic to a group $\bar{G} \supset 1$ and if $E = 1$, then $\bar{G}$ is a P-group and $G_0 = F(G)$. Conversely if G is a P-group, then $\tilde{G} = G/E$ is a P_1-group with $F(\tilde{G}) = \tilde{G}_0$.

Let $p^\alpha q^\beta$ with $p > q$ be the order of G and assume $\alpha > 1$. Then if P is a minimal subgroup of the group S_{p^α}, the group G/P is again a P_1-group, and applying induction we conclude that φ induces a dual-isomorphism on $(G/P)_0 = G_0/P$ and on P and therefore on G_0; moreover $F(G) \leq G_0$. Now if $\mathfrak{L}(F(G))$ is reducible then so would be $\mathfrak{L}(\bar{G}/\Phi(\bar{G}))$ and therefore $\mathfrak{L}(\bar{G})$; but then by prop. VI also $\mathfrak{L}(G)$ would be reducible which is not possible because G is a P_1-group. G_0 and $\bar{G}$ are hence P-groups, and $G_0 = F(G)$. If $\alpha = 1$, the group $F(G)$ is a P-group of order pq. But then $\Phi(G) = 1$ and therefore $F(\bar{G}) = \bar{G}$.

Let A be a normal minimal subgroup of $\bar{G}$, contained in

$\Phi(\bar{G})$ which we assume greater than 1. Then $G \supset A_0 \supseteq F(G)$, therefore A is a P_1-group; but then G/A is a P-group and so $\Phi(\bar{G}) \subseteq \bar{A}$, that is $\Phi(\bar{G}) = A$. Hence the group $\Phi(G)$ has order a prime. $\bar{G}/\Phi(G)$ is a group of order p^2 or pr $(r < p)$, so $\bar{G}$ has order p^3 or r^2p, or rp^2. If $\bar{G}$ is not a p-group, its Sylow subgroups are cyclic; but $F(\bar{G}) \subset G$. If G is a p-group, it must be non abelian, of exponent p, because $F(\bar{G}) = \bar{G}$, and $\bar{G}$ is regular. But then $\bar{G}$ contains $p(p+1)$ subgroups of order p, meanwhile G_0 has only $p+1$ maximal subgroups. Hence $\Phi(\bar{G}) = 1$, $\bar{G}$ is a P-group, and $F(G) = G_0$.

The converse follows from th. 15, III in [1].

PROP. VIII: Let φ be a dual-homomorphism between G and $\bar{G}$, where G is a group of order $p^\alpha q^\beta$ $(\alpha \geq 1,\ \beta \geq 0,\ p > q)$. Then G has a dual and G_0/E_0 is dual-isomorphic to G, if $\bar{G}$ is not cyclic.

We consider the group $\tilde{G} = G_0/E \cap G_0 = G_0/E_0$; φ induces on $\tilde{G}$ a dual-homomorphism φ_1 onto $\bar{G}$ in which $\tilde{G}_0 = \tilde{G}$, $\tilde{E} = 1$. If φ_1 is a dual-isomorphism, then there is nothing to prove. Thus we may assume that φ_1 is a proper dual-homomorphism, that $\mathfrak{L}(G)$ is irriducible by prop. VI and $\Phi(\bar{G}) \supset 1$ by prop. III and IV. Between $F(\tilde{G})$ and $\bar{G}/\Phi(\bar{G})$, φ_1 induces a dual-isomorphism; so $\bar{G}/\Phi(\bar{G})$ and $F(\tilde{G})$ are P-groups, being $\mathfrak{L}(\tilde{G})$ irreducible. If $F(\tilde{G})$ is an abelian P-group, then $\tilde{G}$ and therefore also $\bar{G}$ is a cyclic p-group. Hence let $F(\tilde{G})$ be a non abelian P-group of order rp . We then show that φ_1 can't be a proper dual-homomorphism. We induction on the order of $\tilde{G}$. From our assumptions it follows that the p-Sylow group S_{p^α} is normal in G, on S_{p^α}, φ_1 induces a dual-isomorphism and the r-Sylow groups S_{r^γ} are cyclic and do not contain a normal subgroup. If $\Phi(\tilde{G}) \supset 1$, $\Phi(\tilde{G})$ is contained in S_{p^α}; on $\tilde{G}/\Phi(\tilde{G})$, φ_1 induces a dual-homomorphism ψ with $(\tilde{G}/\Phi(\tilde{G}))_0 = \tilde{G}/\Phi(\tilde{G})$, $E(\tilde{G}/\Phi(\tilde{G})) = 1$. By induction ψ is a dual-isomorphism and so is φ_1 on $\Phi(\tilde{G}) \subset S_{p^\alpha}$; hence φ_1 is a dual-isomorphism, against our assumption. If $\Phi(\tilde{G}) = 1$, then S_{p^α} is elementary abelian, and $F(\tilde{G}) \supset S_{p^\alpha}$ Let P be any fixed minimal subgroup of S_{p^α}; we then show that P is normal in $\tilde{G}$. If $P = S_{\ \alpha}$, then there is nothing to prove. So let be $\alpha > 1$;

by H we denote the maximal subgroup of index r in G. From $H_0 \supseteq F(G) \supset S_{p^\alpha} \supset P$ follows

$$1 \subset \varphi(H_0) \subseteq \Phi(\bar{G}) \subset \varphi(S_{p^\alpha}) \subset \varphi(P).$$

$\Phi(\bar{G})$ is a cyclic group and hence $\varphi(H)$ is normal in G. H_0 is dual-homomorphic to $\bar{G}/\varphi(H_0)$, so by our assumptions it must be a dual-isomorphism. $\bar{G}/\varphi(H)$ is a non abelian P-group and H is a P_1-group (prop. VII); moreover $\Phi(\bar{G}) = \varphi(H)$. If we set $T = S_{r^\gamma} \cap H$, then $P = (T \cup P) \cap S_{p^\alpha}$, $\varphi(T) = \varphi(S_{r^\gamma} \cup \cup \Phi(\bar{G}))$, $\varphi(P) = (\varphi(T) \cap \varphi(P)) \cup \varphi(S_{p^\alpha})$; therefore $\varphi(P) = = [(\varphi(S_{r^\gamma}) \cup \Phi(\bar{G})) \cap \varphi(P)] \cup \varphi(S_{p^\alpha}) = [\Phi(\bar{G}) \cup (\varphi(S_{r^\gamma}) \cap \cap \varphi(P))] \cup \varphi(S_{p^\alpha}) = [\varphi(S_{r^\gamma}) \cap \varphi(P)] \cup \varphi(S_{p^\alpha})$. But then $(S_{r^\gamma} \cup P) \cap S_{p^\alpha} = P$, because φ_1 induces a lattice isomorphism on S_{p^α}. Hence P is normal in $\tilde{G}$. But then we conclude that $\tilde{G}$ is a P_1-group. By prop. VII $\tilde{G} = G_0 = F(\tilde{G})$ and φ_1 is a dual-isomorphism, which is against our assumption. Our prop. is now completely proved. We are now able to prove the following:

THEOREM: *A group $\bar{G}$ is the dual-homomorphic image of a finite group G if and only if $\bar{G}$ is the direct product of groups $\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_n$, where $[\bar{H}_i : 1]$ is relatively prime to $[\bar{H}_j : 1]$ for $i \neq j$ and $\bar{H}_i$ belongs to one of the following types of groups*:

1) *A modular non Hamiltonian p-group*;
2) *A non abelian P-group;*
3) *A simple group with a dual.*

Let φ be a dual-homomorphism between G and $\bar{G}$; if we set $G = \tilde{G}_0/E_0$, then φ induces a dual-isomorphism (prop. V) between $F(\tilde{G})$ and $\bar{G}/\Phi(\bar{G})$. By theorem *I*, $\bar{G}/\Phi(\bar{G}) = = \bar{M}_1 \times \bar{M}_2 \ldots \times M_t$ with $\bar{M}_i$ either a modular p-group, or a P-group or a simple non abelian group with dual, and where $\bar{M}_i$ has order relatively prime to $\bar{M}_j$ if $i \neq j$. From known properties of the Frattini subgroup, it follows that $\bar{G} = \bar{H}_1 \times \bar{H}_2 \ldots \times H_t$ where $\bar{M}_i \simeq \bar{H}_i/\Phi(\bar{H}_i)$ and $\Phi(\bar{G}) = = \Phi(\bar{H}_1) \times \ldots \times \Phi(H_t)$. By prop. VI, we get that $\tilde{G} = G/E = = \tilde{H}_1 \times \tilde{H}_2 \ldots \times \tilde{H}_t$, where $\tilde{H}_i$ is dual-homomorphic to $\bar{H}_i$. If

M_i is not a simple non abelian group, $\bar{H}_i$ is a group of order $p^\alpha q^\beta$ with $\alpha > 0$, $\beta \geq 0$; by prop. VIII and th. 5, IV in [1], H_i is either a modular non-Hamiltonian p-group, or a non abelian P-group. Assume now that $\bar{M}_i$ is a simple group. We have then that φ determines a dual-isomorphism between $F(\tilde{H}_i)$ and $\bar{H}_i/\Phi(\bar{H}_i) \simeq \bar{M}_i$; the group $F(\tilde{H}_i)$ is therefore simple; but then φ determines a dual-isomorphism between $\tilde{H}_i$ and $\bar{H}_i$, and therefore by th. I, $\Phi(\bar{H}_i) = 1$, $M_i \simeq \bar{H}_i$, $F(\tilde{H}_i) = \tilde{H}_i$, and $\bar{H}_i$ is a simple non abelian group with dual. This completes the proof of theorem II.

Theorem II states that if $\bar{G}$ is the dual-homomorphic image of a finite group G, the group G ha a dual $\bar{H}$ where we may assume for $\bar{H}$ the following structure $\bar{H} = \bar{M} \times \bar{T}_1 \times \bar{T}_2 \times \dots \times \bar{T}_m$, where M is a nilpotent Hall subgroup of $\bar{H}$, with dual, and $\bar{T}_i$ is a simple non abelian Hall subgroup with dual. The determination of the finite groups G lattice homomorphic to such a group $\bar{H}$ is a solved problem (see [1] pp. 57) and so we can determine the structure of the finite groups dual-homomorphic to $\bar{G}$.

BIBLIOGRAPHY

[1] Structure of a group and the structure of its lattice of subgroups, by M. Suzuki. Ergeb. der Math. und ihrer Grenzgebiete, Heft 10, Springer Verlag, Berlin.

[2] The theory of groups, by H. Zassenhaus, Chelsea Publishing Company, New York, 1958.

[3] H. Zassenhaus: Über endliche Fastkörper, in Abh. Math. Seminar, Universität Hamburg, vol. 11.