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A NOTE ON LEGENDRE POLYNOMIALS

Nota (*) S. K. CHATTERJEA (*a Calcutta*)

In a recent note [1], B. S. Popov has given an alternative proof of my result [2]:

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^r P_m(x)\cdot\left(\frac{d}{dx}\right)^s P_n(x)dx$$

$$= K\sum_{k=1}^{r}(-)^{k-1}(r-k)!\,(s+k-1)!\,\cdot \tag{1}$$

$$\cdot\binom{n+s+k-1}{s+k-1}\binom{n}{s+k-1}\binom{m+r-k}{r-k}\binom{m}{r-k};$$

where

$$K=[1+(-)^{m+n-r-s}]\cdot 2^{1-r-s}$$

and

$$m-r\geq n-s,\qquad 1\leq r\leq m,\qquad 1\leq s\leq n.$$

The object of this present note is to add some more results to those already obtained in the previous notes [1] and [2].

First we prove

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^r P_m(x)\cdot\left(\frac{d}{dx}\right)^s P_n^{(\alpha,\beta)}(x)dx \tag{2}$$

$$=2^{1-r-s}\sum_{k=1}^{r}(-)^{k-1}\frac{(n+\alpha+\beta+1)_{s+k-1}\cdot(m+r-k)!}{(n-s-k+1)!\,(r-k)!\,(m-r+k)!}\cdot\mu(k)$$

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where

$$\mu(k) = \frac{(1+\alpha)_n}{(1+\alpha)_{s+k-1}} + (-)^{m+n-r-s} \cdot \frac{(1+\beta)_n}{(1+\beta)_{s+k-1}} .$$

and

$$m - r \geq n - s, \quad 1 \leq r \leq m, \quad 1 \leq s \leq n$$

and

$$(\alpha)_n = \alpha \cdot (\alpha+1) \dots (\alpha+n-1), \quad P_n(x) \text{ and } P_n^{(\alpha,\beta)}(x)$$

denote respectively the Legendre and the Jacobi polynomial of degree n.

In proving (2) we shall require the following formulae:

$$(3) \quad \int_{-1}^{1} u(x) \cdot \left(\frac{d}{dx}\right)^n v(x)dx = \left[\sum_{k=1}^{n} (-)^{k-1} \left(\frac{d}{dx}\right)^{k-1} u(x) \cdot \left(\frac{d}{dx}\right)^{n-k} v(x)\right]_{-1}^{1} + (-)^n \int_{-1}^{1} v(x) \cdot \left(\frac{d}{dx}\right)^n u(x)dx,$$

$$(4) \quad \left[\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(x)\right]_{x=1} = \frac{(n+\alpha+\beta+1)_r \cdot (1+\alpha)_n}{2^r (n-r)!\,(1+\alpha)_r},$$

and

$$(5) \quad \left[\left(\frac{d}{dx}\right)^r P_n^{(\alpha,\beta)}(x)\right]_{x=-1} = \frac{(-)^{n-r}(n+\alpha+\beta+1)_r!\,(1+\beta)_n}{2^r (n-r)!\,(1+\beta)_r}.$$

The formula (3) is well-known and the formulae (4) and (5), which are generalization of Grosswald's formula [3]:

$$(6) \quad \left(\frac{d}{dx}\right)^r P_n(1) = \frac{(n+r)!}{2^r r!\,(n-r)!},$$

are obtained by Carlitz [4] and are presented in correct forms by B. R. Bhonsle [5].

Now using (3) we get

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^{r} P_m(x)\cdot\left(\frac{d}{dx}\right)^{s} P_n^{(\alpha,\beta)}(x)dx$$

$$(5)\qquad =\left[\sum_{k=1}^{r}(-)^{k-1}\left(\frac{d}{dx}\right)^{s+k-1} P_n^{(\alpha,\beta)}(x)\cdot\left(\frac{d}{dx}\right)^{r-k} P_m(x)\right]_{-1}^{1}$$

$$+(-)^{r}\int_{-1}^{1} P_m(x)\cdot\left(\frac{d}{dx}\right)^{r+s} P_n^{(\alpha,\beta)}(x)dx;$$

$$1\leq r\leq m,\qquad 1\leq s\leq n.$$

the integral on the right hand member of (7) vanishes, since $m-r\geqslant n-s$.

Thus we easily obtain (2) from (7) with the help of (4), (5) and (6).

Now we like to make some applications of the formula (2). We remark that for $\alpha=\beta$, (2) reduces to

$$\int_{-1}^{1}\left(\frac{d}{dx}\right)^{r} P_m(x)\cdot\left(\frac{d}{dx}\right)^{s} P_n^{(\alpha,\alpha)}(x)dx$$

$$(8)\qquad =K\sum_{k=1}^{r}(-)^{k-1}\frac{(n+2\alpha+1)_{s+k-1}\cdot(1+\alpha)_n}{(1+\alpha)_{s+k-1}}$$

$$\cdot\frac{(m+r-k)!}{(n-s-k+1)!\,(r-k)!\,(m-r+k)!};$$

where

$$K=[1+(-)^{m+n-r-s}]2^{1-r-s}.$$

and

$$m-r\geq n-s,\qquad 1\leq r\leq m,\qquad 1\leq s\leq n.$$

Next putting $\alpha=\lambda-\dfrac{1}{2}$ in (8) and observing that

$$\left(\lambda+\frac{1}{2}\right)_n P_n^{\lambda}(x)=(2\lambda)_n P_n^{(\alpha,\alpha)}(x),$$

where $P_n^\lambda(x)$ denotes the ultraspherical polynomial of degree n and $\alpha = \lambda - \frac{1}{2}$, we obtain

$$\int_{-1}^{1} \left(\frac{d}{dx}\right)^r P_m(x) \cdot \left(\frac{d}{dx}\right)^s P_n^\lambda(x) dx$$

(9)
$$= K \sum_{k=1}^{r} (-)^{k-1} \frac{(n+2\lambda)_{s+k-1} \cdot (2\lambda)_n}{\left(\lambda + \frac{1}{2}\right)_{s+k-1}} \cdot \frac{(m+r-k)!}{(n-s-k+1)!(r-k)!(m-r+k)!}$$

where

$$m - r \geq n - s, \quad 1 \leq r \leq m, \quad 1 \leq s \leq n.$$

Finally putting $\lambda = \frac{1}{2}$ and observing that $P_n^\lambda(x) = P_n(x)$ when $\lambda = \frac{1}{2}$, we obtain from (9)

(10)
$$\int_{-1}^{1} \left(\frac{d}{dx}\right)^r P_m(x) \cdot \left(\frac{d}{dx}\right)^s P_n(x) dx$$

$$= K \sum_{k=1}^{r} (-)^{k-1} \frac{(n+s+k-1)!(m+r-k)!}{(s+k-1)!(r-k)!(n-s-k+1)!(m-r+k)!}$$

where

$$m - r \geq n - s, \quad 1 \leq r \leq m, \quad 1 \leq s \leq n,$$

which is (1).

In precisely the same way, we also obtain

(11)
$$\int_{-1}^{1} \left(\frac{d}{dx}\right)^p P_l(x) \cdot \left(\frac{d}{dx}\right)^r P_m^{(\alpha,\beta)}(x) \cdot \left(\frac{d}{dx}\right)^s P_n^{(\alpha,\beta)}(x) dx$$

$$= 2^{1-p-r-s} \sum_{k=1}^{p} \sum_{\nu=0}^{k-1} (-)^{k-1} \binom{k-1}{\nu} \cdot \eta \cdot \zeta$$

where

$$\eta = \frac{(m + \alpha + \beta + 1)_{r+k-\nu-1}(n + \alpha + \beta + 1)_{s+\nu}}{(m - r - k + \nu + 1)!\,(n - s - \nu)!} \cdot \frac{(l + p - k)!}{(p - k)!\,(l - p + k)!}.$$

$$\zeta = \frac{(1 + \alpha)_m(1 + \alpha)_n}{(1 + \alpha)_{r+k-\nu-1}(1 + \alpha)_{s+\nu}} + (-)^{l+m+n-p-r-s}\frac{(1 + \beta)_m \cdot (1 + \beta)_n}{(1 + \beta)_{r+k-\nu-1} \cdot (1 + \beta)_{s+\nu}}.$$

and

$$1 \leq p \leq l, \quad 1 \leq r \leq m, \quad 1 \leq s \leq n, \quad l - p \geq m + n - r - s.$$

We remark that for $\alpha = \beta = 0$, (11) reduces to a general integral formula, obtained by B. S. Popov, for the product of the derivatives of Legendre polynomials.

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