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Further properties of pseudo-abelian varieties

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FURTHER PROPERTIES OF PSEUDO-ABELIAN VARIETIES

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1. Introduction. - This note adds some results to previous papers on pseudo-Abelian varieties; it also makes an acknowledgment to the work [6] of Dantoni which, regrettably, had been overlooked, and corrects a statement in [12].

After a brief account of Picard and pseudo-Abelian varieties, in so far as they are required here, the following results are established:

(i) *Any (algebraic) variety which admits a finite continuous permutable group of automorphisms and which is neither Picardian nor pseudo-Abelian is birationally equivalent to a ruled variety, i.e. one containing a congruence of linear spaces.* Hence *any such variety has all its plurigenera equal to zero.*

(ii) *If a variety admits the above group and, further, the group leaves transitively invariant a linear system (∞^1 at least) of hypersurfaces which is not compounded of a congruence of subvarieties of any dimension ≥ 1, it is birationally equivalent to a ruled manifold.*

(iii) *On any pseudo-Abelian variety W_p of type q $(1 \leq q \leq \leq p-1)$, the involution cut by the congruence $\{V_q\}$ of trajectories on a member V_{p-q} of the complementary congruence $\{V_{p-q}\}$ is generable by a finite group of automorphisms of*

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V_{p-q} which is either cyclic or Abelian of base k, where k is any integer lying between 2 and $2q$ inclusive.

This result which, as Andreotti has remarked (§ 5), is a simple deduction from a formula given in Andreotti [1], generalises a well known theorem of Chisini [4] concerning elliptic curves, and leads to the analytical representation of W_p in the general case: in previous work only the cyclic case had been considered.

The pseudo-Abelian varieties provide interesting examples of manifolds endowed with torsion of various dimensions. We make some remarks on this subject in § 6. Apart, however, from the one-dimensional torsion, little or nothing has yet been achieved in this field.

Finally we notice a particular kind of para-Abelian variety ([11]) whose chief properties — for instance, the analytical representation and the nature of the various canonical systems — are very similar to those of the pseudo-Abelian variety itself. We also refer briefly to the *special* pseudo-Abelian variety whose group of automorphisms contains invariant subgroups (systems of imprimitivity).

2. Some properties of Picard varieties. - We begin by recalling the chief results concerning Picard varieties which we shall require; for other properties and for further details we refer to [12] and [13].

As usual we denote by V_p a Picard variety free from singular points and exceptional subvarieties, and by $(u_1, u_2, \ldots, u_p)$ its customary parametric representation. V_p admits a completely and simply transitive permutable continuous group of ∞^p automorphisms, consisting of transformations of the first kind, and represented by the equations

$$(1) \qquad u_i' = u_i + a_i \qquad (i = 1, 2, \ldots, p).$$

Conversely, it may be proved ([5]) that any algebraic variety W_p which is free from singularities and which is endowed with a completely transitive permutable continuous group of ∞^p automorphisms is necessarily a Picard variety. It follows from the proof that the group consists of transformations of the first kind on W_p and hence that it is *simply* transitive on W_p. Hence, if a variety W_p admits a continuous permutable ∞^p group which is *multiply* transitive on W_p, then this can only be *generally* transitive, so that W_p is either birational or quasi-Abelian (§ 3); in the latter case W_p is birationally equivalent to the product of a Picard variety V_q $(1 \leq q \leq p-1)$ and a space S_{p-q}.

In this work we are particularly concerned with simple involutions without coincidences which can be carried by V_p. To begin with, it is obvious that any involution generated by transformations of the first kind is free from united points. Conversely it may be proved that, in the case where V_p has general moduli, any simple involution I_n without coincidences carried by V_p is Picardian, and generable by a group, of order n, of transformations of the first kind. Evidently such a group is permutable (Abelian); moreover Andreotti [1] has shown that it may be generated by the transformation

$$u'_i = u_i + \sum_{r=1}^{2p} \mu_r \frac{\omega_{ir}}{\nu_r} \qquad (i = 1, 2, \ldots, p), \tag{2}$$

where (ω_{rs}) denotes the period matrix of V_p, μ_r (≥ 1) are integers, and ν_r (≥ 1) are integers such that, for every r, ν_r divides ν_{r+1}.

While this result holds for all Picard varieties, in the case where V_p is *special,* i.e. has particular moduli, there may exist simple involutions I_n, without coincidences, which are not Picardian; we call these *Abelian of the first species.* It may be shown that the variety which maps I_n has superficial irregularity p' $(0 < p' < p)$ and that I_n is generable by a finite group of *singular transformations* of V_p; any such transformation is reducible to the form

$$\begin{aligned} u'_i &= u_i + b_i \qquad (i = 1, 2, \ldots, p') \\ u'_j &= \varepsilon_j u_j + b_j \qquad (j = p'+1, p'+2, \ldots, p). \end{aligned} \tag{3}$$

where ε_j are roots of unity other than unity itself.

One further property of the Picard variety should be noted: V_p possesses an effective canonical hypersurface of order zero, so that its geometric genus P_g and its plurigenera

P_i are all unity. An Abelian variety of the first species likewise has a canonical hypersurface of order zero, but this may be effective *or* virtual; thus P_g and P_i can assume the values 0 or 1, and there is an infinity of plurigenera equal to unity (C. f. [12]).

3. Pseudo-Abelian and quasi-Abelian varieties. - Since much of our work deals with systems of equivalence, both linear and rational, while other parts depend upon transcendental results, it follows that in all cases we shall require our varieties to be non-singular; further, when considering congruences of subvarieties on the latter, we shall assume that the generic member of each congruence is likewise non-singular. In the present state of knowledge it cannot be said whether these hypotheses are restrictive.

Let W_p be a (non-singular) variety which admits a continuous permutable group $\mathfrak{G}$ of ∞^q automorphisms $(1 \leq q \leq \leq p-1)$; on the hypothesis that these are completely transitive on the trajectories of $\mathfrak{G}$, it follows that the latter form a congruence $\{V_q\}$ of Picard varieties V_q which are free from exceptional manifolds, and that the operations of $\mathfrak{G}$ are transformations of the first kind on V_q. While the generic member of the congruence is of course irreducible, we find that in general there exists a certain aggregate of reducible members each consisting of a Picard variety $V_{q,s}$ counted with a multiplicity s (≥ 2) (see § 6).

We may then prove that W_p contains a complementary congruence $\{V_{p-q}\}$ of ∞^q birationally equivalent varieties V_{p-q} which are transforms of one another under $\mathfrak{G}$. Since the transformations of $\mathfrak{G}$ are of the first kind on V_q, this congruence is Picardian [1]), i.e. representable by the points of a Picard variety, and cuts on the generic V_q an involution i_d $(d \geq 1)$ without coincidences. The number $d = [V_q V_{p-q}]$ is called the *determinant* of W_p; each number s is a divisor of d, including possibly d itself.

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[1]) It is incorrectly stated in [12] that $\{V_{p-q}\}$ can also be Abelian of the first species; this case arises when W_p is a para-Abelian variety of the kind described in § 7.

The variety W_p, which we have elsewhere ([10]) called *pseudo-Abelian of type q,* was first considered by Dantoni [2]) mainly from the transcendental point of view, in connection with simple integrals of the first kind. In the case $d > 1$, we may map W_p on the d-fold product $W_p^* = V_q^* \times V_{p-q}^*$, where V_q^* and V_{p-q}^* are birationally and unexceptionally equivalent to $\{V_{p-q}\}$ and $\{V_q\}$ respectively; thus W_p^* contains a congruence of Picard varieties which we may denote by $\{V_q^*\}$, and is in fact a pseudo-Abelian variety of determinant unity.

From the correspondence between W_p and W_p^* (for the details of which c.f. [12]), we obtain inequalities for the numbers g_i of i-fold integrals of the first kind attached to W_p. In particular we note the equation, due to Dantoni,

$$g_1 = q + q^*, \tag{4}$$

where q^* denotes the superficial irregularity of the congruence $\{V_q\}$.

Conversely, it may be proved that, if a variety W_p contains two congruences $\{V_q\}$ and $\{V_{p-q}\}$ of the kinds specified, it is pseudo-Abelian of type q; the proof, which is the same as in the case of the elliptic surfaces (see [7]), depends on the fact that the involution i_d is generable by a finite group of automorphisms of V_q. Alternatively, a proof by transcendental methods, based on § 5, may be given: see also Dantoni [6].

From the correspondence between W_p and W_p^* we may also deduce equivalences for the canonical varieties $X_h(W_p)$ $(h = q,\ q+1,\ \dots,\ p-1)$; for $h < q$, $X_h(W_p)$ is effective of order zero; see [12].

We conclude this preliminary account by describing a particular pseudo-Abelian variety of determinant 1 which oc-

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2) This work [6] came to the author's notice only after the above-mentioned papers had been published. DANTONI assumes implicitly that the varieties W_p which he considers are such that the associated congruences $\{V_q\}$ contain no reducible members; they are therefore very special subcases. But much of his analysis, including the formula (4), would seem to hold in the general case.

cupies an important position in the theory. If W_p admits a permutable continuous ∞^p group which is only *generally* transitive on W_p, we obtain what Severi has called a *quasi-Abelian* variety; in this case it may be shown that W_p is either birational or birationally equivalent to the product of a Picard variety V_q $(1 \leq q \leq p-1)$ and a space S_{p-q} ([14, 15]).

4. On varieties which admit continuous transformation groups. - We shall now see that the theory of varieties which admit finite continuous groups of automorphisms can be summarised succinctly in a form similar to that assumed by the analogous theory of curves and surfaces ([7]).

We first remark, with Painlevé, that any finite transformation group may, without loss of generality, be supposed algebraic. Next, we may always assume the group to be permutable, since any group contains a subgroup of transformations which permute with one another. In what follows we shall denote by $\mathfrak{G}$ any finite continuous permutable algebraic group of automorphisms operating on a non-singular variety W_n; the dimension of $\mathfrak{G}$ may have any value from 1 onwards. We shall also suppose that $\mathfrak{G}$ has no fixed points; if there were such points, W_n could not be Picardian or pseudo-Abelian.

We shall require the following lemma:

If W_n contains a congruence of subvarieties all of whose plurigenera are zero [3]*, then all the plurigenera of W_n are likewise zero.*

Here, as in the sequel, the congruence $\{W_r\}$ of subvarieties W_r is supposed to be free from base points. The proof is by induction on n. First, let $n = r+1$, so that $\{W_r\}$ is a pencil (rational or irrational) of hypersurfaces; then, if the system $|iX_{n-1}(W_n)|$ were effective (possibly of order zero) for some i, the system $|iX_{r-1}(W_r)|$ would itself be effective, contrary to hypothesis. Next, let $n = r+2$, and consider a rational pencil of hypersurfaces generated by varieties W_r; to each of

[3]) From the vanishing of *any* plurigenus follows that of the geometric genus.

these the previous reasoning applies, so that again W_n has the stated property. And similarly in general.

We now prove that

I. *Any variety W_n which admits a group $\mathfrak{G}$ and which has some plurigenus greater than zero is either Picardian or pseudo-Abelian.*

This result is a simple deduction from Severi 's work on quasi-Abelian varieties. Suppose first that W_n admits a group $\mathfrak{G}$ of dimension n which is either completely or generally transitive. In the former case W_n is a Picard variety; in the latter Severi shows, by transcendental methods, that if $g_1 = 0$, W_n is birational, and that if $g_1 > 0$ $(< n)$ W_n is birationally equivalent to the product of a Picard variety V_q and a space S_{n-q} $(1 \leq q \leq n-1)$, in which case it follows from the lemma that the plurigenera of W_n are all zero. This transcendental proof holds equally well in the case where $\mathfrak{G}$ has dimension greater than n.

Next, suppose that $\mathfrak{G}$ is intransitive, with any dimension from 1 onwards; in this case $\mathfrak{G}$ possesses trajectories W_r of some dimension r which form a congruence $\{W_r\}$. If $\mathfrak{G}$ has dimension r and is completely transitive over W_r, then W_r is a Picard variety and so W_n is pseudo-Abelian. In all other cases W_r is either birational or quasi-Abelian whence, by the lemma, the plurigenera of W_n are all zero.

It should be noted that there exist pseudo-Abelian varieties W_n $(n \geq 3)$ whose plurigenera are all zero.

II. *Any variety W_n which admits a group $\mathfrak{G}$ is either Picardian or pseudo-Abelian or is birationally equivalent to a ruled variety (i.e. one generated by a congruence of linear spaces S_r $(1 \leq r \leq n-1)$).*

This follows incidentally from the work [9] of Hall in which it is shown that any such variety W_n which does not belong to the first or second category is birationally equivalent to the product of a linear space S_r $(1 \leq r \leq n-1)$ and some variety of dimension $n-r$.

Actually theorem I is a consequence of this result and the preceding lemma, but the proof is less direct.

For the characterisation of the pseudo-Abelian varieties by means of their canonical systems, see M. Baldassarri, Annali di Mat., (4) 42 (1956), 227.

III. *Any variety W_n which admits a group $\mathcal{G}$ leaving transitively invariant a linear system (∞^1 at least) of hypersurfaces which is not compounded of a congruence of subvarieties of any dimension ≥ 1, is birationally equivalent to a ruled variety.*

This result is familiar in the case $n = 2$ ([7]). Assuming that $n > 2$, let $|L_{n-1}|$ be the linear system whose members are transformed into one another by $\mathcal{G}$. Then obviously W_n cannot be a Picard variety while at the same time $\mathcal{G}$ is the entire group of transformations of the first kind on W_n; if instead $\mathcal{G}$ is a subgroup, then W_n is to be regarded for present purposes as a pseudo-Abelian variety.

If W_n were a pseudo-Abelian variety of some type q (> 0) it would contain a congruence $\{W_{n-q}\}$, complementary to the congruence $\{V_q\}$ of trajectories, which is transitively invariant under $\mathcal{G}$ (§ 3); and by hypothesis $|L_{n-1}|$ could not be compounded of $\{W_{n-q}\}$. Hence $|L_{n-1}|$ would cut on each variety V_q a linear system $|L_{q-1}|$ which must be transitively invariant under $\mathcal{G}$; and this again is impossible. Thus, by II, W_n is birationally transformable into a ruled variety.

5. Analytical representation of W_p. - Of the varieties considered in § 4 the Picard varieties and those types which are birationally equivalent to ruled varieties have familiar analytical or parametric representations. Here we shall deal with the problem of representing a pseudo-Abelian variety W_p of type q in the general case; only the cyclic case has been considered in previous work.

It follows from § 3 that we have first to obtain the general form of the analytical representation of a Picard variety V_q on a d-ple Picard variety V_p^*; evidently this will depend upon the structure of the (Abelian) group $\mathcal{G}_d$ of automorphisms which generates the involution i_d on V_q whose image is V_q^*. In the case of an elliptic curve ($q = 1$) it was shown by Chisini [4] that $\mathcal{G}_d$ is either cyclic or of base 2. An examination of his work suggests that, in the general case,

$\mathcal{G}_d$ *is either cyclic or of base* k, *where* k *is any integer lying between* 2 *and* $2q$ *inclusive.* The result is contained implicitly in Andreotti's canonical form (§ 2) for a Picardian involution on V_q; this was pointed out to the author by Prof. Andreotti, who also supplied the following proof, which is analogous to Chisini's geometrical treatment of the case $q=1$.

Writing q for p in (2), we observe that we have to determine the nature of the representation of $\mathcal{G}_d$ as a group of translations in the real space S_{2q} $(x_1, x_2, \ldots, x_{2q})$, where $u_s = x_s + ix_{q+s}$ $(s=1, 2, \ldots, q)$. This will be unaltered if, for convenience, we suppose the period matrix (ω_{rs}) to be of the form $(E_r \mid iE_r)$, where (E) denotes a unitary matrix. The equations (2) then become

$$u_s' = u_s + \frac{\mu_s}{\nu_s} + \frac{i\mu_{q+s}}{\nu_{q+s}} \qquad (s=1, 2, \ldots, q)$$

Thus the group of translations in question is given by

$$x_s' = x_s + \frac{\mu_s}{\nu_s} \pmod 1 \qquad (s=1, 2, \ldots, 2q). \tag{5}$$

Here each number $\nu_1, \nu_2, \ldots, \nu_{2q}$ is a divisor of the next. If, then, k denotes the number of ν's which are greater than unity it follows that $\mathcal{G}_d$ can be represented as the direct product of k cyclic groups. The fact that the numbers ν_s can be chosen arbitrarily, subject to the condition stated, follows from the general theory of Riemann matrices. If instead all save one of the ν's are unity, $\mathcal{G}_d$ is simply a cyclic group, of order d.

From this result we obtain the representation of V_q and thence of W_p. Let I_n be the cyclic involution determined on V_q by any one of the above-mentioned groups of automorphisms; and suppose, as is always permissible, that V_q^* is situated in S_{q+1} and endowed with ordinary singularities, and that the equation of V_q^*, in non-homogeneous coordinates, is $F(x_1, x_2, \ldots, x_{q+1}) = 0$. Then I_n is represented by the equations

$$v^n = G(x_1, x_2, \ldots, x_{q+1}) \quad , \quad F(x_1, x_2, \ldots, x_{q+1}) = 0,$$

where G is a polynomial chosen so that the involution thus obtained is irreducible and without coincidences (for the conditions that G must satisfy we refer to [10]).

With a change of notation, let $n_1, n_2, \ldots, n_k$ be the respective orders of the cyclic involutions which form the base for $\mathcal{G}_d$, so that each number n_s divides n_{s+1}, and $n_1 n_2 \ldots n_k = d$; *then V_q is representable in the field of rationality defined by the elements*

$$\{ x_1, x_2, \ldots, x_{q+1}, G_1^{1/n_1}, G_2^{1/n_2}, \ldots, G_k^{1/n_k} \}$$

where $G_1, G_2, \ldots$ are suitably chosen polynomials in $x_1, x_2, \ldots$ and where

$$F(x_1, x_2, \ldots, x_{q+1}) = 0.$$

We may in particular represent V_q by the pair of equations

$$v = G_1^{1/n_1} + G_2^{1/n_2} + \ldots G_k^{1/n_k} \quad , \quad F = 0 \tag{6}$$

The conditions of irreducibility for the variety (6) are similar to those in the case $q = 1$ (c.f. [8]). Corresponding to this choice we have a representation for W_p. For supposing that V^*_{p-q} is situated in S_{p-q+1}, and represented by the equation $f(y_1, y_2, \ldots, y_{p-q+1}) = 0$, we see that W_p is represented by the set of equations

$$v = (G_1 g_1)^{1/n_1} + (G_2 g_2)^{1/n_2} + \ldots + (G_k g_k)^{1/n_k} \quad , \quad F = 0, \; f = 0, \tag{7}$$

where $g_1, g_2, \ldots$ are suitably chosen polynomials in $y_1, y_2, \ldots, y_{p-q+1}$.

6. Questions of classification. Torsion of W_p. The problem of classifying the pseudo-Abenloan varieties W_p, for assigned values of p and q, may be stated as follows: given the appropriate characters of the variety V_{p-q}, to determine all the birationally distinct types of variety W_p. We have thus in each case to find the characters of the associated congruence $\{V_q\}$, the value of the determinant with the corresponding numbers s, and finally the characters of the manifolds $B_h^{(s)}$ generated by the varieties $V_{q,s}$ (see below).

To begin with, we note that $\{V_q\}$ *cuts on each V_{p-p} an involution j_d which is generable by a group G_d of automorphisms of V_{p-q}, and that this group is either cyclic or Abelian of base k $(1 \leq k \leq 2q)$.* The group G_d consists of those transformations of $\mathcal{G}$ which leave each V_{p-q} invariant; the second fact is a consequence of § 5. Hence we have in the first place to determine all those varieties V_{p-q} which admit such groups G_d; for every such group we shall have an involution j_d of known order d and with a set of coincidence loci of various multiplicities and dimensions; these will furnish the numbers s and h. Analytically, this amounts to determining all the varieties V_{p-q} which can be represented by a pair of equations of the form

$$v = g_1^{1/n_1} + g_2^{1/n_2} + \dots + g_k^{1/n_k} \quad , \quad f = 0,$$

where the polynomials $f, g_1, g_2, \dots$ have the same meaning as in § 5. The invariant characters of j_d will furnish those of the congruence $\{V_q\}$.

In the particular case where all the dimensions h of the manifolds $B_h^{(s)}$ are equal to $p-1$, we have correspondence formulae which connect the characters of the canonical systems of V_{p-q} and V^*_{p-q} or, what is the same thing, $\{V_q\}$; these formulae have been given in [12]. They may serve to check the previous results; but they may on occasion be evanescent as, for instance, in the case of a elliptic surface W_2 containing an elliptic pencil of elliptic curves, or an elliptic threefold W_3 containing an elliptic pencil of Picard surfaces. If instead there are values of $h < p-1$, there will be fundamental elements in the correspondence between V_{p-q} and V^*_{p-q}, and formulae for this case are not yet known.

The manifolds $B_h^{(s)}$ which have already been mentioned arise in the following way. In the mapping of W_p on W_p^*, the generic trajectory V_q corresponds to a d-ple Picard variety V_q^*, the representation being without branch points. Hence the branch locus on W_p^* is either lacking altogether or else consists of a number of irreducible varieties each of which is generated by V_q^*'s. To each generator $V^*_{q,s}$, say, of an $(s-1)$-fold component of the branch locus, there corresponds a variety $V_{q,s}$ which is an $(s-1)$-fold element of the coincidence locus on W_p, and which is such that $sV_{q,s} \equiv V_q$. Evidently $V_{q,s}$ is itself a Picard variety, for it is mapped on a d/s-ple Picard variety $V^*_{q,s}$ without the intervention of branch points; it is in fact a trajectory of the group $\mathfrak{G}$.

We denote by $B_h^{(s)}$ an irreducible component of the manifold generated by $V_{q,s}$'s. It may happen that there exist two varieties $B_h^{(s)}$, $B_h'^{(s)}$, corresponding to the same value of s, which are algebraically isolated but such that $sB_h^{(s)} \equiv sB_h'^{(s)}$; in that case W_p has divisor $\sigma_h > 1$, i.e. is endowed with torsion. While the presence of varieties $V_{q,s}$ gives rise to the possibility of torsion of various dimensions, the general question is complicated, as the following remarks suggest.

i) For the elliptic surfaces W_2 Andreotti [2] has given a formula which expresses the divisor σ in terms of the characters s; from this it follows that, if there are no curves $V_{1,s}$ on W_2, then $\sigma = 1$. On the other hand, Gherardelli (see [2]) has shown by an example that the mere presence of two curves $V_{1,s}$ with equal values of s does not necessarily imply that $\sigma > 1$.

ii) It is interesting to note that the first of the above results does *not* in general extend to varieties W_p with $p > 2$. Consider an elliptic W_3 containing an « Enriques » congruence $\{V_1\}$ of trajectories, i.e. a congruence whose elements are mapped by the points of a non-singular surface which is a birational transform of the sextic surface of Enriques ([7]). It is well known that this contains rational pencils $|\mathcal{C}|$ of elliptic curves $\mathcal{C}$ each including two reducible members $2\mathcal{C}_1$, $2\mathcal{C}_2$ such that $\mathcal{C}_1$, $\mathcal{C}_2$ are algebraically disequivalent. It follows that the surfaces belonging to $\{V_1\}$ and corresponding to the curves $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$ behave in a similar fashion and hence that W_3 is endowed with torsion.

Suppose now that the complementary (elliptic) pencil $\{V_2\}$ on W_3 consists of regular surfaces of genera and plurigenera unity of the kind which carry Enriques involutions j_2; since such involutions are free from coincidences ([7]) we see that there can be no curves $V_{1,s}$ in $\{V_1\}$.

An example of such a threefold is easily constructed. With the notation of § 4, let $F(x_1, x_2) = 0$ be the equation of a non-singular plane cubic, and $G(x_1, x_2) = 0$ that of a tritangent conic. Let $f(y_1, y_2, y_3) = 0$, $g(y_1, y_2, y_3) = 0$, denote respectively the equations of an Enriques surface and the tetrahedron whose edges form its double curve; then the equations

(8) $$v^2 = G(x_1, x_2)g(y_1, y_2, y_3), \quad F = 0, \quad f = 0,$$

represent a threefold of the kind required.

iii) On the other hand, it is known from the theory of the Abelian (hyperelliptic) surfaces that certain Picard surfaces can carry Enriques involutions having a finite (non-zero) number of coincidences ([3]). If, then, we suppose an elliptic threefold W_3 with an Enriques congruence $\{V_1\}$ to contain an elliptic pencil $\{V_2\}$ of such Picard surfaces, we shall have a finite number of curves $V_{1,s}$ in $\{V_1\}$.

iv) Again, an elliptic threefold W_3 may possess an apparent torsion due simply to singular features of the analytical representation. Thus suppose that W_3 contains a « Kummer » congruence $\{V_1\}$ of trajectories (i.e. one birationally equivalent to a Kummer quartic surface) and an elliptic pencil $\{V_2\}$ of Picard surfaces bisecant to $\{V_1\}$. Let $f(y_1, y_2, y_3) = 0$ represent the Kummer surface, and $g(y_1, y_2, y_3) = 0$, one of its Rosenhain tetrahedra. Then, with the same interpretation of the polynomials f and g as before, the equations (8) represent a threefold W_3 of the desired type. In this case there are 16 isolated curves $V_{1,2}$ corresponding to the 16 coincidences of the involution j_2. But the « torsion » of the Kummer congruence arises from the fact that the surface $f = 0$ possesses non-ordinary singularities in the shape of 16 isolated nodes.

7. A particular class of para-Abelian variety. Consider a variety W_p containing a congruence $\{V_q\}$ of the kind described in § 3; suppose that W_p contains also a congruence $\{V_{p+q}\}$ of birationally equivalent varieties V_{p+q} which instead of being Picardian is Abelian of the first species (§ 2), and that in addition the involution i_d cut by $\{V_{p+q}\}$ on the generic V_q is cyclic, being generated by a transformation of the form (3), where p is replaced by q, and where b_j, b_j are suitably chosen constants; in fact, by a change of parameters, we can always assume the b_j 's to be zero. The cyclic group which generates i_d is contained in the series Σ of transformations represented by $u_i' = u_i + a_i$, $u_j' = \varepsilon_j u_j + a_j$, where a_j, a_j are arbitrary parameters; and within Σ there is a discontinuous series of transformations which leave i_d invariant, namely those determined by arbitrary values of a_i and by values of a_j such that $a_j(1-\varepsilon_j) \equiv 0$.

By virtue of the presence of the congruences $\{V_q\}$, $\{V_{p+q}\}$, the variety W_p is a special case of the para-Abelian variety defined in [11]. Evidently it is representable on a cyclic multiple variety W_p^* which is constructed as in § 3. The discontinuous series of automorphisms on V_q gives rise to a similar series on W_p. The remaining properties of W_p are closely analogous to those of a pseudo-Abelian variety of type q; they have already been established in [12]. Thus, while for the pseudo-Abelian variety the canonical varieties X_h $(h < q)$ are all effective of order zero, the corresponding canonical varieties of W_p may be either *effective* or *virtual* of order zero. The equivalences for the varieties $X_h(W_p)$ $(h \geq q)$ and the inequalities for the characters $g_i(W_p)$ will also be found in [12]. In fact, W_p is pseudo-Abelian of type less than q.

8. On special pseudo-Abelian varieties. In previous work we have had occasion to consider the special Picard varieties; we recall that a Picard variety V_q is termed *special* if it contains a Picard congruence $\{V_{q_1}\}$ of Picard varieties V $(1 \leq q_1 \leq q-1)$, in which case it must contain a complementary congruence $\{V_{q-q_1}\}$ of Picard varieties V_{q-q_1}. Such a variety may be regarded as pseudo-Abelian either of type q_1 or of type $q-q_1$; for either of the above congruences may be thought of as a congruence of trajectories of an invariant subgroup contained in the complete group of transformations of the first kind on V_q. Similarly we may envisage a variety V_q which contains three or more Picardian congruences of Picard varieties the sum of whose dimensions is q.

In an analogous manner we may consider a variety W_p which can be regarded as pseudo-Abelian in two or more different ways, by reason of the fact that it contains invariant subgroups of the group $\mathfrak{G}$ defined in § 3. Let $(q_1, q_2, \ldots, q_r)$ denote any partition of the number q, and consider the variety $W_p^* = V_{q_1}^* \times V_{q_2}^* \times \ldots V_{q_r}^* \times V_{p-q}^*$, where $V_{q_1}^*$, $V_{q_2}^*$, ..., $V_{q_r}^*$

are Picard varieties and V^*_{p-q} is any given manifold. Then the variety W_p which is mapped on the d-fold variety W^*_p in the usual way contains r congruences $\{V_{q_i}\}$ $(i = 1, 2, ..., r)$ of Picard varieties, each of which is generated by trajectories of an invariant subgroup of $\mathfrak{G}$. We may call W_p a special pseudo-Abelian variety of type $(q_1, q_2, ..., q_r)$ and determinant d. Evidently each congruence $\{V_{q_i}\}$ of trajectories on W_p is d-secant to its complementary congruence $\{V_{p-q_i}\}$. We could of course construct a wider class of special pseudo-Abelian variety for which the intersection numbers $[V_{q_i} V_{p-q_i}]$ are unequal; but the representation of such a variety would be more complicated. We should also be faced with questions of existence which depend in their turn on difficult problems concerning the special Picard varieties.

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