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SOME THEOREMS ON RECURRENT AND RICCI-RECURRENT SPACES

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Introduction. - Ruse [4] and Walker [5] introduced the recurrent spaces for which the curvature tensor satisfies a relation of the form

$$R_{hijk,l} = R_{hijk}\lambda_l \tag{1}$$

where λ_l is a non-zero vector and comma denotes covariant derivative. An n-space of this kind was denoted by K_n. In a recent paper [3] Patterson considered a Riemannian space of more than two dimensions whose Ricci tensor satisfies

$$R_{ij} \neq 0, \quad R_{ij,k} = R_{ij}\lambda_k \tag{2}$$

for some non-zero vector λ_k. He called such a space Ricci-Recurrent and denoted an n-space of this kind by R_n. Obviously every K_n is an R_n but the converse is not true. The question as to when an R_n can be a K_n has been considered in section 1 of this paper. In the remaining sections some necessary conditions for a space V_n $(n > 2)$ to be conformal to a recurrent and to a Ricci-recurrent space have been given.

1. - It is known that the conformal tensor C_{ijkl} of a Riemannian space V_n with metric tensor g_{ij} is given by

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{jk}g_{il} - R_{jl}g_{ik} + R_{il}g_{jk} - R_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{jk}g_{il} - g_{jl}g_{ik}). \tag{1.1}$$

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Now let the V_n be a Ricci-recurrent space R_n specified by a non-zero vector λ_i. Then since

$$R_{ij,p} = R_{ij}\lambda_p \quad \text{and} \quad R,_p = R\lambda_p$$

it follows from (1.1) that

$$(1.2) \qquad \begin{aligned} C_{ijkl,p} &= R_{ijkl,p} + (C_{ijkl} - R_{ijkl})\lambda_p \\ \text{or} \quad C_{ijkl,p} - C_{ijkl}\lambda_p &= R_{ijkl,p} - R_{ijkl}\lambda_p . \end{aligned}$$

Conversely, if (1.2) holds, multiplying both sides of (1.2) by g^{jk} and summing for j and $\underset{l.c.}{k}$ we get

$$C_{il,p} - C_{il}\lambda_p = R_{il,p} - R_{il}\lambda_p$$

which reduces, in virtue of $C_{ij} = 0$, to

$$(1.3) \qquad R_{il,p} = R_{il}\lambda_p$$

From (1.2) and (1.3) we have therefore the following theorem:

THEOREM 1. A necessary and sufficient condition that a V_n be an R_n is that (1.2) holds.

In particular, if the V_n is conformal to a flat space or if $n = 3$ then $C_{ijkl} = 0$. In the first case it follows from (1.2) that the R_n is a K_n. In the second case it follows that the R_3 is a K_3. Thus we have the following theorem:

THEOREM 2. Every R_n $(n > 3)$ is a K_n if it is conformal to a flat space and every R_3 is a K_3.

Again for an R_n specified by a non-zero vector λ_i we have

$$R_{ij,kl} - R_{ij,lk} = (\lambda_{k,l} - \lambda_{l,k})R_{ij}$$

or by Ricci's identity

$$(1.4) \qquad R_{pj}R^p_{ikl} + R_{ip}R^p_{jkl} = (\lambda_{k,l} - \lambda_{l,k})R_{ij} .$$

If λ_i be a gradient then $\lambda_{k,l} = \lambda_{l,k}$ and so the lefthand side of (1.4) is zero. Conversely if the lefthand side of (1.4) is zero then λ_i is a gradient. From this we have the following theorem:

THEOREM 3. A necessary and sufficient condition that the vector of recurrence of an R_n be a gradient is that at every point of the R_n

$$R_{pj}R^p_{ikl} + R_{ip}R^p_{jkl} = 0. \tag{1.5}$$

Since the vector of recurrence of every K_n is a gradient it follows in virtue of Theorem 2 that the relation (1.5) holds at every point of an R_n $(n > 3)$ which is conformal to a flat space and at every point of an R_3.

2. - In this section we deduce a necessary condition that a space V_n $(n > 3)$ with metric tensor g_{ij} be conformal to a recurrent space.

Let $\bar{V}_n$ be a space with metric tensor

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} \tag{2.1}$$

Then since $\bar{C}^t_{ijk} = C^t_{ijk}$ we have

$$\bar{C}_{hijk} = e^{2\sigma} C_{hijk} \tag{2.2}$$

Now suppose $\bar{V}_n$ is a recurrent space specified by a non-zero vector λ_i and let a comma and a semi-colon denote covariant derivatives with respect to g_{ij} and $\bar{g}_{ij}$ respectively. Then from (1.1) we get

$$\bar{C}_{hijk;l} = \bar{C}_{hijk}\lambda_l$$

Writing $\sigma_i = \dfrac{\partial \sigma}{\partial x_i}$, $\sigma^j = g^{ij}\sigma_i$, we obtain from (2.2)

$$\begin{aligned} e^{2\sigma} C_{hijk}\lambda_l &= e^{2\sigma} C_{hijk;l} + 2e^{2\sigma} C_{hijk}\sigma_l \\ \text{or}\quad C_{hijk}\lambda_l &= 2C_{hijk}\sigma_l + C_{hijk;l} \end{aligned} \tag{2.3}$$

But

$$\begin{aligned} C_{hijk;l} = C_{hijk,l} - 4C_{hijk}\sigma_l - (C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j + C_{hijl}\sigma_k) \\ + \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs}). \end{aligned}$$

Therefore, since C_{hijk} possesses the properties of the indices of R_{hijk}, the equation (2.3) reduces to

$$
\begin{aligned}
(2.4)\qquad C_{hijk}\lambda_l = {} & C_{hijk,l} - 2C_{hijk}\sigma_l \\
& - (c_{lijk}\sigma_h + C_{lhkj}\sigma_i + C_{lkhi}\sigma_j + C_{ljih}\sigma_k) \\
& + \sigma^s(g_{hl}C_{sijk} + g_{il}C_{shkj} + g_{jl}C_{skhi} + g_{kl}C_{sjih}).
\end{aligned}
$$

We may therefore state the following theorem:

THEOREM 4. A necessary condition, given in terms of the conformal tensor, that a V_n be conformal to a K_n is that the equation (2.4) holds, provided that the V_n is not conformal to a flat space.

Further, substituting from (2.1) in the last two lines of (2.4) and simplifying we obtain the condition (2.4) as

$$
\begin{aligned}
(2.5)\qquad C_{hijk}\lambda_l = {} & C_{hijk,l} - 2C_{hijk}\sigma_l \\
& - [(R_{lijk}\sigma_h + R_{lhkj}\sigma_i + R_{lkhi}\sigma_j + R_{ljih}\sigma_k) \\
& + \frac{1}{n-2}\{(g_{ik}R_{lj} - g_{ij}R_{lk})\sigma_h + (g_{hj}R_{lk} - g_{hk}R_{lj})\sigma_i \\
& + (g_{ik}R_{lh} - g_{hk}R_{li})\sigma_j + (g_{hj}R_{li} - g_{ij}R_{lh})\sigma_k\}] \\
& + \sigma^s[(g_{hl}R_{sijk} + g_{il}R_{shkj} + g_{jl}R_{skhi} + g_{kl}R_{sjih}) \\
& + \frac{1}{n-2}\{g_{hl}(g_{ik}R_{sj} - g_{ij}R_{sk}) + g_{il}(g_{hj}R_{sk} - g_{hk}R_{sj}) \\
& + g_{jl}(g_{ik}R_{sh} - g_{hk}R_{si}) + g_{kl}(g_{hj}R_{si} - g_{ij}R_{sh})\}].
\end{aligned}
$$

If now the vector σ_i be supposed to be parallel in the space V_n and therefore $\sigma^s R_{sijk} = 0 = \sigma^s R_{sk}$, then the last three lines in the above equation vanish. On the otherhand if the space V_n be supposed to be a recurrent space specified by σ_i then the second line of the above equation reduces to $-2R_{hijk}\sigma_l$.

3. - In this section we deduce a necessary condition that a V_n be conformal to a Ricci-recurrent space.

With reference to the conformal relation (2.1) we have the following equations

$$
\begin{aligned}
(3.1)\qquad e^{-2\sigma}R_{hijk} = {} & R_{hijk} + g_{hk}(\sigma_{i,j} - \sigma_i\sigma_j) + g_{ij}(\sigma_{h,k} - \sigma_h\sigma_k) \\
& - {}_{hj}(\sigma_{i,k} - \sigma_i\sigma_k) - g_{ik}(\sigma_{h,j} - \sigma_h\sigma_j) + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma
\end{aligned}
$$

whence

$$(3.2) \qquad \sigma_{i,j} - \sigma_i\sigma_j = \frac{1}{n-2}(\bar{R}_{ij} - R_{ij})$$

$$- \frac{1}{2(n-1)(n-2)}(\bar{g}_{ij}\bar{R} - g_{ij}R) - \frac{1}{2}\, g_{ij}\Delta_1\sigma\,.$$

Now let $\bar{V}_n$ with metric tensor $\bar{g}_{ij}$ be supposed to be a Ricci-recurrent space. Differentiating (3.2) covariantly with respect to g_{ij} and substituting

$$(3.3) \quad \begin{cases} \bar{R}_{ij,k} = \bar{R}_{ij}\lambda_k + 2\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j - g^{sm}\sigma_m(\bar{R}_{sj}g_{ik} + \bar{R}_{si}g_{jk} \\ \bar{g}_{ij,k} = 2\bar{g}_{ij}\sigma_k \quad , \quad \bar{R}_{,k} = R\lambda_k \end{cases}$$

we get

$$(3.4) \qquad \sigma_{i,jk} - (\sigma_i\sigma_{j,k} + \sigma_j\sigma_{i,k})$$

$$= \frac{1}{n-2}\{ \bar{R}_{ij}\lambda_k + 2\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j$$

$$- g^{sm}\sigma_m(\bar{R}_{sj}g_{ik} + \bar{R}_{si}g_{jk}) - R_{ij,k} \}$$

$$- \frac{g_{ij}}{2(n-1)(n-2)}\{ e^{2\sigma}\bar{R}(\lambda_k + 2\sigma_k) - R_{,k} \} - \frac{1}{2}\, g_{ij}(\Delta_1\, \sigma)_{,k}\,.$$

But from (3.2)

$$\sigma_i\sigma_{j,k} + \sigma_j\sigma_{i,k} - 2\sigma_i\sigma_j\sigma_k = \frac{1}{n-2}\{ \sigma_i(\bar{R}_{jk} - R_{jk}) + \sigma_j(\bar{R}_{ik} - R_{ik}) \}$$

$$- \frac{1}{2(n-1)(n-2)}\{ \sigma_i(\bar{g}_{jk}\bar{R} - g_{jk}R) + \sigma_j(\bar{g}_{ik}\bar{R} - g_{ik}R) \}$$

$$- \frac{1}{2}\, \Delta_1\sigma(\sigma_i g_{jk} + g_j g_{ik})\,.$$

Substituting in (3.4), we obtain

$$(3.5) \qquad \sigma_{i,jk} - 2\sigma_i\sigma_j\sigma_k = \frac{1}{n-2}\{ \bar{R}_{ij}\lambda_k + 2(\bar{R}_{ij}\sigma_k + \bar{R}_{jk}\sigma_i + \bar{R}_{ik}\sigma_j)$$

$$- \sigma_m(\bar{R}^{m}_{j}\bar{g}_{ik} + \bar{R}^{m}_{i}\bar{g}_{jk}) - (R_{ij,k} + R_{jk}\sigma_i + R_{ik}\sigma_j) \}$$

$$-\frac{1}{2(n-1)(n-2)}[e^{2\sigma}\bar{R}\{g_{ij}(\lambda_k+2\sigma_k)+g_{jk}\sigma_i+\sigma_{ik}\sigma_j\}$$

$$-\{g_{ij}R,_k+R(g_{jk}\sigma_i+g_{ik}\sigma_j\}]$$

$$-\frac{1}{2}\{g_{ij}(\Delta_1\sigma),_k+\Delta_1\sigma(g_{jk}\sigma_i+g_{ik}\sigma_j\}.$$

This is therefore a necessary condition to be satisfied by the function σ in order that a V_n with metric tensor g_{ij} be conformal to the Ricci-recurrent $\bar{V}_n$ with metric tensor $\bar{g}_{ij}=$ $=e^{2\sigma}g_{ij}$.

If V_n is supposed to be an Einstein space we have simply to put $R_{ij,k}=R,_k=0$ and $R_{ij}=\frac{R}{n}g_{ij}$ in (3.5).

4. - The condition (3.5) which has been obtained by straightforward method is not simple enough. We look for a simpler condition. For this purpose we first obtain the condition of integrability of the equation (3.2).

Differentiating (3.2) covariantly with respect to x^k, interchanging the indices j and k and subtracting, the lefthand side becomes, after the operation,

l. c.

$$\sigma_{i,jk}-\sigma_{i,kj}+\sigma_{i,j}\sigma_k-\sigma_{i,k}\sigma_j=\sigma_s R^s_{ijk}+\sigma_{i,j}\sigma_k-\sigma_{i,k}\sigma_j.$$

From (3.1) we obtain

$$\sigma_s R^s_{ijk}=\sigma_s[\bar{R}^s_{ijk}-\delta^s_k(\sigma_{i,j}-\sigma_i\sigma_j)-g_{ij}g^{hs}(\sigma_{h,k}-\sigma_h\sigma_k)$$

$$+\delta^s_j(\sigma_{i,k}-\sigma_i\sigma_k)+g_{ik}g^{hs}(\sigma_{h,j}-\sigma_h\sigma_j)-(\delta^s_k g_{ij}-\delta^s_j g_{ik})\Delta_1\sigma]$$

$$=\sigma_s\bar{R}^s_{ijk}-\sigma_k(\sigma_{i,j}-\sigma_i\sigma_j)-\frac{1}{2}g_{ij}(\Delta_1\sigma),_k+g_{ij}\sigma_k\Delta_1\sigma$$

$$+\sigma_j(\sigma_{i,k}-\sigma_i\sigma_k)+\frac{1}{2}g_{ik}(\Delta_1\sigma),_j-g_{ik}\sigma_j\Delta_1\sigma-(\sigma_k g_{ij}-\sigma_j g_{ik})\Delta_1\sigma$$

$$=\sigma\,\bar{R}^s_{ijk}+\sigma_j\sigma_{i,k}-\sigma_k\sigma_{i,j}+\frac{1}{2}\{g_{ik}(\Delta_1\sigma),_j-g_{ij}(\Delta_1\sigma),_k\}.$$

Therefore after the operation we obtain from (3.2)

(4.1)
$$\sigma_s \bar{R}^s_{ijk} + \frac{1}{2}\{ g_{ik}(\Delta_1\sigma)_{,j} - g_{ij}(\Delta_1\sigma)_{,k} \}$$
$$= \frac{1}{n-2}\{ (\bar{R}_{ij,k} - \bar{R}_{ik,j}) - (R_{ij,k} - R_{ik,j}) \}$$
$$- \frac{1}{2(n-1)(n-2)}\{ \bar{R}(\bar{g}_{ij,k} - \bar{g}_{ik,j}) + (\bar{g}_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j})$$
$$- (g_{ij}R_{,k} - g_{ik}R_{,j}) \} - \frac{1}{2}\{ g_{ij}(\Delta_1\sigma)_{,k} - g_{ik}(\Delta_1\sigma)_{,j} \}.$$

Now apply (3.3). Then (4.1) reduces to

$$\sigma_s \bar{R}^s_{ijk} = \frac{1}{n-2}[\bar{R}_{ij;k} - \bar{R}_{ik;j} + \bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma_t \bar{g}^{st}(\bar{g}_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj})$$
$$- \frac{1}{2(n-1)}\{ 2\bar{R}(\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j) + \bar{g}_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j} \}]$$
$$- \frac{1}{n-2}[R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j})].$$

Writing as usual (Eisenhart, 1926, P. 91)

(4.2)
$$\begin{cases} R_{ijk} = R_{ij,k} - R_{ik,j} - \dfrac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j}) \\ \bar{R}_{ijk} = \bar{R}_{ij;k} - \bar{R}_{ik;j} - \dfrac{1}{2(n-1)}(\bar{g}_{ij}\bar{R}_{,k} - \bar{g}_{ik}\bar{R}_{,j}) \end{cases}$$

the above equation reduces to

$$\sigma_s \bar{R}^s_{ijk} - \frac{1}{n-2}[\bar{R}_{ij}\sigma_k - \bar{R}_{ik}\sigma_j + \sigma_t \bar{g}^{st}(\bar{g}_{ij}\bar{R}_{sk} - \bar{g}_{ik}\bar{R}_{sj})$$
$$- \frac{\bar{R}}{n-1}(\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j)] = \frac{1}{n-2}[\bar{R}_{ijk} - R_{ijk}].$$

Now from the conformal tensor (1.1) we get

$$\sigma_h C^h_{ijk} = \sigma_h \bar{C}^h_{ijk} = \sigma_h \bar{R}^h_{ijk}$$

$$+ \frac{1}{n-2} \{ \bar{R}_{ik}\sigma_j - \bar{R}_{ij}\sigma_k + \sigma_h \bar{g}^{ht}(\bar{g}_{ik}\bar{R}_{tj} - \bar{g}_{ij}\bar{R}_{tk}) \}$$

$$+ \frac{\bar{R}}{(n-1)(n-2)} (\bar{g}_{ij}\sigma_k - \bar{g}_{ik}\sigma_j).$$

Substituting this in the above equation we obtain finally

$$\sigma_s C^s_{ijk} = \frac{1}{n-2} (\bar{R}_{ijk} - R_{ijk}) \tag{4.3}$$

This is the condition of integrability of equation (3.2). This reduces to a known result given by Brinkmann [1] in the case when $\bar{V}_n$ is an Einstein space.

Now supposte that $\bar{V}_n$ is a Ricci-recurrent space specified by a vector λ_i. Then from (4.2)

$$\bar{R}_{ijk} = \bar{R}_{ij}\lambda_k - \bar{R}_{ik}\lambda_j + \frac{\bar{R}}{2(n-1)} (\bar{g}_{ik}\lambda_j - \bar{g}_{ij}\lambda_k).$$

Substituting in (4.3) we get

$$\sigma_s C^s_{ijk} + \frac{1}{n-2} R_{ijk} \tag{4.4}$$

$$= \frac{1}{n-2} \left\{ \bar{R}_{ij}\lambda_k - \bar{R}_{ik}\lambda_j + \frac{\bar{R}e^{2\sigma}}{2(n-1)} (g_{ik}\lambda_j - g_{ij}\lambda_k) \right\}.$$

We may therefore state the following theorem:

THEOREM 5. A necessary condition that a space V_n be conformal to a Ricci-recurrent $\bar{V}_n$ is that there exists a function σ satisfying (4.4).

If V_n be an Einstein space, we have simply to put $R_{ijk} = 0$ in (4.4).

And if V_n be a Ricci-recurrent space specified by a vector μ_i we have simply to put in (4.4)

$$R_{ijk} = R_{ij}\mu_k - R_{ik}\mu_j + \frac{R}{2(n-1)}(g_{ik}\mu_j - g_{ij}\mu_k).$$

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