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IMPROPERLY ABELIAN VARIETIES

Nota (*) di Leonard Roth (a Londra)

The present note continues the study of the pseudo-Abelian varieties initiated in (7). We begin by recalling that a pseudo-Abelian variety of type q (1 ≤ q ≤ p − 1) is any non-singular variety $W_p$ which is invariant under a continuous group of $\infty^q$ automorphisms whose trajectories form a congruence ($\infty^{p-q}$ system of index 1) of Picard varieties $V_q$; it is thus a natural generalization of the elliptic surfaces, and of the elliptic and hyperelliptic threefolds previously considered in (8) and (9). Just as the elliptic surfaces contain a subspecies, which we have elsewhere (9) called improperly hyperelliptic, and which map irregular involutions on a Picard surface (necessarily of particular moduli, containing pencils of elliptic curves), so the elliptic and hyperelliptic threefolds include subspecies, namely the improperly Abelian threefolds, the principal types of which have been determined in (10).

In this work the above results are generalized to the case $p > 3$. We first show that any Abelian variety $W_p$ having some plurigenus greater than zero, which maps an involution of superficial irregularity $q (0 < q < p)$ on a Picard variety $V_p$, is pseudo-Abelian of type $q$; then, from the representation of $W_p$ obtained in (7) we deduce that $W_p$ is improperly Abelian, i.e. is representable parametrically by means of Abelian functions of genera less than $p$: more precisely, we show that the coordinates of the general point of $W_p$ are expressible as algebraic functions of Abelian functions of

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genus $q$ and other Abelian functions of genus $p - q$, and that, in many cases, the genera of the functions required for the representation may be lowered still further.

We also remark that the classification of the improperly Abelian varieties is a recursive process depending on the determination of all species of lower dimension which map involutions on Picard varieties that are generable by finite groups of automorphisms, including, it should be added, those species which have all their plurigenera equal to zero. It follows from this that the classical discussion (2, 4) of the hyperelliptic surfaces, which excludes all the rational types, forms an inadequate basis for the treatment of the higher Abelian varieties.

The note concludes with a brief account of the para-Abelian varieties, which bear the same relation to the improperly Abelian varieties as do the paraelliptic surfaces (10) to the improperly elliptic. While these manifolds do not possess the group of automorphisms characteristic of the improperly Abelian varieties, they have certain affinities with the latter: thus, they are in general superficially irregular, and they are representable on multiple product varieties with a particularly simple kind of branch manifold; while their systems of canonical hypersurfaces are compounded of congruences of Picard varieties.

1. The classification of Abelian varieties. - We begin by outlining the process of classifying the Abelian varieties, on which our work depends, referring to (2), (5), (10) for details and illustrations of the method. Let $V_p$ be a Picard variety (Abelian variety of genus $p$ and rank 1) which we assume to be free from singularities and exceptional manifolds; then any Abelian variety $W_p$ of genus $p$ and rank $r > 1$ may be regarded as the image of a simple involution $I_n$ on $V_p$, so that the problem of classifying $W_p$ is equivalent to that of classifying $I_n$. We call $W_p$ properly or improperly Abelian, according as the variety cannot or can be represented parametrically by means of Abelian functions of genus lower than $p$. Thus $V_p$ itself is properly Abelian (in
fact we shall find that it is the only superficially irregular proper type); again, among the proper types, we have the Wirtinger variety (16): this is of rank 2, and maps an involution on $V_p$ which is generated by transformations of the second kind. We then have the following results.

I. The pure canonical and pluricanonical hypersurfaces 1) of $W_p$, if effective, are all of order zero.

For the canonical system of $V_p$, which is of order zero, is the transform of the canonical system of $W_p$, together with the coincidence hypersurface (if any) of $I_n$. It follows that the geometric genus $P_g$ and the plurigenera $P_i$ of $W_p$ satisfy the inequalities $P_g \leq 1$, $P_i \leq 1$.

II. The superficial irregularity $q$ of $W_p$ is at most equal to $p$.

For a generic surface of $W_p$ maps an involution on a surface of $V_p$ which is likewise generic and hence of irregularity $p$, whence the result (2). (Actually, we shall see that $q = p$ if, and only if, $W_p$ is a Picard variety).

III. If $I_n$ possesses $\infty^{p-1}$ coincidences, then the geometric genus and plurigenera of $W_p$ are all zero.

By hypothesis, there is a coincidence hypersurface on $V_p$ and hence a branch hypersurface $B$ on $W_p$. Consider a general linear system $|C|$ of hypersurfaces on $W$; this maps an irreducible system $|C_1|$ on $V_p$, which belongs to $I_n$, and whose adjoint system $|C'_1|$ is equivalent to the sum of the transform of $|C'|$ and the coincidence hypersurface. Now since $V_p$ has a canonical hypersurface of order zero, we have $C'_1 \equiv C_1$, from which it follows that $C \equiv C' + B$: that is to say, $W_p$ possesses an anticanonical system (11, 12). Hence, on $W_p$, the process of successive adjunction always terminates (11), so that $W_p$ must have geometric genus and plurigenera zero.

The group-theoretic method of classifying the varieties $W_p$, which was first applied systematically to the case $p = 2$

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1) That is, varieties of dimension $p - 1$. 
by Bagnera and De Franchis (2), is based on the above concepts, used in conjunction with the following theorem:

IV. If $W_p$ has some plurigenus greater than zero, then $I_n$ can be generated by a finite group $G_n$ of automorphisms of $V_p$.

For, in the first place, it can be shown (3, 1) that, on this hypothesis, $I_n$ cannot possess a united point which is conjugate to a hypersurface; in the second place, it follows from III that $I_n$ can have at most $\infty^{p-2}$ coincidences. Hence, by the transcendental-topological argument due to Bagnera-De Franchis (2) and Andreotti (1), it may be shown that $I_n$ is generable by a group $G_n$ of automorphisms of $V_p$.

It may be noted that the conjecture that, if the involution $I_n$ has at most $\infty^{p-2}$ coincidences, it can be generated by a group $G_n$, was first made by Lefschetz (6) without, however, the additional hypothesis concerning the plurigenera of $W_p$; that this hypothesis is essential may be shown by examples.

The converse theorems suggested by these results are not in general true. Thus, in the case $p=2$, Bagnera-De Franchis have shown that a rational involution $I_n$ may possess a finite (non-zero) number of coincidences, and have also indicated that an involution may be endowed with a coincidence curve and yet be generable by a finite group of automorphisms; while Scorza (13) has actually obtained all the involutions — necessarily rational — of this type. And, generally, it may be shown that there exist anticanonical involutions (i.e. such that $\omega V_p$ possesses an anticanonical system) endowed with $\infty^{p-1}$ coincidences which are generable by finite groups of automorphisms.

In order, however, to obtain a complete classification of the varieties $W_p$ within the limits of the group-theoretic method, it is evidently necessary, in view of III, to exclude those types for which $P_t=0$. In the case $p=2$, this amounts to no more than omitting rational and elliptic scrollar surfaces from the scheme; in the case $p>2$, however, the precise extent of such a limitation is not known. Even so, in order to obtain all the members of this restricted class of
varieties $W_p$, we shall see that it is still necessary to determine those types $W_r$ ($r < p$) with zero plurigenera which map involutions on $V_r$ generable by finite groups of automorphisms.

2. The superficially irregular types. - Supposing that, with the customary notation, the general point of $V_p$ has coordinates $(u_1, u_2, \ldots, u_p)$, let $W_p$ be an Abelian variety which represents any (simple) involution $\Gamma$ generable by a group $\mathcal{G}_n$; then it may be shown (2, 5) that $\mathcal{G}_n$ itself can be generated by a finite set of substitutions of the form

$$u_i' = \sum_{j=1}^p a_{ij} u_j + b_i \quad (i = 1, 2, \ldots, p)$$

where $a_{ij}$ and $b_i$ are constants.

In the case where $W_p$ has superficial irregularity $q > 0$, we may show further (2, 5) that $q$ of the above relations may be taken to be

$$u_i' = u_i + b_i \quad (i = 1, 2, \ldots, q)$$

Lefschetz (5) has remarked that, by modifying suitably the period matrix of $V_p$, the remaining relations of the set (1) may be reduced to the form

$$u_j' = \varepsilon_j u_j + b_j \quad (j = q + 1, q + 2, \ldots, p)$$

The constants $\varepsilon_j$ are called the multipliers of the substitution; and since the group generated by (2) and (3) is finite, they must be roots of unity, other than unity itself.

The first stage in the classification of the varieties $W_p$ consists in determining the finite groups of collineations represented by (2) and (3); the second stage consists in showing that there exist period matrices for $V_p$ of the requisite kinds. In this process an important part is played by the theorem:

V. A necessary and sufficient condition that $W_p$ should have geometric genus unity is that each substitution in $\mathcal{G}_n$ should have modulus unity.
The necessity of the condition is obvious, for $W_p$ must then possess a $p$-ple integral $\int \cdots \int du_1 du_2 \cdots du_p$ of the first kind, from which it follows that the Jacobian $\mathcal{J}(u_1', u_2', \ldots, u_p')/\mathcal{J}(u_1, u_2, \ldots, u_p)$ must be equal to unity. The converse result is likewise true (2).

It is an immediate consequence of (2) that $I_n$ is invariant under a continuous group $G$ of $\infty \varphi$ transformations of the first kind; evidently $G$ is completely transitive and permutable, so that its trajectories $V_q$, which are of course invariant under $G$, must be Picard varieties (5); that is, $W_p$ is a pseudo-Abelian variety of type $q$ (7). Hence

VI. Every Abelian variety $W_p$ of superficial irregularity $q (0 < q < p)$ and with some plurigenus greater than zero is pseudo-Abelian of type $q$.

We may show further that

VII. Every Abelian variety $W_p$ of superficial irregularity $p$ is a Picard variety.

This result is due to Severi (15); a shorter proof, of geometrical character, is as follows. Suppose, if possible, that $W_p$ has geometric genus zero; then (Severi, 14) $W_p$ must contain a congruence of irregularity $p$, and consequently $V_p$ must contain a congruence of irregularity $p$ at least, in contradiction to the known fact that every congruence on $V_p$ has irregularity less than $p$. Thus $W_p$ has geometric genus unity and therefore possesses a pure canonical hypersurface of order zero (I), so that, by (7, § 2), it is a Picard variety.

We have seen that, when $q < p$, $W_p$ contains a congruence $\{V_q\}$ of Picard varieties $V_q$. Now this congruence can arise only from a congruence of varieties $\overline{V}_q$ on $V_p$; from the general theory of Picard varieties we know that this congruence must be of Picard type and that its members are Picard varieties; a variety $V_p$ containing such a congruence is said to be special of type $q$. We know also that $V_p$ must contain a second Picard congruence of Picard varieties $\overline{V}_{p-q}$; evidently these will give rise to a congruence $\{V_{p-q}\}$ of varieties $V_{p-q}$ on $W_p$. Hence
VIII. Every Abelian variety $W_p$ of superficial irregularity $q (0 < q < p)$ and with some plurigenus greater than zero is the image of an involution on a Picard variety $V_p$ which is special of type $q$; in addition to the congruence $|V_q|$ of trajectories $V_q$, $W_p$ contains a congruence $|V_{p-q}|$ of varieties $V_{p-q}$.

In previous work (7) the existence of this second congruence $|V_{p-q}|$ on a pseudo-Abelian variety $W_p$ was established on the assumption that the trajectories of $G$ had general moduli; here, however, such an assumption is not made.

3. Further properties of $W_p$. - Before proceeding it will be convenient to recall the properties of pseudo-Abelian varieties which will be required later. Let $W_p$ be a pseudo-Abelian variety of type $q$ whose congruence of trajectories $V_q$ is $|V_q|$: the varieties $V_{p-q}$ of the associated congruence $|V_{p-q}|$ on $W_p$ are transforms of one another under the group $G$ and are thus birationally equivalent. They cut on the generic $V_q$ an involution $i_q$, where the number $d = |V_q V_{p-q}|$ is called the determinant of $W_p$. This involution is without coincidences, from which it follows incidentally that $|V_{p-q}|$ is necessarily an Abelian congruence of a restricted type. Again, while the generic trajectory $V_q$ is irreducible, there exist in general reducible members of $|V_q|$ of the form $sV_q^s (≡ V_q^q)$ where $V_q^s$ is itself a Picard variety; here the number $s$ may a priori be any divisor of $d$, including $d$. The varieties $V_q^s$ generate a certain number of irreducible manifolds whose dimensions may vary from $q$ to $p - 1$.

In the case where $d = 1$, we may evidently map $W_p$ on the product $V_q \times V_{p-q}$; to obtain a representation in the case where $d > 1$, we first construct the variety $W_p^* = V_q^* \times V_{p-q}^*$, where $V_q^*$ and $V_{p-q}^*$ are birationally equivalent to $|V_q|$ and $|V_{p-q}|$ respectively; then, making correspond the generic point of $W_p^*$ to the set $(V_q V_{p-q})$ we have a mapping of $W_p$ on the $d$-ple variety $W_p^*$. From this representation we deduce that the superficial irregularity $q_2$ of $W_p$ satisfies the inequality
\[
q_z \geq q_z' + q_z'', \text{ where } q_z' \text{ and } q_z'' \text{ denote respectively the irregularities of } |V_q| \text{ and } |V_{p-q}|.
\]

It remains only to recall the nature of the branch variety of the representation. Now the sets \((V_q, V_{p-q})\) define an involution \(I_d\) on \(W_p\) whose coincidence locus is generated by the varieties \(V_{q,s}\), each counted \((s-1)\) times; then, corresponding to \(V_{q,s}\), we have a variety \(V_{q,s}^*\) of the congruence which maps \(|V_q|\), constituting an \((s-1)\)-ple element of the branch locus. Such varieties generate a number of irreducible manifolds which may have any dimension from \(q\) to \(p-1\) inclusive; and any component of the branch locus with dimension less than \(p-1\) will be fundamental in the correspondence between \(W_p\) and \(W^*\).

4. The improperly Abelian varieties. - Suppose now that the variety \(W_p\) has some plurigenus greater than zero, and that it maps a simple involution \(I_n\) of superficial irregularity \(q(0 < q < p)\) on \(V_p\); then the congruence \(|\overline{V_q}|\) on \(V_p\) is mapped, simply or multiply, on the congruence \(|V_q|\) and, in the latter case, there will possibly be \((s-1)\)-ple branch elements \(V_{q,s}\). Hence \(|V_q|\) is an Abelian congruence. Similarly, the congruence \(|\overline{V}_{p-q}|\) is mapped, simply or multiply, on \(|V_{p-q}|\), although here the representation is without branch elements (§ 3); thus, as already remarked, \(|V_{p-q}|\) is an Abelian congruence. It follows that the coordinates of the generic point \(P^*\) of the variety \(W_p^*\) of § 3 are expressible as rational functions of the coordinates of two points, lying on Abelian varieties of genera \(q\) and \(p-q\) respectively; hence the coordinates of the generic point \(P\) of \(W_p\), which is mapped on the \(d\)-ple variety \(W_p^*\), are expressible as algebraic functions of the coordinates of \(P^*\). Thus

IX. Every Abelian variety \(W_p\) of superficial irregularity \(q(0 < q < p)\) and with some plurigenus greater than zero is representable parametrically by means of algebraic functions of Abelian functions of genus \(q\) and other Abelian functions of genus \(p-q\).

It is by virtue of this result that we may call \(W_p\) an improperly Abelian variety. In many cases, one or both of the
congruences $|V_q|$ and $|V_{p-q}|$ are themselves improperly Abelian, and then the genera of the Abelian functions required for the parametric representation can be lowered further. The precise form of the representation will evidently depend on that of the mapping of $W_p$ on $W_{p}^*$; the problem of determining this form has so far been solved completely only for $p = 1$.

5. The classification of improperly Abelian varieties.

It will now become clear that the determination of all possible species of variety $W_p$ is a recursive process depending on the classification of the Abelian varieties $W_r$ ($r < p$) representing involutions which are generable by finite groups of automorphisms. In the case $p = 2$, the solution of this problem is classical (2); for $p = 3$, the procedure to be followed has been outlined in (10): it must be added that our present knowledge of the theory of surfaces does not suffice for a completely geometrical solution in this case.

As an illustration we consider the particularly interesting case where $I_n$ is free from coincidences. We observe first that in this case $W_p$ must be superficially irregular; for the equations (3) always admit solutions and, if $W_p$ were superficially regular, the corresponding set of equations (2) would be absent. Next we remark that $W_p$ and all its submanifolds are free from exceptional varieties; for any such variety contains rational curves and, since the correspondence between $V_p$ and $W_p$ is without branch points, any rational curve on $W_p$ would arise from some rational curve on $V_p$; whereas, as is well known, such curves do not exist.

Suppose, in the first place, that $V_q$ and $V_{p-q}$ both have general moduli, so that $|V_q|$ and $|V_{p-q}|$ are general Picard congruences; then $|V_{p-q}|$ is a Picard congruence of Picard varieties. For $|V_{p-q}|$ is mapped on $|V_{p-q}|$ without branch points, and also $V_{p-q}$ is mapped on $V_{p-q}$ without branch points.

Again, since $|V_p|$ is general, the congruence $|V_p|$ is uniquely specified: thus, when $p - q = 1$, it is a rational pencil; when $p - q = 2$, it is a Kummer surface or its genera-
lization; and, for \( p - q > 2 \), it is mapped by a Wirtinger manifold (16), or its generalization, according as \( \{ V_q \} \), regarded as a Picard variety \( U_{p-q} \), has divisors unity or at least one divisor greater than unity. With an obvious extension of the terminology we may say that \( \{ V_q \} \) is a generalized Wirtinger congruence. In any case it follows from the general theory that the number of varieties \( V_{q,*} \) is finite.

If, in the second place, we allow either or both of \( V_q \) and \( V_{p-q} \) to have particular moduli, a great number of possibilities at once present themselves. Thus, supposing that \( V_{p-q} \) has particular moduli, all that can be asserted about \( V_{p-q} \) is that it maps \( V_{p-q} \) in a correspondence free from coincidences; hence, by the above opening remark, \( V_{p-q} \) is a superficially irregular Abelian variety. Again, if the congruence \( \{ V_{p-q} \} \) has particular moduli, \( \{ V_{p-q} \} \) may be improperly Abelian of a type which maps an involution without coincidences. And \( \{ V_q \} \) may a priori be any Abelian congruence with \( \infty^i \) branch elements \( V_{q,*} \) \( (0 \leq i \leq p - q - 1) \); if, in particular \( i = p - q - 1 \), the congruence is of anticanonical type (§ 1).

The fact that involutions of anticanonical type will actually have to be considered is revealed by the case \( p = 3, q = 1 \) (10); here we may have a \( W_s \) which contains an elliptic pencil \( \{ V_s \} \) of Picard surfaces, and a rational congruence \( \{ V_s \} \) of elliptic curves which possesses \( \infty^i \) curves \( V_{s,*} \), and cuts on each \( V_s \) a (rational) involution endowed with a coincidence curve. As stated in § 1, the involutions of this type have been classified by Scorza.

Continuing the previous discussion, we next observe that the congruence \( \{ V_q \} \) cuts on each \( V_{p-q} \) an involution of order \( d \), having for \((s - 1)\)-ple coincidences the points where \( V_{p-q} \) meets each variety \( V_{q,*} \). Now, by a property of pseudo-Abelian varieties (7), this involution is generable by a finite group \( G_d \) of automorphisms of \( V_{p-q} \): thus, in a geometrical treatment of the problem, we require to determine all possible groups \( G_d \) and their coincidence loci. When \( V_{p-q} \) is a Picard variety, this problem will already have been solved in the recursive process; when, however, \( V_{p-q} \) is Abelian, it may
be reduced to the consideration of compound involutions 2) on a Picard variety $U_{p-q}$.

In conclusion, then, we see that, in order to obtain, by the group-theoretic method, a complete classification of the types $W_p$, it is necessary to determine all the types $W_r$ ($r < p$), even those with plurigenera zero, representing involutions which are generable by finite groups of automorphisms of $V_r$.

6. Para-Abelian varieties. Consider the variety $W_p$ constructed by analogy with an improperly Abelian variety in the following manner. Instead of an Abelian congruence $\{V_q\}$ of trajectories, we suppose that $W_p$ contains a congruence $\{V_q\}$, of arbitrary character, of Picard varieties $V_q$; and that, as before, the irreducible members of the congruence are non-singular and birationally equivalent. We suppose further that $W_p$ contains a second congruence $\{V_{p-q}\}$ which is of Abelian type, and such that its irreducible members are non-singular and birationally equivalent 3). As in § 3, we shall then have on $W_p$ an involution $I_d$ of sets $(V_qV_{p-q})$, where $d = [V_qV_{p-q}]$ is called the determinant of $W_p$.

We now assume that $W_p$ may be mapped on the $d$-ple variety $W_p^* = \overline{V}_p^* \times \overline{V}_{p-q}^*$, where $\overline{V}_q^*$ and $\overline{V}_{p-q}^*$ are birationally equivalent to $\{V_{p-q}\}$ and $\{V_q\}$ respectively, in such a way that the branch locus is generated by varieties $V_p^*$ and $V_{p-q}^*$ corresponding respectively to members of $\{V_q\}$ and $\{V_{p-q}\}$ each counted a certain number of times; and we further assume that the correspondence between $W_p$ and $W_p^*$ possesses no exceptional features other than those which result from these hypotheses. A consequences of these assumptions is that both $\{V_q\}$ and $\{V_{p-q}\}$ will in general contain reducible va-

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2) In this connection we may note Andreotti's result (1) that a compound involution on a Picard variety, even if it has some plurigenera greater than zero, need not be generable by a finite group of automorphisms.

3) The variety $V_{p-q}$ cannot be chosen arbitrarily, since it must contain the involution cut on it by $|V_q|$. 

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rieties $V_{q,s}$ and $V_{p-q,t}$, say, corresponding to generators of
the branch locus, which are respectively $(s-1)$-ple and
$(t-1)$-ple components of the coincidence locus of $I_d$.

We shall call the variety $W_p$ so defined a para-Abelian
variety of type $q$. It is clear that $W_p$ does not admit the
group of automorphisms which characterizes the pseudo-Abel-
lian varieties; nevertheless it will appear that there are
certain resemblances between the two classes of manifold,
except from their representations on multiple product va-
rieties $W_p^*$.

To begin with, we observe that the superficial irregularity
$q_2$ of $W_p$ satisfies the inequality $q_2 \geq q_2' + q_2''$, where $q_2'$
and $q_2''$ denote the irregularities of the congruences on $W^*$
which map $\{V_q\}$ and $\{V_{p-q}\}$ respectively; thus $q_2'$ and $q_2''$
are the respective irregularities of $\{V_q\}$ and $\{V_{p-q}\}$. This
result is strictly analogous to that of § 3.

Again, we remark that every variety belonging to $\{V_q\}$
is para-Abelian (effective or virtual) of type $q$. This follows
at once from the definition.

Next, supposing (as usual), that $W_q$ is free from exceptio-
nal varieties, we see that the virtual canonical system $|X_{p-1}|$
of $W_p$ belongs to $\{V_q\}$; also that $|X_{p-1}|$ contains as fixed
$(s-1)$-ple component every hypersurface generated by va-
rieties $V_{q,s}$, and passes $(s-1)$-ply through every manifold
of lower dimension generated by those varieties.

This proposition may be established by the method al-
ready adopted for the pseudo-Abelian varieties (7); it results
from the fact that the variety $V_q$ has a canonical hypersurface
$X_{q-1}$ of order zero.

It has been shown in (7) that canonical systems $|X_k(W_p)|$
$(k = 0, 1, ..., p - 1)$ of a pseudo-Abelian variety $W_p$ all belong
to the corresponding congruence $\{V_q\}$ or else have order zero.
But the analogous property does not hold for para-Abelian
varieties, as is already clear from an examination of the
cases $p = 2, 3$.

Another essential difference between the pseudo-Abelian and
para-Abelian varieties is as follows. On the former variety the
involution $i_d$ cut by $\{V_q\}$ on the generic $V_{p-1}$ is generable by
a finite group $\mathcal{G}_d$ of automorphisms: on the latter this is not in general the case. We note, however, that there exists a subspecies of para-Abelian variety having the property that $i_d$ is so generable and this type, from the point of view of the analytical representation, is the simplest to deal with; the discussion, in the cyclic case, of the analogous subspecies of parahyperelliptic threefold (10) may, with obvious modifications, be repeated here.

REFERENCES

6. — — Selected topics in algebraic geometry (Washington, 1928), ch. 17.