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S. C. CHAKRABARTI

**On higher differences. Nota II**

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## ON HIGHER DIFFERENCES

Nota II (\*)

di S. C. CHAKRABARTI (*a Jadabpur College, Calcutta*)

### II. - The Relations between the Differential Coefficients and the Higher Differences of a Function.

**1. Introduction.** This paper is in continuation of Note I on the subject of Higher Differences. We make use of the same notations as employed in Note I. The object of this paper is to study the relations between the operators  $F$  and the operators in Differential Calculus.

#### 2. LEMMA (i).

If

$$Z_{sr} = \begin{vmatrix} rO_{r-1} & 1 \\ r+1O_{r-1} & r+1O_r & 1 \\ r+2O_{r-1} & r+2O_r & r+2O_{r+1} \end{vmatrix}, \quad (1)$$

then similarly formed

$$Z_{nr} = {}^{n+r-1}S_n.$$

$$[z_{0r} = 1]$$

$[Z_{sr}]$  evidently develops into

$$\sum_{p=0}^2 (-)^p r+2O_{r+1-p} Z_{2-p}, \quad r$$

Similarly develop  $Z_{nr}$  and then prove the theorem by induction].

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(\*) Pervenuta in Redazione il 26 novembre 1953.

**LEMMA (ii).** If

(2)

$$P_{4c} = \begin{vmatrix} C & 1 & & \\ C^2 & {}^2O_1^- & 1 & \\ C^3 & {}^3O_1^- & {}^3O_2^- & 1 \\ C^4 + (-)^4 {}^4S_4 & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4$$

then

$$P_{4c} = (-)^{n-1} (C)_n {}^n S_n$$

[Here develop  $P_{4c}$  in terms of  $C$ ,  $C^2$  etc. and then apply (1) above and Th. (5), Note I].

**LEMMA (iii).** — If

(3)

$$Q_{4r} = \begin{vmatrix} 1 & 1 & & \\ a^r & {}^2O_1^- & 1 & \\ a^{2r} & {}^3O_1^- & {}^3O_2^- & 1 \\ a^{3r} & {}^4O_1^- & {}^4O_2^- & {}^4O_3^- \end{vmatrix}_4$$

then

$$Q_{4r} = (-)^{n-1} (a^r)_n a^{-r} {}^n S_n + a^{-r} {}^n S_n, \quad Q_{40} = {}^n S_n$$

[Put  $C = a^r$  in Lemma (ii)].

**3.** By Differential Calculus and by § II, Note I,

$$\left. \begin{aligned} F^n u_x &= u_{x+n} = e^{n \frac{d}{dx}} u_x \\ \text{and} \quad F_n u_x &= u_{ax} = e^{anx \frac{d}{dx}} u_0 \end{aligned} \right\} \quad (4)$$

where  $\frac{d^m}{dx^m} u_0$  stands for the value of  $\frac{d^m}{dx^m} u_x$  when  $x$  is replaced by 0.

Thus the operators  $F$  are related to the operators in Differential Calculus.

**4.  $A^n u_x$  may be expressed in terms of the differential coefficients of  $u_x$ .**

**THEOREM.** If  $u_x$  be a rational and integral function of  $x$  of degree  $l$  in  $x$ , then

$$A^n u_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^n O^{1+p} \frac{d^{1+p} u_x}{dx^{1+p}} \quad (5)$$

where  $A^n O^m$  stands for the value of  $A^n x^m$  when  $x = 0$ .

By Th. (26), Note I, we have

$$\left( \sum_{p=0}^n O_p A^p \right) u_x = e^{\frac{d}{dx}} u_x$$

In this equation, putting  $n = 1, 2, 3$  and  $4$ , we have four equations from which eliminating  $Au_x, A^2 u_x$  and  $A^3 u_x$ , we have

$$(-)^4 A^4 u_x = \begin{vmatrix} 1 - e^{\frac{d}{dx}} & 1 \\ 1 - e^{\frac{2d}{dx}} & {}^2 O_1^- \\ 1 - e^{\frac{3d}{dx}} & {}^3 O_1^- \\ 1 - e^{\frac{4d}{dx}} & {}^4 O_1^- \end{vmatrix}_4 u_x$$

$$(-)^3 A^4 u_x = \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \begin{vmatrix} 1^{1+p} & 1 \\ 2^{1+p} & {}^2 O_1^- \\ 3^{1+p} & {}^3 O_1^- \\ 4^{1+p} & {}^4 O_1^- \end{vmatrix}_4 \frac{d^{1+p} u_x}{dx^{1+p}}$$

$$= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ \sum_{m=0}^3 (-)^m (1+m)^{p+1} Z_{3-m, r, 2+m} \right\} \frac{d^{1+p} u_x}{dx^{1+p}},$$

[Lemma (i)]

$$\begin{aligned}
&= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ \sum_{m=0}^3 (-)^m (1+m)^{p+1} {}^4S_{3-m} \right\} \frac{d^{1+p} u_x}{dx^{1+p}} \\
&= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} \left\{ (-)^3 \sum_{m=0}^4 (-)^m (4-m)^{1+p} {}^4S_m \right\} \frac{d^{1+p} u_x}{dx^{1+p}} \\
&= (-)^3 \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^4 O^{1+p} \frac{d^{1+p} u_x}{dx^{1+p}} \\
\therefore A^4 u_x &= \sum_{p=0}^{l-1} \frac{1}{(1+p)!} A^4 O^{1+p} \frac{d^{1+p} u_x}{dx^{1+p}}.
\end{aligned}$$

The general case may be similarly treated.

COR.

$$\begin{aligned}
\Delta^n u_x &= \frac{d^n u_x}{dx^n} + \frac{1}{(n+1)!} \Delta^n O^{n+1} \frac{d^{n+1} u_x}{dx^{n+1}} \\
&\quad + \frac{1}{(n+2)!} \Delta^n O^{n+2} \frac{d^{n+2} u_x}{dx^{n+2}} + \dots
\end{aligned}$$

[This result of Finite Differences, follows from (5) when  $a \rightarrow 1$ , for  $\Delta^n O^1 = \Delta^n O^2 = \Delta^n O^3 = \dots = \Delta^n O^{n-1} = 0$  and  $\Delta^n O^n = n!$ ].

**5.  $A_n u_x$  may also be expressed in terms of the differential coefficients of  $u_x$ .**

**THEOREM.** If  $u_x$  be a rational and integral function of  $x$  of degree  $l$  in  $x$ , then

$$A_n u_x = {}^n S_n \left\{ \sum_{p=0}^{l-n} \frac{x^{n+p}}{(n+p)!} (a^{n+p})_n \frac{d^{n+p}}{dx^{n+p}} \right\} u_0, \quad (6)$$

where  $\frac{d^m u_0}{dx^m}$  denotes what  $\frac{d^m u_x}{dx^m}$  becomes when  $x=0$ .

By Th. (27), Note I, we have

$$\left( \sum_{p=0}^n {}^n O_p^- A_p \right) u_x = e^{a^n x^{\frac{d}{dx}}} u_0$$

If in this equation we put  $n = 1, 2, \dots, n$ , we have  $n$  equations from which eliminating

$A_1 u_x, A_2 u_x, \dots A_{n-1} u_x$ , we have

$$\begin{aligned}
 (-)^n A_n u_x &= \left| \begin{array}{cc} u_x - e^{ax} \frac{d}{dx} u_0 & 1 \\ u_x - e^{a^2 x} \frac{d}{dx} u_0 & {}^2 O_1^- \quad 1 \\ \dots & \dots \quad \dots \quad \dots \\ u_x - e^{a^n x} \frac{d}{dx} u_0 & {}^n O_1^- \quad {}^n O_2^- \dots {}^n O_{n-1}^- \end{array} \right|_n \\
 &= Q_{n,0} u_x - \sum_{p=0}^l \frac{(ax)^p}{p!} Q_{n,p} \frac{d^p u_0}{dx^p} \quad [\text{Lemma (iii)}] \\
 &= {}^n S_{n,0} u_x - \sum_{p=0}^l \frac{(ax)^p}{p!} {}^n S_n \{ (-)^{n-1} (a^p)_n a^{-p} + a^{-p} \} \frac{d^p u_0}{dx^p} \\
 &= {}^n S_{n,0} u_x - \sum_{p=0}^l \frac{x^p}{p!} {}^n S_n \frac{d^p u_0}{dx^p} + (-)^n {}^n S_n \sum_{p=0}^l \frac{x^p}{p!} (a^p)_n \frac{d^p u_0}{dx^p} \\
 &= (-)^n {}^n S_n \sum_{p=0}^l \frac{x^p}{p!} (a^p)_n \frac{d^p u_0}{dx^p} \\
 \therefore A_n u_x &= {}^n S_n \sum_{p=0}^{l-n} \frac{x^{n+p}}{(n+p)!} (a^{n+p})_n \frac{d^{n+p} u_0}{dx^{n+p}}
 \end{aligned}$$

for  $(a^p)_n = 0$  when  $p < n$ .

Cor.

(7)

$$\underset{a \rightarrow 1}{\text{Lt}} \frac{A_n u_x}{(a-1)^n} = x^n \sum_{p=0}^{l-n} \frac{x^p}{p!} \frac{d^{n+p} u_0}{dx^{n+p}}$$

This follows from (6) when  $a \rightarrow 1$ .

6.  $\frac{du_x}{dx}$  may be expressed in terms of  $A^1, A^2, A^3$ , etc.

**THEOREM.** If  $u_x$  be a rational and integral function of  $x$  of degree  $n$  in  $x$ , then

$$\frac{du_x}{dx} = \sum_{r=1}^n \left\{ \sum_{p=r}^n (-)^{p-1} {}^n C_p \frac{{}^p O_r^-}{p} A^r u_x \right\}. \quad (8)$$

By Th. (26), Note I and by (4), § 3, we have

$$\left( \sum_{p=0}^n {}^n O_p^- A^p \right) u_x = e^{\frac{d}{dx}} u_x \quad (9)$$

Let us consider the particular case when  $u_x$  is a rational and integral function of  $x$  of the third degree. If we put  $n=1$  in (9), we have

$$A u_x = \left( \frac{d}{dx} + \frac{1}{2!} \frac{d^2}{dx^2} + \frac{1}{3!} \frac{d^3}{dx^3} \right) u_x$$

Putting  $n=2$  and  $3$ , two similar equations may be obtained. From these three equations, eliminating  $\frac{d^2 u_x}{dx^2}$  and  $\frac{d^3 u_x}{dx^3}$ , we have

$$\begin{aligned} \frac{du_x}{dx} &= \left\{ \sum_{p=1}^3 (-)^{p-1} {}^3 C_p \frac{{}^p O_1^-}{p} \left\{ A^1 u_x - \left\{ \sum_{p=2}^3 (-)^{p-2} {}^3 C_p \frac{{}^p O_2^-}{p} \right\} A^2 u_x \right. \right. \\ &\quad \left. \left. + \left\{ \sum_{p=3}^3 (-)^{p-3} {}^3 C_p \frac{{}^p O_3^-}{p} \right\} A^3 u_x \right\} \right. \end{aligned}$$

i.e

$$\frac{du_x}{dx} = \sum_{r=1}^3 \left\{ \sum_{p=r}^3 (-)^{p-1} {}^3 C_p \frac{{}^p O_r^-}{p} A^r u_x \right\}$$

The general case may be similarly treated

**COR.**

$$\frac{du_x}{dx} = \Delta u_x - \frac{\Delta^2 u_x}{2} + \frac{\Delta^3 u_x}{3} - \dots \quad (10)$$

which is a well-known theorem in F. D.

[If  $a \rightarrow 1$ , the coefficient of  $(-)^{r-1} A^r u_x$  in (8) becomes

$$\sum_{p=r}^n (-)^{p-r} {}^n C_p {}^p C_r / p$$

which may be written

$$\begin{aligned} & \sum_{p=0}^{n-r} (-)^p \frac{1}{r+p} {}^n C_{r+p} {}^{r+p} C_r, \\ & = {}^n C_r \sum_{p=0}^{n-r} (-)^p \frac{1}{r+p} {}^{n-r} C_p \\ & = {}^n C_r \frac{(n-r)! (r-1)!}{n!} \quad \text{by Finite Differences} \\ & = \frac{1}{r} \end{aligned}$$

Hence (10) follows from (8)].