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A MODIFIED ALGORITHM FOR THE STRICT FEASIBILITY PROBLEM

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Abstract. In this note, we present a slight modification of an algorithm for the strict feasibility problem. This modification reduces the number of iterations.

Keywords: Strict feasibility, interior point methods, Ye–Lustig algorithm.

1. Introduction

This paper is concerned with the problem of finding $x \in \mathbb{R}^n$ such that

$$x > 0 \quad \text{and} \quad Ax = b \tag{F}$$

where A is a $m \times n$ real matrix of rank m, $b \in \mathbb{R}^m$ and $0 < m < n$.

This problem, called a *strict feasibility problem*, occurs in many optimization problems in linear or quadratic programming. Such problems are of type

$$\text{Minimize } f(x) \text{ subject to } x \in S = \{x \in \mathbb{R}^n : \ x \geq 0, \ Ax = b\} \cdot \tag{P}$$

Let us denote by $\widetilde{S}$ the following subset:

$$\widetilde{S} = \{x \in \mathbb{R}^n : \ x > 0, \ Ax = b\} \cdot$$

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Interior point methods for solving (P) start from some arbitrary initial point $x^0 \in \widetilde{S}$ and build a sequence $\{x^k\} \subset \widetilde{S}$ expected to converge to some optimal solution x^* of (P). Thus, the first step in interior point methods consists in finding an initial point x^0 in a efficient way. To do that, it is usual to introduce an artificial variable λ and to consider the linear programming problem

$$\text{Minimize}_{x,\lambda}\ \{\lambda\ :\ Ax + \lambda(b - Aa^0) = b,\ x \geq 0,\ \lambda \geq 0\} \qquad (LF)$$

where a^0 is a arbitrary fixed point in the positive orthant of $\mathbb{R}^n$. It is noticed that x^* is a solution of the feasibility problem (F) if and only if $(x^*, 0)$ is an optimal solution of (LF) with $x^* > 0$.

In fact, numerical algorithms provide only approximate optimal solutions for an optimization problem. In our case, an approximate solution of (LF) can be obtained *via* a classical interior method as for instance the Ye–Lustig algorithm that we described below. But before, we precise the notation used in this algorithm: $\varepsilon > 0$ corresponds to the precision of the approximation, $r = \frac{1}{\sqrt{(n+1)(n+2)}}$, $c = (0, 0, \cdots, 0, 1)^t \in \mathbb{R}^{n+1}$, $e_{n+2} = (1, 1, \cdots, 1)^t \in \mathbb{R}^{n+2}$, $\tilde{x} = (x, \lambda)^t \in \mathbb{R}^{n+1}$ and $B = [A, b - Aa^0\,]$ is a $m \times (n+1)$ matrix.

2. The original algorithm and its modification

Let us describe the original algorithm:

The Ye–Lustig algorithm [3]

- a) Initialization: start with $x^0 = a^0$, $\lambda^0 = 1$, $\tilde{x}^0 = (x^0, \lambda^0)^t$ and $k = 0$;
  If $\|Ax^0 - b\| \leq \varepsilon$ **Stop**: x^0 is an ε-approximate solution,
  If not go to b).
- b) **If** $\lambda^k \leq \varepsilon$ **Stop**: x^k is an ε-approximate solution,
  If not go to c).
- c) Set $D^k = \text{diag}(\tilde{x}^k)$ and
  - Compute the projection p^k of the vector $(D^k c, -c^t \tilde{x}^k)^t \in \mathbb{R}^{n+2}$ on the kernel of the $m \times (n+2)$ matrix $B^k = [BD^k, -b]$,
  - Take $y^{k+1} = \frac{e_{n+2}}{n+2} - \alpha^k r \frac{p^k}{\|p^k\|}$, where α^k is obtained by a line search,
  - Take $\tilde{x}^{k+1} = (x^{k+1}, \lambda^{k+1})^t = (y^{k+1}_{n+2})^{-1} D^k y^{k+1}[n+1]$,
- d) do $k = k + 1$ and go back to b).

We propose a slight modification of this algorithm by modifying the stopping criteria.

The modified algorithm

- a') Initialization: start with $x^0 = a^0$, $\lambda^0 = 1$, and $k = 0$;
  If $\|Ax^0 - b\| \leq \varepsilon$ **Stop**: x^0 is an ε-approximate solution.
  If not compute the solution u^0 of the linear system
  $AA^t u^0 = \lambda^0(b - Aa^0)$.
- b') **If** $\lambda^k \leq \varepsilon$ **Stop**: x^k is an ε-approximate solution.
  If not
  - Take $u^k = \lambda^k u^0$,
  - Take $z^k = -\mathrm{diag}[(x^k)]^{-1} A^t u^k$,
  - **If** $\max|z^k|_i < 1$ **Stop**: $x^k + A^t u^k$ is an ε-approximate solution.
    If not go to c')
- c') is identical to c) of the original algorithm.
- d') do $k = k + 1$ and go back to b').

The computation of the vector u^0 occurs once only, it can be performed by a Cholesky method. It remains to prove the validity of the new stopping criteria $\max|z^k|_i < 1$. This is done in the following proposition.

Proposition 2.1. *If* $\max|z^k|_i < 1$ *then* $x^k + A^t u^k > 0$ *and* $A(x^k + A^t u^k) = b$.

Proof. 1) Notice that $-\mathrm{diag}(x^k) z^k = A^t u^k$, then $x^k + A^t u^k = x^k - \mathrm{diag}(x^k) z^k = \mathrm{diag}(x^k)(e_n - z^k) > 0$ because $x^k > 0$ and $|z^k|_i < 1$ for all i.

2) Since $(x^k, \lambda^k)^t$ is a feasible solution of (LF) then $A(x^k + A^t u^k) = Ax^k + AA^t u^k = b - \lambda^k(b - Aa^0) + \lambda^k AA^t u^0 = b - \lambda^k(b - Aa^0) + \lambda^k \lambda^0 (b - Aa^0) = b.$ □

This modification brings a significant improvement in the number of iterations with only a very small increasing in the cost per iteration. We illustrate that by a few examples.

3. Examples

In this examples ε has been taken equal to 10^{-3} or 10^{-6} according to the case.

3.1. Some examples

The following examples are taken form the literature see for instance [1, 4]. In particular, Example 7 is called the Hitac problem.

example	**taille** $m \times n$	**Nbr of iterations Ye–Lustig**	**Nbr of iterations modified algorithm**
1	3×5	4	1
2	3×6	3	1
3	5×11	5	4
4	6×12	3	1
5	11×25	4	3
6	16×27	6	5
7	11×28	7	6

3.2. Cube example

$n = 2m$, $A[i, j] = 0$ if $i \neq j$ *or* $(i + 1) \neq j$
$A[i, i] = A[i, i + m] = 1, b[i] = 2$, for $i, j = 1 \cdots m$.

Dimension $m \times n$	**Nbr of iterations Ye–Lustig**	**Nbr of iterations Modified algorithm**
50×100	3	1
100×200	3	1
150×300	3	1
200×400	3	1

3.3. Hilbert example

$n = 2m$, $A[i, j] = \frac{1}{i+j}, A[i, i + m] = 1$,
$b[i] = \sum_{j=1}^{m} \frac{1}{i+j}$, for $i, j = 1 \cdots m$.

Dimension $m \times n$	**Nbr of iterations Ye–Lustig**	**Nbr of iterations Modified algorithm**
50×100	3	1
100×200	3	1
150×300	3	1
200×400	3	1

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