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RAIRO. Operations Research, tome 34, n° 4 (2000), p. 397-409

http://www.numdam.org/item?id=RO_2000__34_4_397_0

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CONTINUOUS TIME LINEAR-FRACTIONAL PROGRAMMING. THE MINIMUM-RISK APPROACH (*)

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Communicated by Jean-Yves JAFFRAY

1. INTRODUCTION

In this paper, the minimum-risk approach is applied to the stochastic continuous times linear-fractional problem. We note that the minimum-risk model was introduced in stochastic linear programming by Bereanu [5, 6] and Charnes and Cooper [9] (under the name of P-model). This approach was extended by Stancu-Minasian [23], Stancu-Minasian and Tigan [26-30], Tigan and Stancu-Minasian [34] to the stochastic programming with linear-fractional and bilinear fractional objective and by Tigan [31], Tigan and Stancu-Minasian [33] to the continuous time linear and linear-fractional programming.

We consider two classes of continuous time fractional problems, with a linear-fractional objective (see, Sect. 2), respectively with an objective function having a linear-fractional kernel (see, Sect. 4). In the case when the coefficients of the objective functions are simply randomized (*i.e.* are affine functions of a single random variable), we will show that, under some positivity conditions, the stochastic continuous time linear-fractional problem is equivalent with certain deterministic continuous time linear-fractional problem, while the stochastic continuous time fractional problem with an objective function having a linear-fractional kernel is equivalent with a deterministic continuous time nonlinear-fractional problem. Some parametrical procedures are proposed for solving these deterministic equivalent problems.

(*) Received March, 1999.

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2. PROBLEM FORMULATION

The following programming problem, which originated from Bellman's bottleneck problem [3, 4] has received a great amount of attention in the last decades (see, [1, 2, 7, 8, 11-22, 31, 33, 35-38]):

CLP. Find

$$\sup \int_0^T a(t)z(t)dt, \quad (1)$$

subject to

$$B(t)z(t) \leq c(t) + \int_0^T K(t,s)z(s)ds, \quad 0 \leq t \leq T, \quad (2)$$

$$z(t) \geq 0, \quad 0 \leq t \leq T, \quad (3)$$

where $a : [0, T] \rightarrow R^n$ and $c : [0, T] \rightarrow R^m$ are vector-valued continuous known functions and $B : [0, T] \rightarrow R^{m \times n}$, $K : [0, T] \times [0, T] \rightarrow R^{m \times n}$ are matrix-valued continuous known mappings, while $z : [0, T] \rightarrow R^n$ is a vector-valued continuous unknown function. Let denote by S the set of all vector-valued function z satisfying constraints (2) and (3).

In what follows we suppose that S is a non-empty and bounded set.

A partial reference to the earlier works on continuous time programming may be found in Farr and Hanson [12], Singh [21], Bodo and Hanson [7], Hanson and Mond [13], Tyndall [35, 36] and Zalmai [37, 38].

Tyndall [35] has shown that, subject to the following constraint qualifications

$$(i) \{x \in R^n : Bx \leq 0 \text{ and } x \geq 0\} = \{0\},$$

$$(ii) B, K, \text{ and } c(t) \text{ have nonnegative components for } 0 \leq t \leq T,$$

there exists an optimal solution z^* to problem (1-3), in the particular case when B and K are constant.

By adding slack variables, the constraints (2, 3) become

$$B^*(t)z^*(t) = c(t) + \int_0^t K(t,s)z(s)ds, \quad 0 \leq t \leq T, \quad (2')$$

and

$$z^*(t) \geq 0, \quad 0 \leq t \leq T, \quad (3')$$

where $B^*(t) = [B(t), I]$, $z^*(t) = \begin{pmatrix} z(t) \\ z^0(t) \end{pmatrix}$, I is the $m \times m$ identity matrix and $z^0(t)$ is an unknown m -dimensional real vector of slack variables. Therefore, the vector $z^*(t)$ has $m + n$ components.

Involving a condition on the dual associated with (1-3), Johannesson and Hanson [15, 16] showed that the optimum solution to system (2', 3') occurs when at most m of the $m + n$ components of $z^*(t)$ are positive.

Tyndall [35] conjectured that when B and K are independent of time and $c(t)$ is smooth, then the solutions to (1-3) would be piecewise smooth functions.

The usual approach to solving CLP is to form an approximation by discretizing the time interval $[0, T]$ (see, *e.g.*, Buie and Abraham [8]). A number of authors (see, Drews [11], Hartberger [14], Segers [20], Perold [17], and Anstreicher [2]) have attempted to generalize the simplex method to solve instances of CLP without discretizing.

Anderson and Philpott [1] discuss the form of optimal solution for a class of continuous time linear programs called separated continuous linear programs and show that under certain assumption of the problem data the optimal solutions can be taken to be piecewise analytic functions.

Next, we consider a continuous time linear-fractional problem, which extends the continuous linear problem CLP.

CFP. Find

$$\sup_z \frac{\int_0^T a(t)z(t)dt}{\int_0^T b(t)z(t)dt} \quad (4)$$

subject to the constraints (2) and (3), where a , c , B , K are the same as in problem (1-3) and $b : [0, T] \rightarrow R^n$ is a vector-valued continuous known function.

We denote the objective function of CFP by

$$f(z) = \frac{\int_0^T a(t)z(t)dt}{\int_0^T b(t)z(t)dt}, \quad \text{for all } z \in S.$$

In problem (4) we make on the objective function f the following usual assumption:

$$\int_0^T b(t)z(t)dt > 0, \quad \text{for all } z \in S. \quad (5)$$

The problem is to determine a bounded measurable $n \times 1$ vector function $z(t)$ on $[0, T]$ satisfying the constraints (2, 3) and maximizing the objective function (4).

One of the difficulties in solving a stochastic linear programming problem is often a lack of information about the probability distribution function of the random variables involved in the objective function. There exists a special case of stochastic linear programming problem, called stochastic linear programming problem with simple randomization (*i.e.* the random coefficients of which are affine functions of a single random variable), in which under appropriate assumptions, the minimum-risk solution does not depend on the probability distribution function of the random variables.

We assume that in the objective function of problem CFP the vector-valued function $a(t)$ is simply randomized, that is:

$$a(t; \omega) = a'(t) + \tau(\omega)a''(t), \quad t \in [0, T], \quad (6)$$

where $a', a'' : [0, T] \rightarrow R^n$ are vector-valued continuous functions and $\tau(\omega)$ is a random variable on a probability space (Ω, \mathcal{K}, P) with a continuous and strictly increasing distribution function T^* .

Now we consider the following minimum-risk problem associated to the stochastic problem CFP:

CMR. Find

$$\sup_z P \left\{ \omega \mid \frac{\int_0^T a(t; \omega)z(t)dt}{\int_0^T b(t)z(t)dt} \geq \beta \right\} \quad (7)$$

subject to

$$z \in S, \quad (8)$$

where β is a given number that represents a level for the objective function of stochastic problem CFP.

DEFINITION 1: A function $z^* \in S$, is said to be a minimum-risk solution of level β for CMR problem, if for z^* is reached the supremum in (7).

3. DETERMINISTIC EQUIVALENT PROBLEM

In this section we show that for the minimum-risk problem CMR there exists a deterministic equivalent problem, which is a continuous time linear-fractional programming problem.

THEOREM 1: *If*

$$\int_0^T a''(t)z(t)dt > 0, \quad \text{for all } z \in S \tag{9}$$

and the probability distribution function T^* of the random variable $\tau(\omega)$ is continuous and strictly increasing, then every minimum-risk solution of the continuous time linear-fractional programming problem CMR can be found as an optimal solution of the following fractional optimization problem:

FD. *Find*

$$\inf_z \frac{\int_0^T [\beta b(t) - a'(t)]z(t)dt}{\int_0^T a''(t)z(t)dt}, \tag{10}$$

subject to $z \in S$.

Proof: Obviously, by the assumptions (5, 6) and (9), we have

$$\begin{aligned} &P \left\{ \omega \mid \frac{\int_0^T [a'(t) + \tau(\omega)a''(t)]z(t)dt}{\int_0^T b(t)z(t)dt} \geq \beta \right\} \\ &= P \left\{ \omega \mid \int_0^T a'(t)z(t)dt + \tau(\omega) \int_0^T a''(t)z(t)dt \right. \\ &\qquad\qquad\qquad \left. \geq \int_0^T b(t)z(t)dt \right\} \\ &= P \left\{ \omega \mid \tau(\omega) \geq \frac{\int_0^T [\beta b(t) - a'(t)]z(t)dt}{\int_0^T a''(t)z(t)dt} \right\}. \end{aligned} \tag{11}$$

Since T^* is strictly increasing and continuous, from (11), we have

$$\sup_{z \in S} P \left\{ \omega \mid \frac{\int_0^T a(t; \omega)z(t)dt}{\int_0^T b(t)z(t)dt} \geq \beta \right\} = 1 - T^* \left(\inf_{z \in S} \frac{\int_0^T [\beta b(t) - a'(t)]z(t)dt}{\int_0^T a''(t)z(t)dt} \right),$$

which concludes the theorem.

Next we give a sufficient condition, which assure that assumption (9) from Theorem 1 holds.

PROPOSITION 1: *If assumption (5) holds and*

$$a''(t) > 0, \quad \text{for all } t \in [0, T],$$

then

$$\int_0^T a''(t)z(t)dt > 0, \quad \text{for all } z \in S,$$

i.e. the assumption (9) holds.

Proof: Indeed, by assumption (5), it follows that feasible set S didn't contain the null mapping, that is, the vector-valued application

$$z(t) = 0, \quad \text{for all } t \in [0, T].$$

But, this fact together with the continuity of the functions z and a'' implies that (9) holds.

4. THE FRACTIONAL OBJECTIVE KERNEL CASE

Next we consider a continuous time problem with fractional kernel of the objective function, that is:

FP. Find

$$\sup_z \int_0^T \frac{a(t)z(t)}{b(t)z(t)} dt, \quad (12)$$

subject to constraints (2) and (3).

In problem FP, the functions a , b , c , B and K have the same significance as in the problem CFP.

We denote the objective function of problem FP by

$$h(z) = \int_0^T \frac{a(t)z(t)}{b(t)z(t)} dt, \quad \text{for all } z \in S.$$

Moreover, on the objective function h we suppose that

$$b(t)z(t) > 0, \quad \text{for all } t \in [0, T] \text{ and } z \in S. \quad (13)$$

Next we assume that in the objective function of FP problem, the vector-valued mapping $a(t)$ is simply randomized of the form (6).

We can state the following minimum-risk problem corresponding to the level β associated to the stochastic problem FP:

FR. Find

$$\sup_z P \left\{ \omega \mid \int_0^T \frac{a(t; \omega)z(t)}{b(t)z(t)} dt \geq \beta \right\}, \quad (14)$$

subject to $z \in S$.

Similar with Definition 1 we have:

DEFINITION 2: A function $z^* \in S$, is said to be a minimum-risk solution of level β for FR problem, if for z^* is reached the supremum in (14).

Next we show that for the minimum-risk problem FR there exists, under some supplementary assumption, a deterministic equivalent continuous time programming problem with a nonlinear fractional objective function.

THEOREM 2: If

$$\int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt > 0, \quad \text{for all } z \in S, \tag{15}$$

and the probability distribution function T^* of the random variable $\tau(\omega)$ is continuous and strictly increasing, then every minimum-risk solution of the continuous time fractional programming problem FR can be found as an optimal solution of the following deterministic continuous time fractional programming problem:

CFD. Find

$$\inf_z \frac{\int_0^T \frac{[\frac{\beta}{T}b(t)-a'(t)]z(t)}{b(t)z(t)} dt}{\int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt}, \tag{16}$$

subject to $z \in S$.

Proof: Indeed, by (6) and (15), we have

$$\begin{aligned} & P\left\{\omega \mid \int_0^T \frac{[a'(t) + \tau(\omega)a''(t)]z(t)}{b(t)z(t)} dt \geq \beta\right\} \\ &= P\left\{\omega \mid \int_0^T \frac{a'(t)z(t)}{b(t)z(t)} dt + \tau(\omega) \int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt \geq \beta\right\} \\ &= P\left\{\omega \mid \tau(\omega) \geq \frac{\int_0^T \frac{[\frac{\beta}{T}b(t)-a'(t)]z(t)}{b(t)z(t)} dt}{\int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt}\right\}. \end{aligned} \tag{17}$$

Since T^* is strictly increasing and continuous, from (17), we have

$$\sup_{z \in S} P\left\{\omega \mid \int_0^T \frac{a(t;\omega)z(t)}{b(t)z(t)} dt \geq \beta\right\} = 1 - T^*\left(\inf_{z \in S} \frac{\int_0^T \frac{[\frac{\beta}{T}b(t)-a'(t)]z(t)}{b(t)z(t)} dt}{\int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt}\right),$$

which concludes the theorem.

5. ALGORITHMIC REMARKS

The deterministic continuous time programming problems FD and CFD have fractional objectives. For solving these classes of optimization problems the parametric procedure given by Tigan [31] can be used.

We mention that this procedure generalizes the Dinkelbach method [10] for solving nonlinear fractional programming.

Next we present two particularization of the parametrical procedure to the problems FD and CFD respectively.

We associate to problem FD, for every real number q , the following parametrical nonfractional continuous time programming problem:

$$P(q). \inf_{z \in S} \left[\int_0^T [\beta b(t) - a'(t)] z(t) dt - q \int_0^T a''(t) z(t) dt \right].$$

Let $F : R \rightarrow R \cup \{+\infty\}$ be the optimal value function for parametrical linear continuous time programming problem $P(q)$, defined by

$$F(q) = \inf_{z \in S} \int_0^T [\beta b(t) - a'(t) - qa''(t)] z(t) dt, \quad \text{for every } q \in R.$$

A number of properties of function F , which can be proved easily, will be used for finding an algorithm for obtaining the approximate optimal solution of problem FD. We remember that $z' \in S$ is an approximate ε -optimal solution of FD, if $f(z') \leq q^* + \varepsilon$, where f is the objective function of FD, q^* is the optimal value of FD and $\varepsilon \geq 0$ is a given nonnegative real number.

The function F is concave and continuous on R , and it is strictly decreasing, i.e. $q_1, q_2 \in R, q_1 < q_2$ implies $F(q_1) > F(q_2)$. Consequently, the equation $F(q) = 0$ has a unique real solution.

It is well known in the area of fractional programming (see, Stancu-Minasian [24, 25], Tigan [31, 32]) that problem $P(q)$ is closely related to problem FD. The relationships between problems FD and $P(q)$ that will be of immediate interest to us are stated in the following lemma whose simple proof will be omitted.

LEMMA 1: *Let q^* be the optimal value of FD, S a bounded set and let $F(q)$ be the optimal value of $P(q)$ for any fixed $q \in R$. Then the following assertions are valid:*

- (a) *If $q \in f(S)$ then $F(q) \leq 0$.*

- (b) If z^* is an optimal solution of FD, then z^* is an optimal solution of $P(q^*)$ and $F(q^*) = 0$.
- (c) If $P(q')$ has an optimal solution z' and $F(q') = 0$, then z' is an optimal solution of FD and $q' = q^*$.
- (d) Let δ be a given nonnegative real number. If $\int_0^T a''(t)z(t)dt \geq \alpha > 0$, for all $z \in S$, and if $P(q')$ has an optimal solution z' and $F(q') \geq \delta$, then z' is an approximate $\frac{\delta}{\alpha}$ -optimal solution of FD and $q' \leq q^* + \frac{\varepsilon}{\alpha}$.

6. ALGORITHM FOR THE PROBLEM FD

With the help of the equivalence result given by Lemma 1, we can formulate the following algorithm for finding an approximate solution of problem FD.

Algorithm 1

Let δ be a given positive real number.

Step 1. Take $k := 0$ and find $z^0 \in S$.

Step 2. Compute the value V_k of the objective function for the current feasible solution z^k :

$$V_k = \frac{\int_0^T [\beta b(t) - a'(t)]z^k(t)dt}{\int_0^T a''(t)z^k(t)dt}.$$

Step 3. Find

$$Q_{k+1} = \inf_{z \in S} \int_0^T [\beta b(t) - a'(t) - V_k a''(t)]z(t)dt. \quad (18)$$

Let $z^{k+1} \in S$ the optimal solution for the linear continuous time problem (18).

Step 4. i) If $Q_{k+1} < -\delta$, then take $k := k + 1$ and go to Step 2.

ii) If $Q_{k+1} \geq -\delta$, then the algorithm stops. The solution z^k is an approximate optimal solution of problem FD.

7. ALGORITHM FOR THE PROBLEM CFD

The remarks made earlier concerning the relationships between FD and $P(q)$ are, of course, applicable to CFD and its associated parametrical problem:

$$CP(q). \inf_{z \in S} \int_0^T \frac{[\beta b(t) - a'(t) - qa''(t)]z(t)}{b(t)z(t)} dt, \quad q \in R.$$

If F denote, as in the case of problem FD, the optimal value function of problems $CP(q)$, $q \in R$, i.e.

$$F(q) = \inf_{z \in S} \int_0^T \frac{[\beta b(t) - a'(t) - qa''(t)]z(t)}{b(t)z(t)} dt,$$

we can formulate for problem CFD a similar result to Lemma 1.

LEMMA 2: *Let q^* be the optimal value of CFD and let $F(q)$ be the optimal value of $CP(q)$ for any fixed $q \in R$ such that $CP(q)$ has an optimal solution. Then the following assertions are valid:*

- (a) *If $q \in h(S)$ then $F(q) \leq 0$.*
- (b) *If z^* is an optimal solution of CFD, then z^* is an optimal solution of $P(q^*)$ and $F(q^*) = 0$.*
- (c) *If $CP(q')$ has an optimal solution z' and $F(q') = 0$, then z' is an optimal solution of CFD and $q' = q^*$.*
- (d) *Let δ be a given nonnegative real number. If $\int_0^T \frac{a''(t)z(t)}{b(t)z(t)} dt \geq \alpha > 0$, for all $z \in S$, and if $P(q')$ has an optimal solution z' and $F(q') \geq \delta$, then z' is an approximate $\frac{\delta}{\alpha}$ -optimal solution of CFD and $q' \leq q^* + \frac{\delta}{\alpha}$.*

With the help of this equivalence, we can formulate the following algorithm for finding an approximate solution of CFD problem.

Algorithm 2

Let δ be a given positive real number.

Step 1. Take $k := 0$, and find a feasible solution z^0 of the problem CFD.

Step 2. Compute the value V_k of the objective function for the current feasible solution z^k :

$$V_k = \frac{\int_0^T \frac{[\frac{\beta}{T}b(t) - a'(t)]z^k(t)}{b(t)z^k(t)} dt}{\int_0^T \frac{a''(t)z^k(t)}{b(t)z^k(t)} dt}.$$

Step 3. Find

$$Q_{k+1} = \inf_{z \in S} \int_0^T \frac{[\beta b(t) - a'(t) - V_k a''(t)]z(t)}{b(t)z(t)} dt. \tag{19}$$

Let $z^{k+1} \in S$ the optimal solution for the linear-fractional continuous time problem (19).

- Step 4.* i) If $Q_{k+1} < -\delta$, then take $k := k + 1$ and go t Step 2.
- ii) If $Q_{k+1} \geq -\delta$, then the algorithm stops. The solution z^k is an approximate optimal solution of problem CFD.

The numerical resolution of the continuous fractional problems (18) and (19) involves its approximation by a discrete, rather than continuous, optimization problem.

8. CONCLUSIONS

Two classes of stochastic continuous time fractional programming problems were considered. In the case when the denominator of the objective is simply randomized some deterministic equivalent continuous time fractional programming problem has been obtained.

Similar results can be obtained in the case of complete randomization of the fractional objective (*i.e.* the nominator b of the fractional objective function is also random).

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