RAIRO. RECHERCHE OPÉRATIONNELLE

E. BAMPIS
Y. MANOUSSAKIS
I. MILIS

On the parallel complexity of the alternating hamiltonian cycle problem

RAIRO. Recherche opérationnelle, tome 33, n^o 4 (1999), p. 421-437
<http://www.numdam.org/item?id=RO_1999__33_4_421_0>

RAIRO Operations Research
RAIRO Oper. Res. **33** (1999) 421-437

ON THE PARALLEL COMPLEXITY OF THE ALTERNATING HAMILTONIAN CYCLE PROBLEM (*)

by E. BAMPIS (1), Y. MANOUSSAKIS (2) and I. MILIS (2,**)

Communicated by Philippe CHRETIENNE

Abstract. – *Given a graph with colored edges, a Hamiltonian cycle is called alternating if its successive edges differ in color. The problem of finding such a cycle, even for 2-edge-colored graphs, is trivially NP-complete, while it is known to be polynomial for 2-edge-colored complete graphs. In this paper we study the parallel complexity of finding such a cycle, if any, in 2-edge-colored complete graphs. We give a new characterization for such a graph admitting an alternating Hamiltonian cycle which allows us to derive a parallel algorithm for the problem. Our parallel solution uses a perfect matching algorithm putting the alternating Hamiltonian cycle problem to the RNC class. In addition, a sequential version of our parallel algorithm improves the computation time of the fastest known sequential algorithm for the alternating Hamiltonian cycle problem by a factor of* $O(\sqrt{n})$.

1. INTRODUCTION

Last years problems arising in molecular biology are often formulated using colored graphs, *i.e.* graphs with colored edges and/or vertices. Given such a graph, original problems correspond to extracting subgraphs such as Hamiltonian and Eulerian paths or cycles colored in a specified pattern [11–13, 20]. The most natural pattern in such a context is that of alternating coloring, *i.e.* adjacent edges/vertices having different colors. This type of problems is also encountered in VLSI for compacting a programmable logical array [15]. Colored paths and cycles have applications in various other fields, as in cryptography where a color represents a specific type of transmission or in social sciences where a color represents a relation between two individuals and the notion of alternating colored paths and

(*) Received April 1996.
(**) Fellow of the European Commission (Program H.C.&M.). *Present address*: National Technical University of Athens (NTUA), Dept. of Electr. & Comp. Eng., Division of Comp. Sc., Heroon Polytechniou 9, 15773 Zografou - Athens, Greece.
(1) LaMI, Université d'Evry-Val-d'Essonne, 91025 Evry Cedex, France.
(2) L.R.I., bâtiment 490, Université de Paris-Sud, 91405 Orsay Cedex, France.

Recherche opérationnelle/Operations Research, 0399-0559/99/04/$ 7.00

cycles is related to the balance of a graph [9]. Note also that the concept of alternating colored subgraphs is often used implicitly in graph theory; consider for example a given instance of Edmond's maximum matching algorithm [14]: if we consider the edges of the current matching colored blue and any other edge colored red, then the desired augmenting path (in Edmond's terms) is just an alternating colored one.

On the other hand there is a great theoretical interest on problems in colored graphs and a large body of work has been published [3, 5, 6, 8, 21] (a brief review of these works is given in the next section). Motivated by both the applications and the theoretical interest, we consider here complete graphs with edges colored by two colors (2-edge-colored complete graphs) and we search for a Hamiltonian cycle whose successive edges alternate between the two colors. Such a cycle is known as *alternating Hamiltonian cycle* (AHC). Our study is restricted to complete graphs, since the problem is trivially NP-complete in the case of general 2-edge-colored graphs. The same problem can be also stated in the following way: given a graph G and its complement $\overline{G}$ find a Hamiltonian cycle whose edges alternate between those of G and those of $\overline{G}$.

In particular, in this paper we deal with the parallel complexity of the AHC problem. We give first a new characterization for a 2-edge-colored complete graph to admit an AHC. Instead of the up to now known conditions based on vertices' degrees, our characterization relies on terms of connectivity of a specified digraph implied by an alternating factor of the initial colored graph. This new characterization allows us to design a parallel algorithm which takes as input an alternating factor, and either finds an AHC or decides that such an AHC does not exist. The algorithm is on the CRCW-PRAM model where concurrent reads and concurrent writes to the same memory location are allowed. Its parallel complexity is $O(\log^4 n)$ time using $O(n^2)$ processors where n is the order of the graph. An alternating factor can be found using a maximum matching algorithm, putting the whole AHC problem in RNC. In addition, as a byproduct of our parallel algorithm we obtain an $O(n^{2.5})$ sequential algorithm for the problem. This result improves the complexity of the best known $O(n^3)$ sequential algorithm [5].

The paper is organized as follows: in the next section we give a brief review of previous works on the AHC problem. In Section 3, we prove a series of non-trivial lemmas for the parallel contraction of alternating cycles. Given these lemmas the proof of our characterization theorem follows in Section 4 and the parallel algorithm derived is presented in Section 5.

1.1. Notation

Throughout the paper, K denotes a 2-edge-colored complete graph with an even number of vertices. We assume that the edges of K are colored **red** and **blue**. By $r(v)$ and $b(v)$ we denote the red and the blue degree, respectively, of a vertex v. The edge between the vertices u and v is denoted by uv, and its color by $\chi(uv)$. If A_1 and A_2 are subsets of V, then the set of edges between the vertices of A_1 and A_2 is denoted by A_1A_2, while the edges among the vertices of A_1 are denoted by A_1A_1.

The notation $\chi(A_1A_2)$ is used if and only if all the edges A_1A_2 are monochromatic and represents their common color.

An alternating factor $F = \{C_0, C_1, ..., C_{m-1}\}$ of K is a set of pairwise vertex disjoint alternating cycles covering all the vertices of the graph. It is clear that the existence of an alternating factor is a necessary condition for a graph K to admit an alternating Hamiltonian cycle. An alternating factor is said minimum if there is no other one with smaller cardinality. Clearly F becomes an AHC whenever $m = 1$.

It turns out to be convenient for our presentation to divide the vertices of an alternating cycle, C, of length $2p$ into two **classes** $X = \{x_0, x_1, ..., x_{p-1}\}$ and $Y = \{y_0, y_1, ..., y_{p-1}\}$ such that $C = x_0, y_0, x_1, y_1, ..., x_{p-1}, y_{p-1}$ and $\chi(x_iy_i) \neq \chi(y_ix_{i+1})$, for each $i = 0, 1, ..., p-1$ (the indices are considered modulo p). It is clear that classes X and Y can be interchanged w.l.o.g.

Once both classes X and Y are defined, we say that cycle C_1 **dominates** cycle C_2 ($C_1 \Longrightarrow C_2$) if $\chi(X_1C_2) \neq \chi(Y_1C_2)$ (recall that the notation $\chi(A_1A_2)$ is used if and only if all the edges A_1A_2 are monochromatic and by convention we assume that in notation $\chi(X_iC_j)$, X_i represents the class X of C_i and C_j represents the vertex set of C_j).

Obviously, if $C_1 \Longrightarrow C_2$, then $\chi(X_1C_2)$ implies $\chi(Y_1C_2)$, and *vice versa.* By $C_1 \not\Longleftrightarrow C_2$ we denote the fact that neither $C_1 \Longrightarrow C_2$ nor $C_2 \Longrightarrow C_1$.

2. PREVIOUS RESULTS

Erdös first asked for a sufficient condition for the existence of an AHC in a 2-edge-colored complete graph. The first answer to this problem has been presented by Bánkfalvi and Bánkfalvi [3]. In fact, their theorem gives the cardinality of the minimum alternating factor and it is expressed in terms of monochromatic degrees of the vertices.

THEOREM 2.1 [3]: *Let K be a 2-edge-colored complete graph with vertex set $V = \{v_1, v_2, ..., v_{n=2p}\}$. Assume that $r(v_1) \leq r(v_2) \leq ... \leq r(v_n)$. Then,* *the cardinality of the minimum alternating factor F of K is equal to m if and only if there are precisely $m-1$ distinct numbers k_i, $2 \leq k_i \leq p-2$ such that for each i, $i = 1, \ldots, m-1$,*

$$r(v_1) + r(v_2) + \ldots + r(v_{k_i}) + b(v_{2p}) + b(v_{2p-1}) + \ldots + b(v_{2p-k_i+1}) = k_i^2.$$

Next corollary proved in [5], gives a structural translation of Theorem 2.1. In what follows, an alternating factor satisfying Corollary 2.1 and consequently Theorem 2.1 will be referred as a **Bánkfalvi structure**. If F is a Bánkfalvi structure, then K does not contain an AHC.

COROLLARY 2.1 [5]: *Let $F = \{C_0, C_1, \ldots, C_{m-1}\}$, $m \geq 2$, be an alternating factor of K. If $C_i \Longrightarrow C_j$, $\chi(X_iX_i) = \chi(X_iC_j)$ and $\chi(Y_iY_i) = \chi(Y_iC_j)$, $0 \leq i < j \leq m-1$, then F is a minimum alternating factor of K.*

Later, Bollobás and Erdös [6] considered the more general case of k colors and they gave a new sufficient condition based on the relation between the monochromatic degrees of the vertices and the number of vertices in the graph. Chen and Daykin [8] have improved this result. Häggvist and Manoussakis [16] have proven – in a different phrasing – another condition replacing the degree condition of Theorem 2.1 by a more intuitive one.

THEOREM 2.2 [16]: *Let K be a 2-edge-colored complete graph. K contains an AHC if and only if:*

(i) *it admits an alternating factor, and*

(ii) *every pair of vertices is connected with an alternating not necessarily simple path.*

Theorems 2.1 and 2.2 answer to the decision problem, but they do not lead directly to an algorithm for searching an AHC in a 2-edge-colored complete graph K. Such a sequential algorithm of complexity $O(n^3)$ is presented by Benkouar *et al.* [5]. Furthermore, Saad have studied the longest alternating, not necessarily Hamiltonian, cycle problem [21]. However, the algorithms presented in these works are inherently sequential and thus, in a parallel context, a different approach is needed.

We close this section with some known results that will be useful in the parallel study of the problem. Next theorem gives the complexity of an optimal parallel algorithm for finding a Hamiltonian cycle in a strongly connected (3) semicomplete digraph [1]. We recall that a semicomplete

(3) A digraph is called *strongly connected* or simply *strong* if for every pair of vertices u and v there is a directed path from u to v and one directed path from v to u.

digraph is a digraph with no pair of non-adjacent vertices (*i.e.* for every pair v, u of vertices either arc (v, u) or arc (u, v) or both are present).

THEOREM 2.3 [1]: *For any semicomplete digraph T, the strongly connected components of T and a Hamiltonian cycle in each strongly connected component can be found by an $O(\log n)$ time, $O(n^2/\log n)$ processors algorithm within the CRCW-PRAM model.*

Next two lemmas concern the parallel contraction of directed cycles in bipartite tournaments and have been presented in [2].

LEMMA 2.1 [2]: *Let C_1 and C_2 be two vertex disjoint cycles of a bipartite tournament. If there exist at least one arc directed from C_1 to C_2 and another one directed from C_2 to C_1, then C_1 and C_2 can be contracted to a single cycle in $O(\log n)$ time using $O(n^2)$ processors.*

LEMMA 2.2 [2]: *Let $C_0, C_1, ..., C_{m-1}$ be m pairwise vertex disjoint cycles of a bipartite tournament such that all arcs are directed from C_i to C_j, $0 \leq i < j \leq m-1$. Let also C_m be a cycle such that there exist at least two opposite directed arcs between C_0 and C_m, and also between C_{m-1} and C_m. Then, $C_0, C_1, ..., C_{m-1}, C_m$ can be contracted to a single cycle in $O(\log n)$ time using $O(n^2)$ processors.*

3. PARALLEL CONTRACTION OF ALTERNATING CYCLES

Given an alternating factor, $F = \{C_0, C_1, ..., C_{m-1}\}$, of a 2-edge-colored complete graph K, our aim is to reduce it to one of minimum cardinality by contracting its alternating cycles. Following this direction a series of lemmas for the contraction of alternating cycles in specified cases are presented in this section. Lemmas are stated in a parallel context for their later use in the proposed parallel algorithm.

The proofs of lemmas are based on a **transformation** from a given alternating factor F of a 2-edge-colored complete graph K to a bipartite tournament $B = (\mathcal{X} \cup \mathcal{Y}, E(B))$ such that alternating cycles of K correspond to directed cycles in B and *vice versa*. The bipartite tournament B corresponding to F is obtained in the following way:

- for every alternating cycle $C_i \in F$, either $X_i \subset \mathcal{X}$ and $Y_i \subset \mathcal{Y}$, or $X_i \subset \mathcal{Y}$ and $Y_i \subset \mathcal{X}$ depending on the specific domination relations among the cycles in F.
- For every pair, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ of vertices of B if $\chi(xy) =$ red in K, then $(x, y) \in E(B)$; otherwise, if $\chi(xy) =$ blue in K, then $(y, x) \in E(B)$.

In such a transformation, it is clear that an alternating cycle C_i of K corresponds to a directed cycle $\vec{C_i}$ in B and *vice versa*. Consequently, a cycle obtained by contracting directed cycles in B corresponds to an alternating one in K.

LEMMA 3.1: *Let C_1, C_2 be two vertex disjoint alternating cycles. Then, either $C_1 \Longrightarrow C_2$ or $C_2 \Longrightarrow C_1$ or C_1 and C_2 can be contracted to a single alternating cycle in $O(\log n)$ time using $O(n^2)$ processors.*

Proof: Let us first examine, whether there exist at least two edges e_1, e_2 such that $e_1, e_2 \in X_1Y_2$, or $e_1, e_2 \in Y_1X_2$, and $\chi(e_1) \neq \chi(e_2)$. If this is the case, we consider the bipartite tournament B with $\mathcal{X} = X_1 \cup X_2$ and $\mathcal{Y} = Y_1 \cup Y_2$. Then, the edges e_1, e_2 are replaced by two arcs such that one of them is oriented from $\vec{C_1}$ to $\vec{C_2}$ and the other one from $\vec{C_2}$ to $\vec{C_1}$. By Lemma 2.1, $\vec{C_1}$ and $\vec{C_2}$ can be contracted to a single directed cycle that implies an alternating one.

If there are at least two edges e_1, e_2 such that $e_1, e_2 \in X_1X_2$ or $e_1, e_2 \in Y_1Y_2$ and $\chi(e_1) \neq \chi(e_2)$, then we consider the bipartite tournament B with $\mathcal{X} = X_1 \cup Y_2$ and $\mathcal{Y} = Y_1 \cup X_2$. As before C_1 and C_2 can be contracted to a single alternating cycle.

Therefore, either C_1 and C_2 can be contracted to a single alternating cycle, or $\chi(X_1X_2)$, $\chi(X_1Y_2)$, $\chi(Y_1X_2)$, and $\chi(Y_1Y_2)$.

If there are two edges such that $e_1 \in X_1X_2$, $e_2 \in Y_1Y_2$ and $\chi(e_1) = \chi(e_2)$, then we consider the bipartite tournament B with $\mathcal{X} = X_1 \cup Y_2$ and $\mathcal{Y} = Y_1 \cup X_2$. Otherwise, if there are $e_1 \in X_1Y_2$, $e_2 \in Y_1X_2$ such that $\chi(e_1) = \chi(e_2)$, then we consider the bipartite tournament B with $\mathcal{X} = X_1 \cup X_2$ and $\mathcal{Y} = Y_1 \cup Y_2$. In both cases, C_1 and C_2 can be contracted.

Therefore, either C_1 and C_2 can be contracted to a single alternating cycle, or $\chi(X_1C_2) \neq \chi(Y_1C_2)$ *i.e.* $C_1 \Longrightarrow C_2$, or $\chi(C_1Y_2) \neq \chi(C_1X_2)$ *i.e.* $C_2 \Longrightarrow C_1$.

Notice that in all cases where C_1 and C_2 can be contracted to a single alternating cycle this can be done, by Lemma 2.1, in $O(\log n)$ time using $O(n^2)$ processors. □

LEMMA 3.2: *Let $C_0, C_1, C_2, \ldots, C_{m-1}$ be m pairwise vertex disjoint alternating cycles. These cycles can be contracted to a single alternating cycle C, in $O(\log n)$ time using $O(n/\log n)$ processors if for every $i = 0, 1, \ldots, m-1$ one of the following holds (all indices are considered modulo m):*

(i) $C_i \Longrightarrow C_{i+1}$.

(ii) *Either* $C_i \Longrightarrow C_{i+1}$, *or else if for some* i, $C_i \not\Longleftrightarrow C_{i+1}$, *then* $C_{i+1} \Longrightarrow C_{i+2}$.

(iii) *Either* $C_i \Longrightarrow C_{i+1}$ *and* $\chi(X_iC_{i+1}) =$ *red (resp. blue), or* $C_{i+1} \Longrightarrow C_i$ *and* $\chi(X_{i+1}C_i) =$ *blue (resp. red).*

(iv) $C_i \Longrightarrow C_j$, $0 \le i < j \le m-1$, *and there is an edge* $e \in X_0X_0$ *(resp.* $e \in Y_0Y_0$*) such that* $\chi(e) \neq \chi(X_iC_j)$ *(resp.* $\chi(e) \neq \chi(Y_iC_j)$*).*

Proof:

(i) Assume w.l.o.g. that $\chi(X_iC_{i+1}) =$ red, $0 \le i \le m-1$. (If for some i, $\chi(X_iC_{i+1}) = blue$, then we may interchange X_i and Y_i.)

We consider the bipartite tournament $B = (\mathcal{X} \cup \mathcal{Y}, E(B))$, with $\mathcal{X} = \bigcup_{i=0}^{m-1} X_i$ and $\mathcal{Y} = \bigcup_{i=0}^{m-1} Y_i$. Then, all arcs between $\vec{C}_i$ and $\vec{C}_{i+1}$ in B are oriented from $\vec{C}_i$ to $\vec{C}_{i+1}$, for $i = 0, 1, ..., m-1$.

A single directed cycle, that implies a single alternating one, can be found by breaking each cycle in any vertex from a fixed partition class and connecting the resulting paths cyclically.

(ii) Assume again that $\chi(X_iC_{i+1}) =$ red for every i, $0 \le i \le m-1$, for which $C_i \Longrightarrow C_{i+1}$.

We consider the following bipartite tournament B:

if $C_i \Longrightarrow C_{i+1}$, then X_i is added to $\mathcal{X}$ and Y_i is added to $\mathcal{Y}$.

If $C_i \not\Longleftrightarrow C_{i+1}$, then we proceed similarly with Lemma 3.1. Let e_1, e_2 be two edges having one endpoint in C_i and the other in C_{i+1} which destroy the domination relation between C_i and C_{i+1}, *i.e.* $\chi(e_1) \neq \chi(e_2)$. The construction of the bipartite tournament B depends on the classes to which the endpoints of e_1 and e_2 belong to. We distinguish between three main subcases:

- if $e_1, e_2 \in X_iY_{i+1}$ or $e_1, e_2 \in Y_iX_{i+1}$, then X_i is added to $\mathcal{X}$ and Y_i is added to $\mathcal{Y}$.
- If $e_1, e_2 \in X_iX_{i+1}$ or $e_1, e_2 \in Y_iY_{i+1}$, then X_i is added to $\mathcal{Y}$ and Y_i is added to $\mathcal{X}$.
- If e_1 (resp. e_2) $\in X_iX_{i+1}$ and e_2 (resp. e_1) $\in Y_iY_{i+1}$, then X_i is added to $\mathcal{Y}$ and Y_i is added to $\mathcal{X}$.

If $C_i \Longrightarrow C_{i+1}$, then all arcs between $\vec{C}_i$ and $\vec{C}_{i+1}$ in B are oriented from $\vec{C}_i$ to $\vec{C}_{i+1}$ and the two cycles can be contracted as in Case (i).

If $C_i \not\Longleftrightarrow C_{i+1}$, then there exist arcs from $\vec{C}_i$ to $\vec{C}_{i+1}$ and oppositely. Therefore, there is at least one arc (u_i, v_{i+1}) from $\vec{C}_i$ to $\vec{C}_{i+1}$ for every

$i = 0, 1, ..., m - 1$. In order to construct a single cycle, we do not break $\vec{C}_i$ and $\vec{C}_{i+1}$ arbitrarily, but cycle $\vec{C}_i$ in the vertex u_i and $\vec{C}_{i+1}$ in the predecessor of v_{i+1}. Recall that by the hypothesis $C_{i+1} \Longrightarrow C_{i+2}$ and all arcs between $\vec{C}_{i+1}$ and $\vec{C}_{i+2}$ in B are oriented from $\vec{C}_{i+1}$ to $\vec{C}_{i+2}$. Therefore, the predecessor of v_{i+1} always dominates all vertices of $\vec{C}_{i+2}$.

(iii) Let $\mathcal{X} = \bigcup_{i=0}^{m-1} X_i$ and $\mathcal{Y} = \bigcup_{i=0}^{m-1} Y_i$.

- If $C_i \Longrightarrow C_{i+1}$ and $\chi(X_i C_{i+1}) =$ red (resp. blue), then $\chi(X_i Y_{i+1}) =$ red (resp. blue) and $\chi(Y_i X_{i+1}) =$ blue (resp. red).
- If $C_{i+1} \Longrightarrow C_i$ and $\chi(X_{i+1} C_i) =$ blue (resp. red), then $\chi(X_{i+1} Y_i) =$ blue (resp. red) and $\chi(Y_{i+1} X_i) =$ red (resp. blue).

Therefore, $\chi(X_i Y_{i+1}) =$ red (resp. blue) and $\chi(Y_i X_{i+1}) =$ blue (resp. red) for every $i = 0, 1, ..., m - 1$.

Thus, all arcs in B are oriented from (resp. to) $\vec{C}_i$ to (resp. from) $\vec{C}_{i+1}$, for every $i = 0, 1, \ldots, m - 1$, and we conclude as in Case (i).

(iv) Assume w.l.o.g. that $\chi(X_i C_j) =$ red, $0 \leq i < j \leq m - 1$. Let us denote by $x_0^i y_0^i x_1^i y_1^i ... x_{t_i}^i y_{t_i}^i$ the cycle C_i and let $x_i^0 x_j^0$ be a blue $X_0 X_0$ edge.

Then, the desired cycle is: $y_{j-1}^0 (x_0^1 y_0^1 x_1^1 y_1^1 ... x_{t_1}^1 y_{t_1}^1)(x_0^2 y_0^2 x_1^2 y_1^2 ... ... x_{t_2}^2 y_{t_2}^2) ... (x_0^{m-1} y_0^{m-1} x_1^{m-1} y_1^{m-1} ... x_{t_{m-1}}^{m-1} y_{t_{m-1}}^{m-1})(y_{i-1}^0 x_{i-1}^0 ... x_j^0 x_i^0 y_i^0 x_{i+1}^0 ... ... y_{j-1}^0)$.

In all cases (i - iv) the desired cycle can be constructed in $O(\log n)$ time with $O(n/\log n)$ processors using the list ranking technique [10]. □

LEMMA 3.3: *Let $C_0, C_1, ..., C_{m-1}$ be m pairwise vertex disjoint alternating cycles forming a Bánkfalvi structure, and C_m be another alternating cycle such that $C_0 \not\Longleftrightarrow C_m$ and $C_{m-1} \not\Longleftrightarrow C_m$. Then $C_0, C_1, ..., C_{m-1}, C_m$ can be contracted to a single alternating cycle in $O(\log n)$ time using $O(n^2)$ processors.*

Proof: Let us consider the corresponding directed cycles $\vec{C}_0, \vec{C}_1, \vec{C}_2, ..., \vec{C}_{m-1}, \vec{C}_m$ of a bipartite tournament B. The construction of B is as following: since $C_i \Longrightarrow C_j$, $0 \leq i < j \leq m - 1$, we can define w.l.o.g. the classes X_i, Y_i of C_i's, $0 \leq i < m - 1$, such that in B all arcs go from $\vec{C}_i$ to $\vec{C}_j$.

Note that this definition of classes does not fix the classes of C_{m-1} and allows us to interchange them. We define first the classes of C_m and then those of C_{m-1}.

Since $C_0 \not\Longleftrightarrow C_m$ we can define the classes X_m and Y_m of C_m in order to have in B arcs from $\vec{C}_0$ to $\vec{C}_m$ and oppositely.

Similarly, since $C_{m-1} \not\Leftrightarrow C_m$, we can also define the classes X_{m-1} and Y_{m-1} of C_{m-1} in order to have in B arcs from $\vec{C}_{m-1}$ to $\vec{C}_m$ and oppositely.

We consider now the bipartite tournament, B with $\mathcal{X} = \bigcup_{i=0}^{m} X_i$ and $\mathcal{Y} = \bigcup_{i=0}^{m} Y_i$. All arcs go from $\vec{C}_i$ to $\vec{C}_j$, $0 \leq i < j \leq m-1$, and there are also arcs from $\vec{C}_m$ to both $\vec{C}_0$ and $\vec{C}_{m-1}$, and oppositely.

Lemma 2.2 can be therefore applied to find a single directed cycle covering the vertices of $\vec{C}_0, \vec{C}_1, \vec{C}_2, ..., \vec{C}_{m-1}, \vec{C}_m$ in $O(\log n)$ time using $O(n^2)$ processors, and consequently a single alternating cycle with the same vertex set. □

4. MAIN RESULT

Before stating our main theorem, let us introduce two definitions.

DEFINITION 4.1: *Let $F = \{C_0, C_1, ..., C_{m-1}\}$ be an alternating factor of K. The* **underlying digraph** *of F is defined as the semicomplete digraph D with vertex set $V(D) = \{c_0, c_1, ..., c_{m-1}\}$ corresponding to the cycle set of F (each cycle C_i is contracted to a single vertex c_i) and arc set $E(D)$ defined as follows:*

- *if $C_i \Longrightarrow C_j$, then the arc $(c_i, c_j) \in E(D)$.*
- *Otherwise, if $C_i \not\Leftrightarrow C_j$, then both $(c_i, c_j) \in E(D)$ and $(c_j, c_i) \in E(D)$. (In this case we say that c_i and c_j are connected by a* **symmetric arc***.)*

In what follows, $C_i \in F$ is called the *underlying cycle* of vertex $c_i \in V(D)$. We show, in Theorem 4.1, that if D is strongly connected then K admits an AHC. If the underlying digraph D is not strongly connected, let $D_0, D_1, ..., D_{k-1}$ be its strongly connected components, ordered such that for every pair of vertices $v \in D_i$, $u \in D_j$, $0 \leq i < j \leq k-1$, no arc (u, v) exists. We focus our interest on its first strongly connected component D_0. Let us denote by C_{D_0} the alternating cycle resulting from the contraction of the cycles involved in D_0, and by $X_{C_{D_0}}$ and $Y_{C_{D_0}}$ the obtained partition classes. By C_i^r let us denote the underlying cycle of some vertex $c_i \in D_r$, $0 \leq r \leq k-1$.

Remark: In the next definition, we assume, without loss of generality, that the classes X and Y of the alternating cycles are defined such that $\chi(X_i^{r-1} C_s^r)$=red for every cycle C_s^r in D_r, $1 \leq r \leq k-1$. (In fact, if $\chi(X_i^{r-1} C_s^r)$=blue then it suffices to interchange the classes X_i^{r-1} and Y_i^{r-1}.)

DEFINITION 4.2: *The first component D_0 of D is a* **nice component** *if one of the following holds:*

(i) *there is a cycle* $C^r_{i'}$, $2 \leq r \leq k-1$, *such that for some cycle* C^0_i, $\chi(X^0_i C^r_{i'})$ *= blue.*

(ii) $X_{C_{D_0}} \neq \cup_{C_i \in D_0} X_i$ *(that also implies* $Y_{C_{D_0}} \neq \cup_{C_i \in D_0} Y_i$*).*

(iii) *There is a blue (resp. red)* $X_{C_{D_0}} X_{C_{D_0}}$ *(resp.* $Y_{C_{D_0}} Y_{C_{D_0}}$*) edge.*

THEOREM 4.1: *A 2-edge-colored complete graph* K *admits an AHC if and only if it admits an alternating factor and either (a)* D *is strongly connected or (b)* D_0 *is nice.*

Proof: We prove first the <u>if direction</u>.

(a) By induction on the number m of cycles of the factor F. For $m = 2$, since D is strongly connected, $C_0 \not\Leftrightarrow C_1$ and therefore C_0 and C_1 can be contracted into a single one by Lemma 3.1. Assume therefore that $m \geq 3$. If D contains a strongly connected proper subdigraph D' with $m' < m$ vertices, it can be contracted into a single cycle C', by induction. The new digraph induced by the cycles of $D - D'$ plus C' remains strongly connected and contains $m - m' + 1 < m$ cycles. Using induction once more the proof is completed. If there is no such a subdigraph, then D is an almost transitive tournament except the arc (c_{m-1}, c_0). In this case the Hamiltonian cycle $c_0, c_1, \ldots, c_{m-1}, c_0$ in D implies, by Lemma 3.2(i), an alternating Hamiltonian cycle in K.

(b) Three cases can be distinguished according to Definition 4.2. The idea is to contract, if possible, some cycles of F in order to obtain a new alternating factor whose underlying digraph is strongly connected. Then, we can conclude using Case (a).

(i) *There is a cycle* $C^r_{i'}$, $2 \leq r \leq k-1$, *such that for some cycle* C^0_i, $\chi(X^0_i C^r_{i'})$ *= blue.*

Assume first that $r = k - 1$. Let $C^1_{i''}$ be a cycle in D_1. We consider the three cycles C^0_i, $C^1_{i''}$ and $C^{k-1}_{i'}$. Clearly, $C^0_i \Longrightarrow C^1_{i''} \Longrightarrow C^{k-1}_{i'} \Longleftarrow C^0_i$. By remark preceding Definition 4.2, $\chi(X^0_i C^1_{i''})$=red and by the hypothesis of this case $\chi(X^0_i C^{k-1}_{i'})$ = blue.

If $\chi(X^1_{i''} C^{k-1}_{i'})$ = red, then by Lemma 3.2(iii), we contract the three cycles into a single one. If $\chi(X^1_{i''} C^{k-1}_{i'})$=blue, then by interchanging the classes of C^0_i we obtain $\chi(X^0_i C^1_{i''})$ = blue and $\chi(X^0_i C^{k-1}_{i'})$ = red, and thus again by Lemma 3.2(iii), we contract the three cycles into a single one.

In both cases, one cycle from the first and one cycle of the last strongly connected component of D participate in the resulting cycle. Therefore, the new underlying digraph becomes strongly connected.

Assume next that $r \neq k-1$. Let now $C_{i''}^{k-1}$ be a cycle in D_{k-1}. Clearly, $\chi(X_i^0 C_{i''}^{k-1})$ = red, for otherwise $r = k-1$. We consider the three cycles $C_i^0 \Longrightarrow C_{i'}^r \Longrightarrow C_{i''}^{k-1} \Longleftarrow C_i^0$ and we conclude as before.

(ii) $X_{C_{D_0}} \neq \cup_{C_i \in D_0} X_i$.

By (a) the cycles included in the first component D_0 of D can be contracted into a single cycle C_{D_0}. By the hypothesis of this case, the contraction leads to a mixing of classes of the included F's cycles, and therefore the new cycle does not dominate any cycle in each strongly connected component D_r, $1 \leq r \leq k-1$. This yields a new strongly connected D.

(iii) *There is a blue (resp. red)* $X_{C_{D_0}} X_{C_{D_0}}$ *(resp.* $Y_{C_{D_0}} Y_{C_{D_0}}$*) edge.*

We consider the cycle C_{D_0} and one cycle from each strongly connected component of D. This collection of cycles satisfies the hypothesis of Lemma 3.2(iv) and can be contracted to a single cycle. Since a cycle from each strongly connected component of D participates in the new cycle, the new D is strongly connected.

We complete the proof by the only if direction. Assume by contradiction that K admits an AHC but none of the conditions of Theorem 4.1 holds. By (a), D is not strongly connected. By (b), the first component of D is not nice *i.e.* $\chi(X_{C_{D_0}} X_{C_{D_0}}) = \chi(X_i^0 C_{i'}^r)$ = red and $\chi(Y_{C_{D_0}} Y_{C_{D_0}}) = \chi(Y_0^1 C_{i'}^r)$ = blue, for every cycle $C_{i'}^r$ in D_r, $1 \leq r \leq k-1$. In this case, by Corollary 2.1, C_{D_0} is a cycle of a Bánkfalvi structure and its vertices can not be contained in any alternating Hamiltonian cycle of K, a contradiction. □

5. A PARALLEL ALGORITHM

The existence of an alternating factor is a necessary condition for a 2-edge-colored complete graph to admit an AHC by Theorem 4.1. In order to find such an alternating factor, we have to apply two times a maximum matching algorithm: find a red maximum matching M_r in the graph induced by the red edges, and a blue one M_b in the graph induced by the blue edges. If either M_r or M_b is not perfect, it is clear that K has no alternating factor. Otherwise, an alternating factor can be constructed by considering the union of M_r and M_b.

Next algorithm follows the cases of Definition 4.2 and according the proof of Theorem 4.1 either decides that an AHC does not exist or reduces the AHC problem in a 2-edge-colored complete graph K to the same problem in the case where K has a strongly connected underlying digraph (then an AHC exists by Case Th. 4.1(a)). A procedure called FIND-AHC (D strong) is used for this latter problem. The details of this procedure are given in the next section.

ALGORITHM HAMILTONIAN CYCLE(K)

1. Find an alternating factor $F = \{C_0, C_1, ..., C_{m-1}\}$ of K.
2. If F does not exist then STOP $\{K$ has no AHC$\}$.
3. If $m = 1$ then STOP $\{F$ is an AHC$\}$.
4. Construct D.
5. If D is strongly connected then FIND-AHC(D strong); STOP.
6. Find the strongly connected components $D_0, D_2, ..., D_k$ of D and fix the classes X and Y of the alternating cycles such that $\chi(X_i^{r-1} C_s^r) =$ red for every cycle C_s^r in D_r, $2 \leq r \leq k$.
7. If Definition 4.2(i) holds, then using Theorem 4.1(b-i) construct new D and FIND-AHC (D strong); STOP.
8. FIND-AHC(D_0 strong).
9. If Definition 4.2(ii) holds, then using Theorem 4.1(b-ii) construct new D and FIND-AHC (D strong); STOP.
10. If Definition 4.2(iii) holds, then using Theorem 4.1(b-iii) construct new D and FIND-AHC (D strong); STOP.
11. STOP $\{K$ has no AHC$\}$.

The complexity of the above algorithm is determined by the complexity of finding an alternating factor in a 2-edge-colored complete graph. By [19], a perfect matching of a graph containing $n(= 2p)$ vertices and m edges can be found by a randomized algorithm in $O(\log^2 n)$ time using $O(n^{3.5} m)$ processors. Since the number of red or blue edges of a 2-edge-colored complete graph is $O(n^2)$, an alternating factor can be found by a randomized algorithm in $O(\log^2 n)$ time using $O(n^{5.5})$ processors.

If an alternating factor is given, then the complexity of the algorithm is determined by the complexity of the procedure FIND-AHC (D strong).

5.1. Procedure FIND-AHC (D strong)

Recall that the underlying digraph D is a semicomplete digraph and, by Theorem 2.3, we can find in parallel a Hamiltonian cycle in D, since it is strongly connected. Unfortunately, this cycle does not help us to construct in parallel an AHC in K because of the symmetric arcs that may arise in this cycle. We can isolate these arcs as follows: Consider the undirected graph induced by the vertices of D which are extremities of at least one symmetric arc in D. Let M be a maximal matching in this graph (we denote by $V(M)$ the set of vertices covered by edges in M). Then for every $c_i c_j \in M$, we can contract the cycles C_i and C_j using Lemma 3.1. Furthermore, no both extremities of a symmetric arcs belong to $V(D) - V(M)$, for otherwise M

is not maximal. Therefore, the directed graph induced by $V(D) - V(M)$ is a tournament.

In next lemma, we consider an alternating factor having a tournament as underlying digraph, and we prove that its reduction to a minimum one can be efficiently parallelized.

LEMMA 5.1: *Let $C_0, C_1, ..., C_{m-1}$ be a collection of alternating cycles whose underlying digraph is a tournament T. This collection can be reduced either to a single alternating cycle or to a Bánkfalvi structure, in $O(\log n)$ time using $O(n^2/\log n)$ processors.*

Proof: By Theorem 2.3, we can find a Hamiltonian cycle in each strongly connected component of T in $O(\log n)$ time using $O(n^2/\log n)$ processors. If T is strongly connected, then a single Hamiltonian cycle is obtained and the alternating cycles $C_0, C_1, ..., C_{m-1}$ can be contracted to a single alternating cycle using Lemma 3.2(i).

If T is not strongly connected, we first find a Hamiltonian path $H = \{c_0, c_1, ..., c_{m-1}\}$ in T. This can be done in $O(\log n)$ time using $O(n)$ processors, by [4]. This path implies a sequence of the alternating cycles $C_0, C_1, ..., C_{m-1}$, where C_i dominates C_{i+1} for $0 \leq i \leq m-2$. Assume w.l.o.g. that $\chi(X_i C_{i+1}) =$ red and $\chi(Y_i C_{i+1}) =$ blue, $0 \leq i \leq m-2$. We define now a new tournament T' with the same vertex set as T and arc set as follows: if $C_i \Longrightarrow C_j$ and $\chi(X_i C_j) =$ red (resp. blue), then there is an arc (c_i, c_j) in T' (resp. (c_j, c_i)). Clearly, the Hamiltonian path H of T remains unchanged in the new tournament T'.

By Theorem 2.3, we find a Hamiltonian cycle in each strongly connected component of T' in $O(\log n)$ time using $O(n^2/\log n)$ processors. Then, using Lemma 3.2(iii), we contract in parallel the alternating cycles involved in each strongly connected component of T' into a single one.

If T' is strongly connected, then a single alternating cycle is obtained and the proof is completed.

If T' is not strongly connected, then we obtain a new collection of alternating cycles $C'_0, C'_1, ..., C'_{m'-1}$, each one corresponding to a strongly connected component of T'. The contraction of cycles by Lemma 3.2(iii) gives a new cycle with classes that are unions of the corresponding classes of the cycles involved, *i.e.* there is no mixing of classes. Taking into account the construction of T', we conclude that $\chi(X'_i C'_j) =$ red and $\chi(Y'_i C'_j) =$ blue, $0 \leq i < j \leq m' - 1$. Finally, we find the minimum p, $0 \leq p \leq m' - 1$, for which there is at least one $X'_p X'_p$ blue edge, or a $Y'_p Y'_p$ red edge and using

Lemma 3.2(iv), we contract the cycles $C'_{p+1}, C'_{p+2}, \ldots, C'_{m'-1}$, into a single one. By Corollary 2.1, no further reduction is possible. Therefore either a single alternating cycle (if $p = 0$) or a Bánkfalvi structure (otherwise) is obtained. □

Given Lemma 5.1, we can reduce an alternating factor with strongly connected underlying digraph into a Bánkfalvi structure plus some alternating cycles. Since the underlying digraph of a Bánkfalvi structure is a transitive tournament we are in the situation of the next lemma proved in [2].

LEMMA 5.2 [2]: *Let S be a strongly connected semicomplete digraph which contains as an induced subgraph a transitive tournament T. Let $t_1, t_2, \ldots, t_k$ be the Hamiltonian path of T and $Q = \{q_1, q_2, \ldots, q_m\}$ the set of vertices in $V(S) - V(T)$. Then there is in S a cycle C: $t_i, t_{i+1}, \ldots, t_j, q_1, q_2, \ldots, q_l, t_i$, such that $P = t_j, q_1, q_2, \ldots, q_l, t_i$ is a minimal path from t_j to t_i in $V(Q) \cup \{t_i, t_j\}$, and C covers at least $\lceil k/2 \rceil$ vertices of T. Such a cycle C can be found in $O(\log n)$ time using $O(n^2)$ processors.*

Notice that, since path P in the above lemma is a minimal one, the arc between any two non-consecutive vertices of this path has a backward orientation. Notice also that if S is the underlying semicomplete digraph of K, then Lemma 5.2 gives a method of $O(\log n)$ depth to contract a set of alternating cycles, defined by C. The way to contract the cycles defined by C will be explained in Step 8 of the procedure below.

PROCEDURE FIND-AHC (D strong)

1. Construct an undirected graph U, where $V(U) = V(D)$ and $E(U) = \{c_i c_j | c_i c_j$ is a symmetric arc in $D\}$.
2. Find a maximal matching M in U.
3. In parallel, for every $c_i c_j \in M$, contract the corresponding cycles C_i and C_j using Lemma 3.1.
4. Construct new D; If $|V(D)| = 1$ then STOP.
5. Reduce the tournament $V(D) - V(M)$ either to a single cycle or to a Bánkfalvi structure with at most m alternating cycles using Lemma 5.1.
6. Construct new D; If $|V(D)| = 1$ then STOP.
7. Apply Lemma 5.2 in D and find the cycle C and the path $P = t_j, q_1, q_2, \ldots, q_l, t_i$ as defined in this lemma.
8. Contract the involved in C alternating cycles as following:
   8.1 if P does not contain consecutive symmetric arcs, then apply Lemma 3.2(ii) to contract all alternating cycles involved in C and go to Step (9);
   8.2 consider each symmetric arc of P as undirected and find all maximal directed subpaths of the form $q_r, q_{r+1}, \ldots, q_s$, $r \geq 1, s \leq l$ in P;
   8.3 in parallel, for each such maximal directed path in P:
   – if $s - r \geq 2$, then contract the involved in maximal directed path alternating cycles by using Lemma 3.2(ii) (notice that the arc (q_s, q_r) exists by the minimality of P and the fact

that D is a semicomplete digraph);
- if $s - r = 1$ (*i.e.* the maximal directed path is a single arc (q_r, q_s)), then contract the alternating cycles corresponding to the end vertices of the symmetric arc (q_s, q_{s+1}) in P, by using Lemma 3.1 (if $q_{s+1} = t_i$ then we contract the alternating cycles corresponding to the end vertices of the symmetric arc (q_{r-1}, q_r));

8.4 after step 8.3 all arcs in P are symmetric; by repeating applications of Lemma 3.1 contract P to a path of length two (*i.e.* a single vertex q' connected with symmetric arcs to both t_i and t_j);

8.5 apply Lemma 3.3 to contract the alternating cycles corresponding to q' and the remaining vertices of C into a single one.

9. Construct new D; FIND-AHC (D strong).

Given the complexity of the lemmas used in the procedure FIND-AHC, its complexity is determined by the complexity of the required maximal matching algorithm. The fastest known parallel algorithm for the maximal matching problem has been proposed in [17]. On a CRCW-PRAM this algorithm works in $O(\log^3 n)$ time using $O(n^2)$ processors because the Euler tour can be found on this model in $O(\log n)$ time.

Notice that either the maximal matching M of Step 2 or the Bánkfalvi structure of Step 5 covers at least the half of the cycles of K. However in Step 3 the number of cycles is divided by two and in Step 8 the size of the Bánkfalvi structure is also divided by two. Consequently, after $O(\log m)$ calls Procedure FIND-AHC is completed (m is the initial number of cycles in K). Since m can be a function of n, then the complexity of the whole procedure becomes $O(\log^4 n)$ time using $O(n^2)$ processors.

We notice that for a randomized version of procedure FIND-AHC, the maximal matching algorithm used in Step 2 can be replaced by a randomized maximum matching algorithm and thus, in this case, its complexity becomes $O(\log^3 n)$ time using $O(n^{5.5})$ processors.

Therefore, we can state the next theorem.

THEOREM 5.1: *An AHC in a 2-edge-colored complete graph can be found on a CRCW-PRAM*

- *by a deterministic $O(\log^4 n)$ time, $O(n^2)$ processors algorithm, if an alternating factor is given, and*
- *by a randomized $O(\log^3 n)$ time, $O(n^{5.5})$ processors algorithm, if an alternating factor has to be found.*

Our parallel algorithm implies also a sequential algorithm for the AHC problem. The complexity of this sequential algorithm is dominated by the complexity of the fastest known sequential maximum matching algorithm [18, 22].

THEOREM 5.2: *There is an $O(n^{2.5})$ sequential algorithm for finding an AHC in a 2-edge-colored complete graph.*

REFERENCES

1. E. BAMPIS, M. EL HADDAD, Y. MANOUSSAKIS and M. SANTHA, *A parallel reduction of Hamiltonian cycle to Hamiltonian path in tournaments*, J. Algorithms, 1995, *19*, p. 432-440.
2. J. BANG-JENSEN, M. EL HADDAD, Y. MANOUSSAKIS and T. M. PRZYTYCKA, *Parallel algorithms for the Hamiltonian cycle and Hamiltonian path problems in semicomplete bipartite digraphs*, Algorithmica, 1997, *97*, p. 67-87.
3. M. BÁNKFALVI and Z. BÁNKFALVI, *Alternating Hamiltonian circuit in two-colored complete graphs*, Theory of Graphs (Proc. Colloq. Tihany 1968), Academic Press, New York, p. 11-18.
4. A. BAR-NOY and J. NAOR, *Sorting, minimal feedback sets and Hamiltonian paths in tournaments*, SIAM J. Discr. Math., 1990, *3*, p. 7-20.
5. A. BENKOUAR, Y. MANOUSSAKIS, V. Th. PASCHOS and R. SAAD, *On the complexity of some Hamiltonian and Eulerian problems in edge-colored complete graphs*, RAIRO-Oper. Res., 1996, *30*, p. 417-438.
6. B. BOLLOBÁS and P. ERDÖS, *Alternating Hamiltonian cycles*, Israel J. Math., 1976, *23*, p. 126-131.
7. R. BRENT, *The parallel evaluation of general arithmetic expressions*, J. ACM 1974, *21*, p. 201-206.
8. C. C. CHEN and D. E. DAYKIN, *Graphs with Hamiltonian cycles having adjacent lines different colors*, J. Combin. Theory Ser. B, 1976, *21*, p. 135-139.
9. D. CARTWRIGHT and F. HARARY, *Structural balance: A generalization of Heider's theory*, Psychological Rev., 1956, *73*, p. 277-293.
10. R. COLE and U. VISHKIN, *Aproximate and exact parallel scheduling with applications to list tree and graph problems*, In Proc. 27th FOCS, 1986, p. 478-491.
11. D. DORNIGER, *On permutations of chromosomes*, In Contributions of General Algebra, *5*, Verlag Hölder-Pichler-Tempsky, Wien, and Teubner-Verlag, Stuttgart, 1987, p. 95-103.
12. D. DORNIGER, *Hamiltonian circuits determining the order of chromosomes*, Discrete Appl. Math., 1994, *50*, p. 159-168.
13. D. DORNIGER and W. TIMISCHL, *Geometrical constraints on Bennett's predictions of chromosome order*, Heredity, 1987, *58*, p. 321-325.
14. J. EDMONDS, *Paths, trees and flowers*, Canad. J. Math., 1965, *17*, p. 449-467.
15. T. C. HU and Y. S. KUO, *Graph folding and programmable logical arrays*, Networks, 1987, *17*, p. 19-37.
16. R. HÄGGVIST and Y. MANOUSSAKIS, *Cycles and paths in bipartite tournaments with spanning configurations*, Combinatorica, 1989, *9*, p. 33-38.
17. A. ISRAELI and Y. SHILOACH, *An improved parallel algorithm for maximal matching*, Inform. Process. Lett., 1986, *22*, p. 57-60.
18. S. MICALI and V. V. VAZIRANI, *An $O(|V|^{\frac{1}{2}}|E|)$ algorithm for finding maximum matchings in general graphs*, In Proc. 21th FOCS, 1980, p. 17-23.
19. K. MULMULEY, U. V. VAZIRANI and V. B. VAZIRANI, *Matching is as easy as matrix inversion*, Combinatorica, 1987, *7*, p. 105-113.

20. P. A. Pevzner, *DNA Physical mapping and alternating Eulerian cycles in colored graphs*, Algorithmica, 1995, *13*, p. 77-105.

21. R. Saad, *Finding a longest alternating Hamiltonian cycle in an edge colored complete graph is not hard*, Rapport de Recherche No. 691, Laboratoire de Recherche en Informatique, Université de Paris-Sud XI, Sep. 1991, to appear in Combinatorics, Probability and Computing.

22. V. V. Vazirani, *A theory of alternating paths and blossoms for proving correctness of the* $O(\sqrt{V}E)$ *general graph maximum matching algorithm*, Combinatorica, 1994, *14*, p. 71-109.