RAIRO. RECHERCHE OPÉRATIONNELLE

D. BHATIA

PANKAJ KUMAR GARG

(V, ρ) invexity and non-smooth multiobjective programming

RAIRO. Recherche opérationnelle, tome 32, nº 4 (1998), p. 399-414

<http://www.numdam.org/item?id=RO_1998__32_4_399_0>

Recherche opérationnelle/Operations Research
(vol. 32, n° 4, 1998, pp. 399-414)

(V, ρ) INVEXITY AND NON-SMOOTH MULTIOBJECTIVE PROGRAMMING (*)

by D. BHATIA ([1]) and PANKAJ KUMAR GARG ([2])

Communicated by Jean Pierre CROUZEIX

Abstract. – *The concept of (V, ρ) invexity has been introduced for non-smooth vector functions and is used to establish duality results for multiobjective programs.* © Elsevier, Paris

Keywords: (V, ρ)-Invexity, duality, multiobjective programming.

Résumé. – *Le concept de (V, ρ)-invexité est introduit pour les fonctions vectorielles non lisses, et est utilisé pour établir des résultats de dualité pour les programmes à plusieurs objectifs.* © Elsevier, Paris

Mots clés : (V, ρ)-invexité, dualité, programmation multiobjectif.

1. INTRODUCTION

Hanson [6] introduced the concept of invexity as a very broad generalization of convexity. Jeyakumar [8] introduced ρ-invex functions and studied various results for a single objective non-linear programming problem. Mond and Jeyakumar [9] have introduced the notion of V-invexity for vector function f and discussed its application to a class of multiobjective programming problems. Jeyakumar [9] established the equivalence between saddle points and optima, and duality theorems for a class of non-smooth non-convex problems in which functions are locally Lipschitz and satisfying invex type conditions of Hanson and Craven.

Recently, Bector *et al.* [2] developed sufficient optimality conditions and established duality results under V-invexity type of assumptions on the objective and constraint functions.

(*) Received April 1997.

([1]) Department of Mathematics, S.G.T.B. Khalsa College, University of Delhi, Delhi, 110007, India.

([2]) Department of Mathematics, University of Delhi, Delhi, 110007, India.

In all the above references the authors worked under differentiability assumptions. In the present paper, we have defined (V, ρ) invexity for non-smooth functions. Duality results for multiobjective programmes are established under these restrictions.

2. PRELIMINARIES AND DEFINITIONS

Here we consider the following multiobjective non-linear program:

(VOP) Minimize $[f_1(x), f_2(x), \ldots, f_p(x)]$

Subject to:

$$g_j(x) \leq 0 \qquad j = 1, 2, \ldots, m \tag{1}$$

$$x \in X \tag{2}$$

where functions

$$f_i : R^n \to R \quad i = 1, 2, \ldots, p$$

$$g_j : R^n \to R \quad j = 1, 2, \ldots, m$$

and X is an open subset of R^n. Also $f_i, i = 1, 2, \ldots, p, g_j, j = 1, 2, \ldots, m$ are locally Lipschitz functions around a point of X.

DEFINITION 1: A feasible point $\overline{x} \in X$ is said to be efficient solution for (VOP) if there is no other feasible solution x such that for some $r \in \{1, 2, \ldots, p\}$

$$f_r(x) < f_r(\overline{x})$$

and

$$f_i(x) \leq f_i(\overline{x}) \quad \text{for all} \quad i = 1, 2, \ldots, p \qquad i \neq r$$

DEFINITION 2: Let X be an open subset of R^n, the function $h : X \to R$ is locally Lipschitz around $x \in X$ if there exists a positive constant k and a positive number ε such that

$$| h(x_1) - h(x_2) | \leq K \| x_1 - x_2 \| \qquad \forall x_1, x_2 \in x + \varepsilon B$$

where $x + \varepsilon B$ is the open ball of radius ε about x.

DEFINITION 3: [4] If $h : X \to R$, the directional derivative of h at $x \in X$ in the direction of $v \in R^n$ denoted by $h'(x; v)$ is defined as follows:

$$h'(x, v) = \lim_{\lambda \to 0} \frac{h(x + \lambda v) - h(x)}{\lambda}.$$

DEFINITION 4: [4] If $h : X \to R$ is locally Lipschitz around $x \in X$, the generalized derivative of h at $x \in X$ in the direction of $v \in R^n$, denoted by $h^0(x, v)$ is given by

$$h^0(x; v) = \lim_{\lambda \downarrow 0} \sup_{y \to x} \left[\frac{h(y + \lambda v) - h(y)}{\lambda}\right].$$

The Lipschitz condition on the function guarantees that the above limit is a well defined quantity as $|h^0(x; v)| \leq K \| v \|$ where K is a Lipschitz constant.

DEFINITION 5: [4] The generalized gradient of h at $x \in X$, denoted by $\partial h(x)$ is defined as follows

$$\partial h(x) = [\xi \in R^n : h^0(x; v) \geq \xi^T v \quad \forall v \in R^n].$$

DEFINITION 6: [4] The function $h : X \to R$ is said to be regular at $x \in X$ provided that

(i) For all v, the usual one-sided directional derivative $h'(x; v)$ exists.

(ii) For all v, $h'(x, v) = h^0(x; v)$.

Now, we introduce the following definitions:

A vector function $f : X \to R^p$ is locally Lipschitz around $u \in X$ if every component f_i, $i = 1, 2, \ldots, p$, is locally Lipschitz around $u \in X$.

DEFINITION 7: A vector function $f : X \to R^p$, locally Lipschitz at $u \in X$, is said to be (V, ρ)-invex at u if there exist functions $\eta, \psi : X \times X \to R^n$, a real number ρ and $\theta_i : X \times X \to R^+ \backslash \{0\}$ $i = 1, 2, \ldots, p$ such that for all $x \in X$ and for $i = 1, 2, \ldots, p$ $f_i(x) - f_i(u) \geq \theta_i(x, u) \xi_i^T \eta(x, u) + \rho \| \psi(x, u) \|^2$ for every $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$.

If

(7a) $\rho > 0$, then the function is strongly V-invex at u

(7b) $\rho = 0$ then the function is V-invex at u

(7c) $\rho < 0$ then the function is weakly V-invex at u

(7d) $\forall x \in X, \quad x \neq u$ and for $i = 1, 2, \ldots, p$

$$f_i(x) - f_i(u) > \theta_i(x, u)\, \xi_i^T \eta(x, u) + \rho \| \psi(x, u) \|^2$$

for every $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$ then f is called strictly (V, ρ) invex at u.

DEFINITION 8: A vector function $f : X \to R^p$ locally Lipschitz at $u \in X$, is said to be (V, ρ) pseudoinvex at u if there exist functions $\eta, \psi : X \times X \to R^n$, a real number ρ and $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ such that for all $x \in X$

$$\sum_{i=1}^{p} \xi_i^T \eta(x, u) + \rho \| \psi(x, u) \|^2 \geq 0$$

$$\Rightarrow \quad \sum_{i=1}^{p} \phi_i(x, u)\, f_i(x) \geq \sum_{i=1}^{p} \phi_i(x, u)\, f_i(u)$$

for every $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$.

If

(8a) $\rho > 0$, then the function is strongly V-pseudoinvex at u

(8b) $\rho = 0$ then the function is V-pseudoinvex at u

(8c) $\rho < 0$, then the function is weakly V-pseudoinvex at u

(8d) $\forall x \in X, \quad x \neq u$

$$\sum_{i=1}^{p} \xi_i^T \eta(x, u) \geq -\rho \| \psi(x, u) \|^2$$

$$\Rightarrow \sum_{i=1}^{p} \phi_i(x, u)\, f_i(x) > \sum_{i=1}^{p} \phi_i(x, u)\, f_i(u)$$

for every $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$ then the function is strictly (V, ρ) pseudoinvex at u.

DEFINITION 9: A vector function $f : X \to R^p$, locally Lipschitz at $u \in X$, is said to be (V, ρ) quasiinvex at u if there exist functions $\eta, \psi : X \times X \to R^n$, a real number ρ, $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ such that for all $x \in X$

$$\sum_{i=1}^{p} \phi_i(x, u) f_i(x) \leq \sum_{i=1}^{p} \phi_i(x, u) f_i(u)$$

$$\Rightarrow \quad \sum_{i=1}^{p} \xi_i^T \eta(x, u) \leq -\rho \| \psi(x, u) \|^2$$

for every $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$

(9a) $\rho > 0$ then the function is strongly V-quasiinvex at u

(9b) $\rho = 0$ then the function is V-quasiinvex at u

(9c) $\rho < 0$ then the function is weakly V-quasiinvex at u

If f is (V, ρ) invex at each $u \in X$ then the function is (V, ρ) invex on X. Similar is the definition of other functions.

It is evident that every (V, ρ) invex function is both (V, ρ) pseudoinvex and (V, ρ) quasiinvex with $\theta_i = 1/\phi_i$ and

$$\sum_{i=1}^{p} \phi_i(x, u) = 1.$$

From the definitions it is clear that every strictly (V, ρ)-pseudoinvex function on X is (V, ρ)-quasiinvex on X.

Example 1: Let f_1 and f_2 be real valued functions defined on an interval $X_0 = [-1, 1]$ as follows:

$$f_1(x) = \begin{cases} -6x^2 & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases} \text{ and } f_2(x) = \begin{cases} 7x^2 + 9x^6 & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases}$$

Here,

$$\partial f_1(0) = \partial f_2(0) = \{\xi : 0 \leq \xi \leq 1\}$$

Define

$$\eta : X_0 \times X_0 \to R \quad \text{as}$$

$$\eta(x, u) = 1 - 2x^2 + u$$

$$\psi : X_0 \times X_0 \to R \quad \text{as}$$

$$\psi(x, u) = \sqrt{1 - 2(x^2 + u^2)}$$

$$\phi_1 : X_0 \times X_0 \to R \quad \text{as}$$

$$\phi_1(x, u) = x^2 + 1$$

and

$$\phi_2 : X_0 \times X_0 \to R \quad \text{as}$$

$$\phi_2(x, u) = u^2 + 1.$$

For $\rho = 1$, the vector function $f(x) = [f_1(x), f_2(x)]$ is (V, ρ) pseudoinvex at $u = 0$ but not (V, ρ) quasiinvex as at $u = 0$ and $x = -\sqrt{1/3}$.

$$\phi_1(x, u) f_1(x) + \phi_2(x, u) f_2(x) = \phi_1(x, u) f_1(u) + \phi_2(x, u) f_2(u)$$

but

$$(\xi_1 + \xi_2)\, \eta(x, u) + \rho \,\|\, \psi(x, u) \,\|^2 > 0$$

for every $\xi_1 \in \partial f_1(0)$ and $\xi_2 \in \partial f_2(0)$

Hence one can say that there exist non differentiable functions which are (V, ρ) pseudoinvex but not (V, ρ) quasiinvex.

Example 2: Let f_1 and f_2 be real valued functions defined on an interval $X_0 = (-1, 1)$ as follows:

$$f_1(x) = \begin{cases} x^2 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} -3x^2 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

Here,

$$\partial f_1(0) = \partial f_2(0) = \{\xi : 0 \le \xi \le 1\}$$

Define

$$\eta : X_0 \times X_0 \rightarrow R \quad \text{as}$$

$$\eta(x, u) = x^2 - 1 + u$$

$$\psi : X_0 \times X_0 \rightarrow R \quad \text{as}$$

$$\psi(x, u) = \sqrt{x^2 - 1 - u^2}$$

$$\phi_1 : X_0 \times X_0 \rightarrow R^+ \backslash \{0\} \quad \text{as}$$

$$\phi_1(x, u) = x^2 + 1$$

and

$$\phi_2 : X_0 \times X_0 \rightarrow R^+ \backslash \{0\} \quad \text{as}$$

$$\phi_2(x, u) = u^2 + 1.$$

For $\rho = 1$, the function is (V, ρ) quasiinvex at $u = 0$ but f is not (V, ρ) pseudoinvex as at $u = 0$, $x = -1$

$$\xi \eta(x, u) + \rho \| \psi(x, u) \|^2 = 0 \text{ for every } \xi_1 \in \partial f_1(0) \text{ and } \xi_2 \in \partial f_2(0)$$

but

$$\phi_1(x, u) f_1(x) + \phi_2(x, u) f_2(x) < \phi_1(x, u) f_1(u) + \phi_2(x, u) f_2(u)$$

Thus there exists a class of non differentiable functions which are (V, ρ) quasiinvex but not (V, ρ) pseudoinvex.

LEMMA 1: [3] *$\overline{x}$ is an efficient solution for (VOP) is and only if $\overline{x}$ solves*

$P_r(\overline{x})$ *Minimize* $f_r(x)$

Subject to

$$f_i(x) \leq f_i(\overline{x}) \quad i \neq r, \quad i = 1, 2, \ldots, p$$
$$g_j(x) \leq 0 \qquad j = 1, 2, \ldots, m$$

for each $r = 1, 2, \ldots, p$.

The following scalar optimization problem:

(P1) *Minimize* $p(x)$

Subject to

$$g_j(x) \leq 0, \quad j = 1, 2, \ldots, m$$

where

$$p : R^n \to R \quad g_j : R^n \to R \quad j = 1, 2, \ldots, m$$

are locally Lipschitz around $\overline{x}$ *and regular at* $\overline{x}$, *for* $s \in R^m$ *is associated to the following problem:*

(P2) *Minimize* $p(x)$

Subject to

$$g_j(x) \leq s_j, \quad j = 1, 2, \ldots, m$$

by using the following definition:

DEFINITION 13: [12] Problem (P1) is said to be calm at $\overline{x} \in R^n$ if for all sequences $x^k \to \overline{x}$ with $s^k \to 0$ such that x^k is feasible for (P2) with $s = s^k$, we have

$$\frac{p(\overline{x}) - p(x^k)}{\| s^k \|} \leq M \quad \text{for some constant } M.$$

Noting again that if $\overline{x}$ is an efficient solution of (VOP), then by Lemma 1, $\overline{x}$ solves $P_i(\overline{x})$ for all $i \in P$, the following result holds:

THEOREM 1: (Necessary Conditions) [4, Proposition 6.4.4]. *If* $P_i(\overline{x})$ *is calm at* $\overline{x}$ *for at least one* i, *say* $i = r$ *then* $\exists\ \hat{\lambda}_i \in R_+$, $i = 1, 2, \ldots, p$, $i \neq r$ $\hat{y} \in R^m_+$ *such that*

$$0 \in \partial f_r(\overline{x}) + \sum_{\substack{i=1 \\ i \neq r}}^{p} \hat{\lambda}_i\, \partial f_i(\overline{x}) + \sum_{j=1}^{m} \hat{y}_j\, \partial g_j(x)$$

$$\hat{y}_j\, g_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m$$

where $i \in P$, $P = \{1, 2, \ldots, p\}$.

3. WOLFE VECTOR DUALITY

In this section we obtain weak and strong duality relations between (VOP) and the following Wolfe vector dual:

(D_1VOP) Maximize $[f_1(u) + y^T g(u), \ldots, f_p(u) + y^T g(u)]$

Subject to

$$0 \in \sum_{i=1}^{p} \lambda_i \, \partial f_i(u) + \sum_{j=1}^{m} y_j \, \partial g_j(u) \tag{3}$$

$$\lambda^T e = 1 \tag{4}$$

$$y \geq 0, \quad \lambda \geq 0 \tag{5}$$

$$\lambda \in R^p, \quad y \in R^m$$

and where $e = (1, 1, \ldots, 1) \in R^p$.

THEOREM 2: (Weak Duality) *For all feasible x for (VOP) and all feasible (u, λ, y) for $(D_1 VOP)$, if any one of the following holds with $\rho \geq 0$.*

(*a*) $[\lambda_1 f_1(\cdot), \lambda_2 f_2(\cdot), \ldots, \lambda_p f_p(\cdot)]$ *and*

$$[\lambda_1 y^T g(\cdot), \lambda_2 y^T g(\cdot), \ldots, \lambda_p y^T g(\cdot)]$$

are (V, ρ)-pseudoinvex at u for common η, $\psi : X \times X \to R^n$ and $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ and $\lambda_i > 0$, $i = 1, 2, \ldots, p$.

(*b*) $[\lambda_1 f_1(\cdot), \lambda_2 f_2(\cdot), \ldots, \lambda_p f_p(\cdot)]$ *and*

$$[\lambda_1 y^T g(\cdot), \lambda_2 y^T g(\cdot), \ldots, \lambda_p y^T g(\cdot)]$$

are strictly (V, ρ)-quasiinvex at u for common η, $\psi : X \times X \to R^n$ and $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ and $\lambda_i > 0$, $i = 1, 2, \ldots, p$ then the following cannot hold:

$$f_i(x) \leq f_i(u) + y^T g(u) \quad \textit{for all } i \in P, \quad i \neq r \tag{6}$$

$$f_r(x) < f_r(u) + y^T g(u) \quad \textit{for some } r \in P. \tag{7}$$

Proof: Since (u, λ, y) is feasible for $(D_1\ VOP)$ therefore from (3), we have

$$0 \in \sum_{i=1}^{p} \lambda_i \, \partial f_i(u) + \sum_{j=1}^{m} y_j \, \partial g_j(u)$$

$$\Rightarrow \quad 0 = \sum_{i=1}^{p} \lambda_i \, \xi_i + \sum_{j=1}^{m} y_j \, \beta_j \tag{8}$$

where $\xi_i \in \partial f_i(u)$, $i \in P$ and $\beta_j \in \partial g_j(u)$, $j = 1, 2, \ldots, m$.

Using vector notation (8) can be rewritten as

$$0 = \lambda^T \xi + y^T \beta. \tag{9}$$

Now, contrary to the result of the theorem, let (6) and (7) hold.

As x is feasible for (VOP) and $y \geq 0$, (6) and (7) imply.

$$f_i(x) + y^T g(x) \leq f_i(u) + y^T g(u) \quad \forall i \in P, \quad i \neq r \tag{10}$$

and

$$f_r(x) + y^T g(x) < f_r(u) + y^T g(u) \quad \text{for some } r \in P \tag{11}$$

Now from (10) and (11), in case hypothesis (*a*) holds, there exists a real number ρ, functions η, $\psi : X \times X \to R^n$ and $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ such that for all $x \in X$

$$\sum_{i=1}^{p} \phi_i(x, u) [\lambda_i [f_i(x) + y^T g(x)]] < \sum_{i=1}^{p} \phi_i(x, u) \{\lambda_i [f_i(u) + y^T g(x)]\}$$

$$\Rightarrow \quad \left\{ \sum_{i=1}^{p} \lambda_i [\xi_i + y^T \beta] \right\}^T \eta(x, u) < -2\rho \, \| \psi(x, u) \|^2 \tag{12}$$

for $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$ and $\beta_j \in \partial g_j(u)$, $j = 1, 2, \ldots, m$ using $\lambda^T e = 1$, (12) can be rewritten as

$$(\lambda^T \xi + y^T \beta)^T \eta(x, u) < -2\rho \, \| \psi(x, u) \|^2 \tag{13}$$

As $\rho \geq 0$, using it in (13)

$$(\lambda^T \xi + y^T \beta)^T \eta(x, u) < 0$$

a contradiction to (9).

Again from (10) and (11), when hypothesis (*b*) holds, there exist a real number ρ, functions $\eta, \psi : X \times X \to R^n$ and $\phi_i : X \times X \to R^+ \backslash \{0\}$, $i = 1, 2, \ldots, p$ such that for all $x \in X$

$$\sum_{i=1}^{p} \phi_i(x, u)[\lambda_i[f_i(x) + y^T g(x)]] \leq \sum_{i=1}^{p} \phi_i(x, u)\{\lambda_i[f_i(u) + y^T g(x)]\}$$

$$\Rightarrow \quad \left\{\sum_{i=1}^{p} \lambda_i[\xi_i + y^T \beta]\right\}^T \eta(x, u) < -2\rho\,|\psi(x, u)|^2 \qquad (14)$$

for $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$ and $\beta_j \in \partial g_j(u)$, $j = 1, 2, \ldots, m$. Again using $\lambda^T e = 1$ and $\rho \geq 0$ relation (14) can be rewritten as

$$(\lambda^T \xi + y^T \beta)^T \eta(x, u) < 0 \qquad (15)$$

a contradiction to (9).

Hence the proof of the theorem is complete.

COROLLARY 1: *Let* $(\overline{u}, \overline{\lambda}, \overline{y})$ *be a feasible solution for* $(D_1 VOP)$ *such that* $\overline{y}^T g(\overline{u}) = 0$ *and assume that* $\overline{u}$ *is feasible* (VOP). *If the weak duality theorem holds between* (VOP) *and* $(D_1 VOP)$ *then* $\bar{u}$ *is efficient for* (VOP) *and* $(\overline{u}, \overline{\lambda}, \overline{y})$ *is efficient for* $(D_1 VOP)$.

THEOREM 3: *(Strong Duality). Let* $\overline{x}$ *be a feasible solution for* (VOP) *and assume that*

(i) $\overline{x}$ *is an efficient solution for* (VOP).

(ii) *for at least one* $i \in P$, *problem* $P_i(\overline{x})$ *is calm at* $\overline{x}$ *then there exist* $\overline{\lambda} \in R_+^p$, $\overline{y} \in R_+^m$ *such that* $(\overline{x}, \overline{\lambda}, \overline{y})$ *is feasible for* $(D_1 VOP)$. *and* $\overline{y}^T g(\overline{x}) = 0$.

Further if weak duality theorem 2 holds between (VOP) and $(D_1 VOP)$. then $(\overline{x}, \overline{\lambda}, \overline{y})$ is efficient for $(D_1 VOP)$.

Proof: Since $\overline{x}$ is efficient for (VOP) from Lemma 1, $\overline{x}$ solves $P_i(\overline{x})$ is calm at $\overline{x}$ for at least one i, say for $i = r$, it therefore follows from Theorem 1 that there exists $\hat{\lambda}_i \in R_+$, $i \in P$, $i \neq r$ $\hat{y} \in R^m_+$ such that

$$0 \in \partial f_r(\overline{x}) + \sum_{i=1}^{p} \hat{\lambda}_i \, \partial f_i(\overline{x}) + \sum_{j=1}^{m} \hat{y}_j \, \partial g_j(\overline{x})$$

$$\hat{y}_j \, g_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m.$$

Set

$$\overline{\lambda}_i = \frac{\hat{\lambda}_i}{1 + \sum\limits_{\substack{i=1 \\ i \neq r}}^{p} \hat{\lambda}_i}$$

$$\overline{\lambda}_r = \frac{1}{1 + \sum\limits_{\substack{i=1 \\ i \neq r}}^{p} \hat{\lambda}_i} \quad \text{for all } j = 1, 2, \ldots, m$$

It follows that $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible to $(D_1 VOP)$ and $\overline{y}^T g(\overline{x}) = 0$.

Efficiency of $(\overline{x}, \overline{\lambda}, \overline{y})$ for $(D_1 VOP)$ follows from Corollary 1.

4. MOND-WEIR VECTOR DUALITY

In this section duality results are established between (VOP) and the following Mond-Weir dual of the problem (VOP):

$(D_2 VOP)$ Maximize $[f_1(u), \ldots, f_p(u)]$

Subject to

$$0 \in \sum_{i=1}^{p} \lambda_i \, \partial f_i(u) + \sum_{j=1}^{m} y_j \, \partial g_j(u) \tag{16}$$

$$\lambda^T g(u) \geq 0 \tag{17}$$

$$\lambda^T e = 1 \tag{18}$$

$$y \geq 0 \quad \lambda \geq 0 \tag{19}$$

THEOREM 4: (Weak Duality). *Let x be feasible for (VOP) (u, λ, y) be feasible for (D_2VOP) and $(y_1g_1, \ldots, y_mg_m)$ is (V, ρ) quasiinvex at u with respect to η, ψ, with $\rho \geq 0$ and if any one of the following holds.*

(i) $(\lambda_1 f_1, \ldots, \lambda_p f_p)$ is strictly (V, ρ')-pseudoinvex at u with respect to same η, ψ and $\lambda_i > 0$, $i = 1, 2, \ldots, p$ and $\rho' \geq 0$.

(ii) $(\lambda_1 f_1, \ldots, \lambda_p f_p)$ is (V, ρ') pseudoinvex at u with respect to same η, ψ, and $\lambda_i > 0$, $i = 1, 2, \ldots, p$ and $\rho' \geq 0$

then the following cannot hold:

$$f_i(x) \leq f_i(u) \quad \text{for all } i \in P, \quad i \neq r \tag{20}$$

$$f_r(x) < f_r(u) \quad \text{for some } r \in P. \tag{21}$$

Proof: Since x is feasible for (VOP) and (u, λ, y) is feasible for $(D_2 VOP)$ therefore from (16)

$$\Rightarrow \quad 0 = \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} y_j \beta_j \tag{22}$$

where

$$\xi_i \in \partial f_i(u) \quad i = 1, 2, \ldots, p$$
$$\beta_j \in \partial g_j(u) \quad j = 1, 2, \ldots, m.$$

Also

$$\begin{aligned} g_j(x) \leq 0 \quad \text{and as} \quad y_j \geq 0 \quad j = 1, 2, \ldots, m \\ y_j g_j(x) \leq 0 \end{aligned} \tag{23}$$

Using (17) and (23) we get

$$y_j g_j(x) \leq y_j g_j(u) \quad j = 1, 2, \ldots, m \tag{24}$$

Now as $(y_j g_j, \ldots, y_m g_m)$ is (V, ρ) quasiinvex at u with respect to η, ψ there exists a real number ρ, η, $\psi : X \times X \to R^n$ and $\phi_j : X \times X \to R^+ \backslash \{0\}$ such that for all $x \in X$

$$\sum_{j=1}^{m} \phi_j y_j g_j(x) \leq \sum_{j=1}^{m} \phi_j y_j g_j(u)$$

$$\Rightarrow \quad \sum_{j=1}^{m} (y_j \beta_j)^T \eta(x, u) \leq -\rho \| \psi(x, u) \|^2 \tag{25}$$

for $\beta_j \in \partial g_j(u)$ $j = 1, 2, \ldots, m$. As $\rho \geq 0$. Using it in (25)

$$\sum_{j=1}^{m} (y_j, \beta_j)^T \eta(x, u) \leq 0 \tag{26}$$

For (22) and (26) implies that

$$\sum_{i=1}^{p} (\lambda_i, \xi_i)^T \eta(x, u) \geq 0 \tag{27}$$

Now contrary to the results of the theorem, let (20) and (21) hold

From (20) and (21) and $\lambda_i \geq 0$, in case (a) holds, there exist functions $\eta, \psi : X \times X \to R^n$, a real number ρ' and $\delta_i : X \times X \to R^+\backslash\{0\}$ such that for all $x \in X$

$$\sum_{i=1}^{p} \delta_i(x, u) \lambda_i f_i(x) \leq \sum_{i=1}^{p} \delta_i(x, u) \lambda_i f_i(u)$$

$$\Rightarrow \quad \sum_{i=1}^{p} (\lambda_i \xi_i)^T \eta(x, u) < -\rho' \| \psi(x, u) \|^2 \tag{28}$$

for $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$ which implies

$$\sum (\lambda_i \xi_i)^T \eta(x, u) < 0 \quad (\text{as } \rho' \geq 0)$$

a contradiction to (27).

Again, from (20) and (21) in case hypothesis (b) holds, there exist functions $\eta, \psi : X \times X \to R^n$, a real number ρ' and $\delta_i : X \times X \to R^+\backslash\{0\}$ such that for all $x \in X$

$$\sum_{i=1}^{p} \delta_i(x, u) \lambda_i f_i(x) < \sum_{i=1}^{p} \delta_i(x, u) \lambda_i f_i(u)$$

$$\Rightarrow \quad \sum_{i=1}^{p} (\lambda_i \xi_i)^T \eta(x, u) < -\rho' \| \psi(x, u) \|^2$$

for $\xi_i \in \partial f_i(u)$, $i = 1, 2, \ldots, p$.

Again we have

$$\sum_{i=1}^{p} (\lambda_i \xi_i)^T \eta(x, u) < 0 \quad (\text{as } \rho' \geq 0)$$

a contradiction to (27).

This completes the proof.

COROLLARY 2: *Assume weak duality holds between* (VOP) *and* $(D_2 VOP)$. *If* $(\overline{x}, \overline{\lambda}, \overline{y})$ *is feasible to* $(D_2 VOP)$ *such that* $\overline{u}$ *is feasible for* (VOP) *then* $\overline{u}$ *is efficient for* (VOP) *and* $(\overline{u}, \overline{\lambda}, \overline{y})$ *is efficient for* $(D_2 VOP)$.

THEOREM 5: (Strong Duality). *Let* $\overline{x}$ *be feasible for* (VOP) *and assume*

(a) $\overline{x}$ *is efficient for* (VOP)

(b) for at least one $i \in P$, *problem* $P_i(x)$ *is calm at* $\overline{x}$ *then there exist* $\overline{\lambda} \in R_+^p$, $\overline{y} \in R_+^m$ *such that* $(\overline{x}, \overline{\lambda}, \overline{y})$ *is feasible for* $(D_2 VOP)$.

Further if also weak duality theorem 4 holds between (VOP) and $(D_2 VOP)$ then $(\overline{x}, \overline{\lambda}, \overline{y})$ is efficient for $(D_2 VOP)$.

Proof: The proof runs on the lines as that of theorem 3 and is hence omitted.

ACKNOWLEDGEMENT

The authors wish to express their deep gratitude to Prof. R. N. Kaul for his inspiring guidance throughout the preparation of the paper. The authors are grateful to the unknown referees for their comments.

REFERENCES

1. D. BHATIA and P. JAIN, Generalized (F, ρ)-Convexity and Duality for Non-smooth Multiobjective Programming, *Optimization,* 1994, *31*, pp. 153-164.
2. C. R. BECTOR, S. CHANDRA and V. KUMAR, Duality for Minmax Programming Involving V-Invex Functions, *Optimization,* 1994, *30*, pp. 93-103.
3. V. CHANKONG and Y. Y. HAIMES, Multiobjective Decision Making, Theory and Methodology, *North-Holland,* New York.
4. F. H. CLARKE, Optimization and Non-smooth Analysis, Wiley-Interscience, New York, *Numerical Analysis and Application Sciences*, 1983, pp. 514-550.
5. R. R. EGUDO, Efficiency and Generalized Convex Duality for Multiobjective Programs, *Journal of Mathematical Analysis and Applications,* 1989, *138*, pp. 184-194.
6. M. M. HANSON, On Sufficiency of Kuhn-Tucker Conditions, *Journal of Mathematical Analysis and Applications,* 1981, *80*, pp. 544-550.

7. M. A. Hanson and B. Mond, Furhter Generalization of Convexity in Mathematical Programming, *Journal Information and Optimization Science,* 1982, *4*, pp. 25-32.
8. V. Jeyakumar, Strong and Weak Invexity in Mathematical Programming, *Method Oper. Res.,* 1985, *55,* pp. 109-125.
9. V. Jeyakumar, Equivalence of Saddle Points and Optima, and Duality for a Class of Non-convex Problems, *Journal of Mathematical Analysis and Application,* 1988, *130,* pp. 334-343.
10. V. Jeyakumar and B. Mond, On Generalized Convex Mathematical Programming, *Journal of Austral. Math. Soc. (Ser. B),* 1992, *34*, pp. 43-53.
11. V. Preda, On Efficiency and Duality for Multiobjective Programs, *Journal of Mathematical Analysis and Application,* 1992, *166*, pp. 365-377.
12. Y. Tanaka M. Fukushima and T. Ibaraki, On Generalized Pseudo Convex Functions, *Journal of Math. Analysis and Application,* 1989, *144*, pp. 342-355.
13. P. Wolfe, A Duality Theorem for Nonlinear Programming, *Quarterly Application Math.,* 1961, *19*, pp. 239-244.