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OPTIMAL NUMBER OF SPARE CASHBOXES FOR UNMANNED BANK ATMS (*)

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Abstract. – *There are many unmaned ATMs (Automatic Tellers Machines) where customers can withdraw even on holidays. When all the cash in the ATM has been drawn out, the cashbox is replaced by the new spare one. This paper proposes a stochastic model of ATM and considers the problem how many number of cashboxes should be provided. Using the theory of probability, the expected total cost is derived and an optimal number which minimizes it is discussed. Finally, numerical examples are given when the distribution function of the total cash drawn a day is exponential and normal.* © Elsevier, Paris

Keywords: ATM of bank, Spare cashboxes, Expected cost, Optimal number.

Résumé. – *UN DAB (Distributeur Automatique de Billets) est une machine automatique d'où, en l'absence de tout employé, les clients peuvent à tout moment retirer des billets de banque. Quand tout l'argent du DAB est épuisé, la boîte à billets est remplacée par une nouvelle boîte tenue en réserve. Nous proposons dans cet article un modèle stochastique de DAB, et considérons la question de savoir combien de boîtes à billets doivent être fournies pour la réserve. Utilisant la théorie des probabilités, nous donnons une formule pour l'espérance du coût et nous examinons le nombre optimal qui la minimise. Finalement, nous donnons des exemples numériques lorsque la fonction de distribution de l'argent total retiré dans la journée est exponentielle ou normale.* © Elsevier, Paris

Mots clés : DAB bancaire, boîtes à billets en réserve, espérance de coût, nombre optimal.

1. INTRODUCTION

Banks offer many kinds of services for customers. Recently, most banks have set up many unmanned ATMs where customers can deposit or withdraw money even on Saturdays, Sundays and holidays. There are some small cashboxes which hold cash in an ATM.

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There are usually two different types of boxes in an ATM. One is a box only for deposit and the other is a box only for withdrawing. According to customers' demand, a constant amount of cash is kept beforehand in each cashbox. If all the cash in a box has been drawn out, the operation of the ATM is stopped. Then, the service of the ATM is restarted by replacing an empty box with a new one.

When a cashbox becomes empty on weekdays, a banker can usually replenish it with cash. However, when all the cash in an ATM has been drawn out on holidays, a guard company receives the information from the control center which continuously monitores the ATMs, and replaces it quickly with a new one. Thereafter, the service for drawing cash begins again. This company provides in advance some new spare cashboxes for such situations. Such replacements may be repeated during a day.

One important problem arising in the above situation is how many spare casboxes per each branch have been previously provided to the guard company. As one method of answering this question, we introduce some costs and propose a stochastic model. This is one modification of discrete replacement models [1, 2]. We derive an expected cost, using the theories of probability, and determine an optimal number of spare cashboxes which minimizes it.

Finally, we give numerical examples and compute the optimal number under suitable conditions.

2. EXPECTED COST

We examine an ATM in the branch of Bank N. In this paper, we consider the model with only one ATM. However, in the case where there are several ATMs in the branch, when all the cash of the boxes has been withdrawn from all ATMs, the company replaces them with only one new box. Thus, we use the word "ATM" in this paper without "ATMs".

The costs might be mainly incurred in the following three cases for the continuous operation of the ATM:

1) The cash in spare cashboxes is surplus funds and brings no profit if it is not used.

2) When all the cash in the ATM has been drawn out, customers would draw cash from other ATMs of Bank N or other banks. In this case, if customers would use an ATM of other banks, not only they have to pay the extra commission, but also Bank N has to allow some commission to other

banks. Conversely, if customers of other banks use this ATM, Bank N can receive a commission from both customers and other banks.

3) Bank N has to pay a fixed contract deposit, and also, pay a constant commission to the guard company, whenever the company delivers a spare cashbox and exchanges it to an empty one.

From the above viewpoints 1), 2) and 3), we introduce the following three costs: All the cash provided in spare cashboxes incurs an opportunity cost c_1 per unit of cash, and when customers use other banks, this incurs a cost c_2 per unit of cash. Further, a cost c_3 is needed for each exchange of one box, whenever the guard company delivers a spare cashbox.

It is assumed that A is the first amount of cash which is stored in the ATM of the branch, and a is the amount of cash which is stored in one cashbox. Evidently, if there are j ATMs in the branch, then we have that $A = ja$ $(j = 1, 2, \ldots)$.

Next, we define that $F(x)$ is the distribution function of the total amount of cash which is drawn a day from the ATM in the branch, and its *mean* is $\mu = \int_0^\infty \overline{F}(x)\,dx$. Let $\alpha\,(0 \le \alpha < 1)$ be the rate that customers of other banks have drawn cash from the ATM in the branch, and $\beta\,(0 \le \beta < 1)$ be the probability that customers of Bank N give up drawing cash or use other ATMs of Bank N, *i.e.*, $1 - \beta$ is the probability that they use the ATM of other banks, when all ATMs in the branch were stopped.

Suppose that n is the number of spare cashboxes for the ATM in the branch. Then, the total cost required for providing n spare cashboxes is

$$c_1\, na. \tag{1}$$

The cost for the case where customers use the ATM of other banks, when the ATM in the branch was stopped, is

$$c_2\,(1-\alpha)\,(1-\beta) \int_{A+na}^{\infty} (x - A - na)\, dF(x). \tag{2}$$

Conversely, the profit that customers of other banks use the ATM in the branch is

$$\begin{aligned} &- c_2\,\alpha \left[(A + na)\,\overline{F}(A + na) + \int_0^{A+na} x dF(x) \right] \\ &= -c_2\,\alpha \int_0^{A+na} \overline{F}(x)\,dx, \end{aligned} \tag{3}$$

where $\overline{F} \equiv 1 - F$. This also includes the commission that customers pay to the branch. The above formulations of (1), (2) and (3) are cost functions similar to a "Newspaper sellers' problem" [3-7].

Further, the total cost required for the guard company which delivers spare cashboxes is

$$c_3 \left[\sum_{i=0}^{n-1} (i+1) \int_{A+ia}^{A+(i+1)\,a} dF\,(x) + n \int_{A+na}^{\infty} dF\,(x) \right] = c_3 \sum_{i=0}^{n-1} \overline{F}\,(A + ia), \tag{4}$$

where $\sum_{i=0}^{-1} \equiv 0$.

Summing up the equations (1), (2), (3) and (4), and arranging it, the expected total cost $C\,(n)$ is given by

$$C\,(n) = nc_1\,a + c_2 \left\{ [1 - (1-\alpha)\,\beta] \int_{A+na}^{\infty} \overline{F}\,(x)\,dx - \alpha\mu \right\} + c_3 \sum_{i=0}^{n-1} \overline{F}\,(A+ia)\;(n = 0,\,1,\,2, \ldots). \tag{5}$$

3. OPTIMAL NUMBER

We find an optimal number n^* of spare cashboxes which minimizes $C\,(n)$ in (5). It is seen that

$$C\,(0) = c_2 \left\{ [1 - (1-\alpha)\,\beta] \int_{A}^{\infty} \overline{F}\,(x)\,dx - \alpha\mu \right\}, \tag{6}$$

$$C\,(\infty) \equiv \lim_{n \to \infty} C\,(n) = \infty. \tag{7}$$

Thus, there exists at least one finite n^* which minimizes $C\,(n)$.

Forming the inequality $C\,(n+1) - C\,(n) \geq 0$ to seek an optimal number n^*, we have

$$\frac{c_1\,a + c_3\,\overline{F}\,(A+na)}{\displaystyle\int_{A+na}^{A+(n+1)\,a} \overline{F}\,(x)\,dx} \geq c_2\,\gamma, \tag{8}$$

where $\gamma \equiv 1 - (1-\alpha)\,\beta$.

Assume that a density $f(x)$ of the distribution $F(x)$ exists, *i.e.*, $f(x) \equiv dF(x)/dx$. Let denote the left-hand side of (8) by $L(y)$ where $y \equiv A + na$, and investigate the properties of $L(y)$. From (8),

$$L(y) = \frac{c_1 a + c_3 \overline{F}(y)}{\int_y^{y+a} \overline{F}(x)\, dx} \qquad (y \geq A). \tag{9}$$

Evidently,

$$L(A) = \frac{c_1 a + c_3 \overline{F}(A)}{\int_A^{A+a} \overline{F}(x)\, dx}, \tag{10}$$

$$L(\infty) \equiv \lim_{y \to \infty} L(y) = \infty, \tag{11}$$

$$L'(y) = \frac{F(y+a) - F(y)}{\int_y^{y+a} \overline{F}(x)\, dx} \left[\frac{c_1 a + c_3 \overline{F}(y)}{\int_y^{y+a} \overline{F}(x)\, dx} - \frac{c_3 f(y)}{F(y+a) - F(y)} \right]. \tag{12}$$

If $F(x)$ has the property of IFR [1], then

$$\frac{f(x)}{\overline{F}(x)} \geq \frac{f(y)}{\overline{F}(y)} \qquad (y \leq x \leq y + a). \tag{13}$$

Hence, we have

$$\begin{aligned} \frac{c_1 a + c_3 \overline{F}(y)}{\int_y^{y+a} \overline{F}(x)\, dx} &\geq \frac{c_1 a + c_3 \overline{F}(y)}{\int_y^{y+a} f(x) \frac{\overline{F}(y)}{f(y)}\, dx} \\ &= \frac{\left[\frac{c_1 a}{\overline{F}(y)} + c_3 \right] f(y)}{F(y+a) - F(y)} > \frac{c_3 f(y)}{F(y+a) - F(y)}. \end{aligned} \tag{14}$$

Thus, if $F(x)$ is IFR then $L(y)$ is strictly increasing from $L(A)$ in (10) to infinity.

Therefore, we can give the following optimal number n^* when $F(x)$ is IFR:

(i) If $L(A) < c_2 \gamma$ then there exists a unique minimum n^* $(n^* \geq 1)$ which satisfies (8).

(ii) If $L(A) \geq c_2 \gamma$ then $n^* = 0$, *i.e.*, we should provide no spare cashbox.

We can explain the reason why the distribution $F(x)$ has the property of IFR in reliability theory: The total amount of cash on ordinary days is almost constant, however, a lot of money is drawn out at the end of the month just after most workers have received their salaries. It is well-known that its amount is about 1.75 times more than that of ordinary days. Thus, we might consider that the drawing-rate of cash is constant or increases with the total amount of cash. It is noted that exponential and normal distributions given in numerical examples have the property of IFR.

4. NUMERICAL EXAMPLES

We consider two cases where distribution function $F(x)$ is exponential and normal.

4.1. Exponential case

Suppose that $F(x) = 1 - e^{-\lambda x}$ $(x \geq 0)$. Since $f(x)/\overline{F}(x) \equiv \lambda$ for $x \geq 0$, it follows that $F(x)$ is IFR [1]. Then, equations (5) and (8) are rewritten as, respectively:

$$\begin{aligned} C(n) =& n c_1 a + \frac{c_2}{\lambda} [\gamma e^{-\lambda (A + n a)} - \alpha] \\ &+ c_3 e^{-\lambda a} \frac{1 - e^{-n \lambda a}}{1 - e^{-\lambda a}}, \end{aligned} \tag{15}$$

$$c_1 a e^{\lambda (A + n a)} + c_3 \geq c_2 \gamma \frac{1 - e^{-\lambda a}}{\lambda}. \tag{16}$$

Therefore, the optimal policies are:

(i) If $c_1 a e^{\lambda A} + c_3 < c_2 \gamma \frac{1 - e^{-\lambda a}}{\lambda}$ then there exists a unique minimum n^* which satisfies (16).

(ii) If $c_1 a e^{\lambda A} + c_3 \geq c_2 \gamma \frac{1 - e^{-\lambda a}}{\lambda}$ then $n^* = 0$.

In case of (i), solving (16) with respect to n, we get

$$n^* = \left[\frac{1}{\lambda a} \ln \left(\frac{c_2 \gamma}{c_1 a} \cdot \frac{1 - e^{-\lambda a}}{\lambda} - \frac{c_3}{c_1 a}\right) - \frac{A}{a}\right] + 1, \tag{17}$$

where $[x]$ denotes the greatest integer contained in x. Further, it is evidently seen from (16) or (17) that an optimal n^* is increasing with c_2/c_1, and inversely decreasing with c_3/c_1.

In case of (ii), it is noted that if

$$c_1 a (1 + \lambda A) + c_3 \geq c_2 a \gamma \tag{18}$$

then $n^* = 0$.

TABLE I

Optimal number n^* and expected total cost $C(n^*)$ for c_1 and c_2 when $A = a = 20.0$, $\alpha = 0.4$, $\beta = 0.6$, $1/\lambda = 25.0$ and $c_3 = 0.01$.

c_2	c_1			
	0.000068		0.000164	
	n^*	$C(n^*)$	n^*	$C(n^*)$
0.001	0	–0.00325	0	–0.00325
0.002	1	–0.00753	0	–0.00650
0.003	2	–0.01456	1	–0.01194
0.004	2	–0.02205	1	–0.01828
0.005	3	–0.02983	2	–0.02572

It is assumed that $A = a = 20.0$ where a denominator of money is 1 million yen $\fallingdotseq$ \$9,000. The yield on investment $c_1 = 2.5$ or 6.0 percents per one year, *i.e.*, $c_1 = 0.025/365 = 0.000068$ or $0.06/365 = 0.000164$ per one day, and $c_3 = 10,000$ yen$= 0.01$.

Further suppose that the *mean* of the total cash a day is $1/\lambda = 25.0$. Then, from (18) and (16), we can compute n^* and the resulting cost $C(n^*)$ for given c_2, respectively. Table I shows the computing results for $c_2 = 100 \sim 500$ yen per 100,000 yen, *i.e.*, $c_2 = 0.001 \sim 0.005$. For example, when $c_1 = 2.5\%$ per a year, $c_2 = 300$ yen and $c_3 = 10,000$ yen, $n^* = 2$ and $C(n^*) = -14,560$ yen. In this case, we should provide two spare cashboxes and the branch of Bank N gains a profit of 14,560 yen a day. Further, the probability that the total cash has been drawn out is given by $e^{-(20.0+2\times 20.0)/25.0} \doteqdot 0.0907$.

It is evidently seen that n^* is increasing with c_2, and $C(n^*)$ is negative and decreasing with c_2, because it yields a profit for customers of other banks to use the ATM. If c_1 is large, $C(n^*)$ increases, *i.e.*, the profit of Bank N decreases.

TABLE II
Upper number $\overline{n}$ for ε and $1/\lambda$ when $A = a = 20.0$.

ϵ	$1/\lambda$	
	25.0	30.0
	$\overline{n}$	$\overline{n}$
0.20	2	2
0.10	2	3
0.05	3	4

Moreover, we are also interested in an upper spare number $\overline{n}$ in which the probability that the total cash has been drawn out a day from the ATM is less than or equal to a small ε. This upper number $\overline{n}$ is given by

$$\overline{F}(A+na) = e^{-\lambda(A+na)} \leq \varepsilon. \tag{19}$$

Table II gives the upper number $\overline{n}$ for $1/\lambda = 25.0$, 30.0 and $\varepsilon = 0.20$, 0.10, 0.05 when $A = a = 20.0$. It can be seen from this table that how many number of spare cash boxes should be provided for given ε.

4.2. Normal case

Suppose that $1 - F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_x^\infty e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$. Then, from (8), an optimal number n^* is given by a minimum number which satisfies

$$\frac{c_1 a + \dfrac{c_3}{\sqrt{2\pi}\sigma} \displaystyle\int_{A+na}^{\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt}{\dfrac{1}{\sqrt{2\pi}\sigma} \displaystyle\int_{A+na}^{A+(n+1)a} \int_x^\infty e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt\, dx} \geq c_2 \gamma. \tag{20}$$

Table III gives the optimal number n^* and the resulting cost $C(n^*)$ for $c_1 = 0.000068$, 0.000164 and $c_2 = 0.001 \sim 0.005$ when $A = a = 20.0$, $\alpha = 0.4$, $\beta = 0.6$, $\mu = 25.0$, $\sigma = 20.0$ and $c_3 = 0.01$.

It can be seen that the values of n^* in Table III are greater than those in Table I and the resulting costs $C(n^*)$ are greater when c_2 is small and are less when c_2 is large than those in Table I. However, two tables have the similar tendencies. It should be estimated from actual data which distribution is suitable for the distribution $F(x)$.

We can compute an upper spare number $\overline{n}$ by the method similar to Table II. This upper number $\overline{n}$ is given by

$$\overline{F}(A + na) = \frac{1}{\sqrt{2\pi}\sigma} \int_{A+na}^{\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \le \varepsilon. \tag{21}$$

Table IV gives the upper number $\overline{n}$ for $\mu = 250$, 30.0 and $\varepsilon = 0.20$, 0.10, 0.05 when $A = a = 20.0$ and $\sigma = 20.0$.

TABLE III

Optimal number n^* and expected total cost $C(n^*)$ for c_1 and c_2 when $A = a = 20.0$, $\alpha = 0.4$, $\beta = 0.6$, $\mu = 25.0$, $\sigma = 20.0$ and $c_3 = 0.01$.

c_2	c_1			
	0.000068		0.000164	
	n^*	$C(n^*)$	n^*	$C(n^*)$
0.001	0	0.16163	0	0.16163
0.002	3	–0.00666	3	–0.00090
0.003	3	–0.01637	3	–0.01061
0.004	3	–0.02608	3	–0.02033
0.005	4	–0.03580	3	–0.03004

TABLE IV

Upper number $\overline{n}$ for ε and μ when $A = a = 20.0$ and $\sigma = 20.0$.

ϵ	μ	
	25.0	30.0
	$\overline{n}$	$\overline{n}$
0.20	2	2
0.10	2	2
0.05	2	3

The values of $\overline{n}$ in Table IV are not greater than those in Table II. This would be probably from the reasons that an exponential distribution has a randomness property and its standard deviation is greater than that of a normal distribution.

5. CONCLUSIONS

We have proposed the stochastic model of ATM in Bank N. We have considered the cost provided for spare cashboxes, the loss cost at which customers use ATMs of other banks and the cost required for exchanges of spare cashboxes, and have derived the expected total cost. Using the theory of probability, we have obtained the optimal number of spare cashboxes. Finally, to understand the results easily, we have given the numerical examples.

We have defined the total amount of cash a day as the distribution function $F(x)$ and assumed that $F(x)$ is exponential and normal. In this paper, we did not touch on how to estimate a distribution function and its parameters. In future, it would be necessary to gather sufficient data and to analyze it, using the statistical theory. Further, we need to examine from actual data why $F(x)$ is IFR.

Actually, the operation of an ATM is sometimes stopped by mechanical failure or human error. It would be important to consider such failures and to formulate a stochastic model. Further studies for such cases would be expected.

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