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CONE CONVEXITY OF MEASURED SET VECTOR FUNCTIONS AND VECTOR OPTIMIZATION (*)

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Abstract. – *With the absence of a linear structure on a σ-algebra of sets, an accepted concept of convexity on it is defined through Morris sequences. Some known results on such a convexity for scalar valued set functions are generalized to cone convexity of vector valued measured set functions. Finally these notions are exploited in a study of vector optimization involving the latter functions, a study in which cone-optimal and proper cone-optimal solutions are characterized in a duality setting.*

Keywords: Cone-convex set function, vector optimization, cone-optimal solution, proper cone-optimal solution, primal problem, dual problem.

Résumé. – *En l'absence d'une structure linéaire sur une σ-algèbre d'ensembles, un concept de convexité accepté dessus est défini à travers les suites de Morris. Quelques résultats connus sur une telle convexité de fonctions scalaires d'ensemble mesuré sont généralisés à la cône-convexité de fonctions vectorielles d'ensemble mesuré. Finalement ces notions sont exploitées dans l'étude de l'optimisation vectorielle comportant ces dernières fonctions, étude dans laquelle des solutions cône-optimales et proprement cône-optimales sont caractérisées en situation de dualité.*

Mots clés : Fonction cône-convexe d'ensemble, optimisation vectorielle, solution cône-optimale, solution proprement cône-optimale, problème primal, problème dual.

1. INTRODUCTION

Throughout the paper, $(\Omega, \mathcal{A}, \mu)$ is a measure space where Ω is a non empty set, $\mathcal{A}$ a σ-algebra of subsets of Ω and μ a finite, positive non-null and atomless measure. L_μ the space of real μ-integrable functions on Ω is supposed to be separable. It's topological dual $L_\mu^* = L_\mu^\infty$ is supposed to be equipped with the weak-*-topology to which we simply refer to as the weak topology. By identifying any $A \in \mathcal{A}$ to its indicator function I_A and any subset $\mathcal{S}$ of $\mathcal{A}$ to $I_{\mathcal{S}} = \{I_A : A \in \mathcal{S}\}$, it can be shown (Lemma 3.3

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in [6]) that for any $A \in \mathcal{A}$ and $\alpha \in [0, 1]$, $\alpha I_A \in \overline{\mathcal{A}} \subset L_\mu^\infty$ where $\overline{\mathcal{A}}$ is the weak closure of $\mathcal{A}$ in L_μ^∞. Furthermore, for any A and $B \in \mathcal{A}$, $\alpha \in [0, 1]$ and any two sequences (A_n) and (B_n) in $\mathcal{A}$ such that (s.t.) $\operatorname{Lim} I_{A_n} = \alpha I_{A\backslash B}$ and $\operatorname{Lim} I_{B_n} = (I - \alpha) I_{B\backslash A}$ weakly, $A\backslash B$ and $B\backslash A$ being the set differences, we have $\operatorname{Lim} I_{A_n \cup B_n \cup (A\cap B)} = \alpha I_A + (1 - \alpha) I_B$ weakly (Proposition 3.2 in [6]) and the result holds in the absence of the separability of L_μ. The sequence (Z_n) with $Z_n = A_n \cup B_n \cup (A \cap B)$ is called a Morris sequence associated with $(\alpha, A, B) : (Z_n) \sim (\alpha, A, B)$. Morris in [6] defines a numerical function H on $\mathcal{A}$ as being convex if it is s.t. for any $\alpha \in [0, 1]$, A and $B \in \mathcal{A}$ and $(Z_n) \sim (\alpha, A, B)$, $\operatorname{Lim\,sup} H(Z_n) \leq \alpha H(A) + (1 - \alpha) H(B)$. We recall that because of the separability hypothesis on L_μ, for any $\alpha \in [0, 1]$, A and $B \in \mathcal{A}$ there exists (*a* Morris sequence) $(Z_n) \sim (\alpha, A, B)$ (i.e. $(Z_n) \subset \mathcal{A}$ and $\lim I_{Z_n} = \alpha I_A + (I - \alpha) I_B$ weakly).

Chou, Hsia and Lee in [1] generalized Morris procedure the following way.

Let $\mathcal{S}$ be and remain a non empty subset of $\mathcal{A}$. When $\mathcal{S}$ is identified to $I_\mathcal{S} \subset L_\mu^\infty$, $\overline{\mathcal{S}}$ is its weak closure, $\operatorname{co}(\mathcal{S})$ its (usual) convex hull and, $\overline{\operatorname{co}}(\mathcal{S})$ its weakly closed convex hull. $\mathcal{S}$ (non identified to $I_\mathcal{S}$) is said to be *a* convex subset of $\mathcal{A}$ if for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$ there exists a subsequence (Z_{n_k}) in $\mathcal{S}$. If $\mathcal{S}$ is convex (in the sense just given), a numerical function H on $\mathcal{S}$ is said to be convex if for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$ there exists a subsequence (Z_{n_k}) in $\mathcal{S}$ verifying $\limsup H((Z_{n_k}) \leq \alpha H(A) + (1 - \alpha) H(B)$. Results related to this convexity are given in [1] along with the fact that if $\mathcal{S}$ is convex in the sense above, then identified to $I_\mathcal{S} \subset L_\mu^\infty$, $\overline{\mathcal{S}} = \overline{\operatorname{co}}(\mathcal{S})$ so that $\overline{\mathcal{S}} \subset L_\mu^\infty$ is convex in the usual sense (Proposition 3.5 in [1]). It follows that in L_μ^∞, $\overline{\mathcal{A}} = \{f \in L_\mu^\infty : 0 \leq f \leq 1\}$ (Corollary 3.6 in [1]). It follows again that $\overline{\mathcal{A}}$ is nowhere dense in L_μ^∞ since any non-empty weakly open set in L_μ^∞ is unbounded. Further, since $\overline{\mathcal{A}}$ is weakly compact in L_μ^∞ and L_μ separable, $\overline{\mathcal{A}}$ is metrizable (Remark 3.7 in [1]).

Now for any $f \in \overline{\mathcal{A}}$, if $\mathcal{N}(f)$ is the set of weak neighborhoods of f in $\overline{\mathcal{A}}$ and H is a numerical function on $\mathcal{S}$, then the weak closure or weak lower semi-continuous (resp. weak upper semi-continuous) hull of H is the numerical function $\overline{H}$ (resp. $\hat{H}$) on $\overline{\mathcal{S}}$ s.t. for any $f \in \overline{\mathcal{S}}$,

$$\overline{H}(f) = \sup_{V \in \mathcal{N}(f)} \inf_{A \in V \cap \mathcal{S}} H(A) \,(\text{resp.}\, \hat{H}(f) = \inf_{V \in \mathcal{N}(f)} \sup_{A \in V \cap \mathcal{S}} H(A)).$$

H is weakly lower semi-continuous (resp. weakly upper semi-continuous, weakly continuous) if $H = \overline{H}$ (resp. $H = \hat{H}$, $H = \overline{H} = \hat{H}$) (Definition 3.8 in [1]).

In so doing $\mathcal{S}$ has been identified to $I_{\mathcal{S}} \subset L_\mu^\infty$ and H considered as a function on $I_{\mathcal{S}}$. In any case $\overline{H}$ (resp. $\hat{H}$) is weakly lower semi-continuous (resp. weakly upper semi-continuous) on $\overline{\mathcal{S}}$, $\overline{H} \leq H \leq \hat{H}$ on $\mathcal{S}$ and if H is weakly continuous on $\mathcal{S}$ then $\overline{H} = \hat{H}$ on $\overline{\mathcal{S}}$ and $\overline{H}$ is the unique extension of H to a weakly continuous function on $\overline{\mathcal{S}}$. We may also observe that H is weakly continuous if and only if (iff) for any $f \in \overline{\mathcal{S}}$ and all sequences (A_n) in $\mathcal{S}$ converging weakly to f, the sequences $(H(A_n))$ have the same limit. To close the paragraph, the scalar product of two Euclidean vectors x and y of the same dimension is noted xy, $x \geq y$ (resp. $x > y$) means $x_i \geq y_i$ (resp. $x_i > y_i$) for all i, the product of x with a real matrix U is $x\,U$ or $U\,x$ depending on the one allowed (where in $x\,U$, x is taken as a row vector while in $U\,x$ it is a column vector). Finally e will always be an Euclidean vector of appropriate dimension, the components of which are all unity and if S_1 and S_2 are two subsets of an Euclidean space, then

$$S_1 + S_2 = \{x+y \,:\, x \in S_1,\, y \in S_2\}, \quad S_1 - S_2 = \{x - y \,:\, x \in S_1,\, y \in S_2\},$$

$-S_1 = \{-x \,:\, x \in S_1\}$, and if $S_1 = \{z\})$ then

$$S_1 + S_2 = z + S_2, \quad S_2 - S_1 = S_2 - z.$$

In the second paragraph below, we consider convex set vector functions and among other results, generalizations to such functions of properties of scalar convex set functions. All those notions are used in the third paragraph on vector optimization involving set vector functions.

2. CONE-CONVEXITY OF SET VECTOR FUNCTIONS

The measure space $(\Omega, \mathcal{A}, \mu)$ and L_μ remain as above and $\mathcal{S}$ is a non-empty convex subfamily of $\mathcal{A}$. We let $C \neq \{0\}$ be a closed, convex cone in $\mathbb{R}^p$ with apex $0 \in C$ and positive polar

$$C^* = \{a \in \mathbb{R}^p \,:\, ax \geq 0 \text{ for all } x \in C\} \text{ s.t. interior of } C^*, \ \text{int } C^* \neq \phi$$

so that int $C^* = \{a \in \mathbb{R}^p : ax > 0 \text{ for all } x \in C\backslash\{0\}\}$ and $C \cap -C = \{0\}$ i.e. C is pointed. We suppose that $\mathbb{R}_+^p \subset C$ and that C is s.t. if (y^k) and (α^k) are sequences in $\mathbb{R}^p$ with $\operatorname{Lim} \alpha^k = \alpha \in \mathbb{R}^p$ and $\alpha^k \in y^k + C$ for all k then

$\alpha \in \operatorname{Lim\,sup} y^k + C$ where Lim sup $y^k = (\operatorname{Lim\,sup} y_1^k, \ldots, \operatorname{Lim\,sup} y_p^k)$. Let us observe that $C = \mathbb{R}_+^p$ satisfies all those conditions.

Let $F = (F_1, \ldots, F_p)$ be a set vector function from $\mathcal{S}$ to $\mathbb{R}^p$.

In generalization of the convexity, the weak continuity and the epigraph of a scalar set function, we have:

1. *Définitions:* a) F is said to be C-convex if for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$, there exists a subsequence (Z_{n_k}) in $\mathcal{S}$ s.t. $\alpha F(A) + (1-\alpha) F(B) \in \operatorname{Lim\,sup} F(Z_{n_k}) + C$.

b) F is said to be weakly continuous if each F_i is weakly continuous.

c) The C-epigraph of F is the subset epi F of $\mathcal{A} \times \mathbb{R}^p$ s.t.

$$\operatorname{epi} F = \{(A, \alpha) \in \mathcal{S} \times \mathbb{R}^p : \alpha \in F(A) + C\}. \quad \square$$

2. *Remarks:* a) If F is weakly continuous then $\overline{F} = (\overline{F}_1, \ldots, \overline{F}_p)$ is the unique extension of F to a weakly continuous function on $\overline{\mathcal{S}}$ and since $C \supset \mathbb{R}_+^p$, hence $C^* \subset \mathbb{R}_+^p$, then $a F$ is weakly continuous $\forall a \in C^*$.

b) Let F be C-convex and weakly continuous.

α) For any real $\alpha \geq 0$, it is easily checked that αF is weakly continuous and C-convex.

β) Let H be C-convex and weakly continuous from $\mathcal{S}$ to $\mathbb{R}^p$. Then for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$, there exists a subsequence (Z_n^1) in $\mathcal{S}$ s.t. $\alpha F(A) + (1-\alpha) F(B) \in \operatorname{Lim} F(Z_n^1) + C$. As $(Z_n^1) \sim (\alpha, A, B)$, there exists a subsequence (Z_n^2) of (Z_n^1) s.t. $\alpha H(A) + (1-\alpha) H(B) \in \operatorname{Lim} H(Z_n^2) + C$.

Consequently $\alpha[F(A) + H(A)] + (1-\alpha)[F(B) + H(B)] \in \operatorname{Lim}[F(Z_n^2) + H(Z_n^2)] + C$, so $F + H$ is C-convex and it is obviously weakly continuous.

c) Saying that F is weakly continuous amounts to saying that for any $f \in \overline{\mathcal{S}}$ and all sequences (A_n) in $\mathcal{S}$ converging weakly to f (i.e. $\operatorname{Lim} I_{A_n} = f$ weakly in L_μ^∞), all sequences $(F(A_n))$ have the same limit (this is the definition of weak continuity of F in [2, 3]).

d) If F is weakly continuous, then F is C-convex iff for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{S}$ s.t. $(Z_n) \sim (\alpha, A, B)$, we have $\alpha F(A) + (1-\alpha) F(B) \in \operatorname{Lim} F(Z_n) + C$. $\square$

For the remaining of the paper, we suppose F weakly continuous. Here also $L_\mu^\infty \times \mathbb{R}^p$ has the weak $*$-topology (simply called weak topology) product of the weak topology of L_μ^∞ and the topology of $\mathbb{R}^p$.

Any subset W of $\mathcal{A} \times \mathbb{R}^p$ identified to $J_W = \{I_A \times \alpha \,:\, A \times \alpha \in W\}$ becomes a subset of $L_\mu^\infty \times \mathbb{R}^p$ and we can therefore define in a similar fashion as in the preceding section the subsets $\overline{W}$, $\operatorname{co}(W)$ and $\overline{\operatorname{co}}(W)$ of $L_\mu^\infty \times \mathbb{R}^p$. We also set:

2. *Definition:* A subset W of $\mathcal{A} \times \mathbb{R}^p$ (non identified to $J_W \subset J_{\mathcal{A}\times\mathbb{R}^p} \subset L_\mu^\infty \times \mathbb{R}^p$) is said to be convex if for any $\alpha \in [0, 1]$, (A, x) and $(B, y) \in W$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$, there exists a subsequence (Z_{n_k}) of (Z_n) and a sequence (z_k) in $\mathbb{R}^p$ verifying $\lim z_k = \alpha\, x + (1-\alpha)\, y$ and $(Z_{n_k}, z_k) \in W$ for all k. $\square$

This is obviously a generalization of the convexity of subsets of $\mathcal{A} \times \mathbb{R}$.

3. Proposition: *F is C-convex iff epi F is convex.* $\square$

Proof: Let F be C-convex, $\alpha \in [0, 1]$, (A, x) and $(B, y) \in \operatorname{epi} F$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$. Since $\mathcal{S}$ is convex, F C-convex and weakly continuous, there exists a subsequence (Z_{n_k}) in $\mathcal{S}$ s.t. $\alpha\, F(A) + (1-\alpha)\, F(B) \in \operatorname{Lim} F(Z_{n_k}) + C$. On the other hand $\alpha\, x + (1-\alpha)\, y \in \alpha\, F(A) + (1-\alpha)\, F(B) + C$, thus $\alpha\, x + (1-\alpha)\, y \in \operatorname{Lim} F(Z_{n_k}) + C$ and $\operatorname{Lim} F(Z_{n_k}) = \alpha\, x + (1-\alpha)\, y - c$ for some $c \in C$. It follows that for each k there exists Q_k a member of (Z_{n_k}) s.t. $F(Q_k) \leq \alpha\,.\,x + (1-\alpha)\, y - c + (1/k)\, e$, that is $\alpha\, x + (1-\alpha)\, y + (1/k)\, e \in F(Q_k) + c + \mathbb{R}_+^p$, so $\alpha\, x + (1-\alpha)\, y + (1/k)\, e \in F(Q_k) + C$ since $\mathbb{R}_+^p \subset C$. With $t_k = \alpha\, x + (1-\alpha)\, y + (1/k)\, e$, we have $\operatorname{Lim} t_k = \alpha\, x + (1-\alpha)\, y$ and for each k, $(Q_k, t_k) \in \operatorname{epi} F$, thus epi F is convex.

Conversely let epi F be convex, $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$, $(A, F(A))$ and $(B, F(B)) \in \operatorname{epi} F$, therefore, there exist a subsequence (Z_{n_k}) in $\mathcal{S}$, a sequence (z_k) converging to $\alpha\, F(A) + (1-\alpha)\, F(B)$ and $(Z_{n_k}, z_k) \in \operatorname{epi} F$ for each k. Consequently $z_k \in F(Z_{n_k}) + C$ and $\alpha\, F(A) + (1-\alpha)\, F(B) \in \operatorname{Lim} F(Z_{n_k}) + C$ according to the conditions on C so that F is C-convex. $\square$

4. Proposition: *Let W be a convex subset of $\mathcal{A} \times \mathbb{R}^p$. Then $\overline{W} = \overline{\operatorname{co}}(W)$.* $\square$

Proof: If (A, x) and $(B, y) \in W$, $\alpha \in [0, 1]$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$, then there exists a subsequence (Q_n) of (Z_n) s.t. (I_{Q_n}) converges weakly to $\alpha\, I_A + (1-\alpha)\, I_B$, also there exists a sequence (z_n) in $\mathbb{R}^p$ converging to $\alpha\, x + (1-\alpha)\, y$ and $(Q_n, z_n) \in W$ for each n. It follows that $\operatorname{co}(W) \subset \overline{W}$ and so $\overline{W} \subset \overline{\operatorname{co}}(\mathrm{W}) \subset \overline{\mathrm{W}}$. $\square$

5. *Remark*: It could also be concluded from the proposition above that $\overline{\mathcal{S}} = \overline{\operatorname{co}}(\mathcal{S})$ and that for $p = 1$, $\overline{\mathcal{A}} = \{f \in L_\mu^\infty : 0 \leq f \leq 1\}$. $\square$

6. Definitions: a) The C-epigraph of $\overline{F}$ is epi $\overline{F} = \{(f, \alpha) \in \overline{\mathcal{S}} \times \mathbb{R}^p : \alpha \in \overline{F}(f) + C\}$.

b) $\overline{F}$ is said to be C-convex if epi $\overline{F}$ is convex (in the usual sense) in $L_\mu^\infty \times \mathbb{R}^p$. □

7. *Remark:* Let (f_k, α_k) be a sequence in epi $\overline{F}$ converging to (f, α). Then because of the conditions on C, $\alpha \in \operatorname{Lim\,sup} \overline{F}(f_k) + C$ and since F is weakly continuous, $\overline{F}$ is weakly continuous, therefore $(f, \alpha) \in$ epi $\overline{F}$ and epi $\overline{F}$ is closed. □

8. Proposition: $\overline{\operatorname{epi} F} = \operatorname{epi} \overline{F}$. □

Proof: $F = \overline{F}$ on $\mathcal{S}$ implies epi $F \subset \operatorname{epi} \overline{F}$ and since $\operatorname{epi} \overline{F}$ is closed, $\overline{\operatorname{epi} F} \subset \operatorname{epi} \overline{F}$. Now let $(f, \alpha) \in \operatorname{epi} \overline{F}$, then $\alpha \in \overline{F}(f) + C$ with $f \in \overline{\mathcal{S}}$, so $\alpha = \overline{F}(f) + c$ for some $c \in C$. Now $f \in \overline{\mathcal{S}} \subset \overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ being metrizable, there exists a sequence (A_n) in $\mathcal{S}$ s.t. $\operatorname{Lim} I_{A_n} = f$ weakly. Since $\overline{F}$ is weakly continuous, $\overline{F}(f) = \overline{F}(\operatorname{Lim} I_{A_n}) = \operatorname{Lim} F(A_n)$. Consequently there exists a subsequence (A_{n_k}) of (A_n) s.t. $\alpha + (1/k)\, e \geq F(A_{n_k}) + c$ for each k, so $\alpha + (1/k)\, e \in F(A_{n_k}) + c + \mathbb{R}_+^p \subset F(A_{n_k}) + C$ and $(A_{n_k}, \alpha + (1/k)\, e) \in \operatorname{epi} F$, thus $(f, \alpha) \in \overline{\operatorname{epi} F}$ and $\operatorname{epi} \overline{F} \subset \overline{\operatorname{epi} F}$. □

9. Lemma: 1. *The following cases* a), b) *and* c) *are equivalent:*

a) $\overline{F}$ *is C-convex* (Definitions 6);

b) *For any* f *and* $g \in \overline{\mathcal{S}}$ *and* $\alpha \in [0, 1]$, $\alpha \overline{F}(f) + (1 - \alpha) \overline{F}(g) \in \overline{F}(\alpha f - (1 - \alpha) g) + C$;

c) *For any* $\alpha \in C^*$, $a \overline{F}$ *is convex.*

2. *In the hypothesis of any of the above three equivalences,* $\overline{F}(\overline{\mathcal{S}})$ *is C-convex (i.e.* $\overline{F}(\overline{\mathcal{S}}) + C$ *is convex) and C-closed (i.e.* $\overline{F}(\overline{\mathcal{S}}) + C$ *is closed) and* $\overline{F}(\overline{\mathcal{S}}) = \overline{F(\mathcal{S})}$.

3. F *is C-convex on* $\mathcal{S}$ *iff* $\overline{F}$ *is C-convex on* $\overline{\mathcal{S}}$.

Proof: Since $\overline{\mathcal{S}}$ as a subset of L_μ^∞ is convex in the ordinary sense and $C^{**} = C$, then $\overline{F}$ is C-convex iff for any f and $g \in \overline{\mathcal{S}}$ and $\alpha \in [0, 1]$, $(\alpha f + (1 - \alpha) g,\ \alpha \overline{F}(f) + (1 - \alpha) \overline{F}(g)) \in \operatorname{epi} \overline{F}$ (i.e. $\alpha \overline{F}(f) + (1 - \alpha) \overline{F}(g) \in \overline{F}(\alpha f + (1 - \alpha) g) + C$) iff for any $a \in C^*$, $a \overline{F}$ is convex in the usual sense in which case (Corollary 3.2 in [10]), $\overline{F}(\overline{\mathcal{S}})$ is C-convex i.e. $\overline{F}(\overline{\mathcal{S}}) + C$ is convex.

Let us also observe that F hence $\overline{F}$ being weakly continuous and $\overline{F}$ being the unique extension of F to a weakly continuous function on $\overline{\mathcal{S}}$, we have $\overline{F}(\overline{\mathcal{S}}) = \overline{F(\mathcal{S})}$. Furthermore $\overline{\mathcal{S}}$ being weakly compact,

$\overline{F}(\overline{\mathcal{S}}) = \overline{F(\mathcal{S})}$ is compact and since C is closed, $\overline{F}(\overline{\mathcal{S}}) + C = \overline{F(\mathcal{S})} + C$ is closed. We have F weak continuous. Now F C-convex implies epi F convex (Proposition 3) which in turn implies $\overline{\text{epi}\, F}$ convex in the usual sense (Proposition 4), thus epi $\overline{F}$ is convex in the usual sense (Proposition 8). It follows that $\overline{F}$ is C-convex (Definition 6) so $\overline{F}(\overline{\mathcal{S}}) + C = \overline{F(\mathcal{S})} + C$ is a closed and convex set.

In fact since F is weakly continuous, it is equivalent to say F is C-convex or $\overline{F}$ is C-convex. For $\overline{F}$ is weakly continuous and if we suppose $\overline{F}$ C-convex then for any $\alpha \in [0, 1]$, A and $B \in \mathcal{S}$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$ there exists a subsequence (Q_n) of (Z_n) in $\mathcal{S}$ s.t. $\lim I_{Q_n} = \alpha I_A + (1 - \alpha) I_B$ (since $\mathcal{S}$ is convex) and $\alpha \overline{F}(I_A) + (1 - \alpha) \overline{F}(I_B) \in \overline{F}(\alpha I_A + (1 - \alpha) I_B) + C$ (since $\overline{F}$ is C-convex). We also have $\overline{F}(\alpha I_A + (1 - \alpha) I_B) = \overline{F}(\text{Lim}\, I_{Q_n}) = \text{Lim}\, \overline{F}(I_{Q_n})$ (since $\overline{F}$ is weakly continuous), so that $\alpha \overline{F}(I_A) + (1 - \alpha) \overline{F}(I_B) \in \text{Lim}\, \overline{F}(I_{Q_n}) + C$. As A, B, $Q_n \in \mathcal{S}$, we conclude that F is C-convex. □

We may therefore state the following comprehensive type result:

10. PROPOSITION: *The following cases are equivalent*:

a) *F is C-convex;*

b) *$\overline{F}$ is C-convex;*

c) *$a \overline{F}$ is convex (in the usual sense) for all $a \in C^*$;*

d) *$a F$ is convex for all $a \in C^*$.* □

Proof: a)-b) and b)-c) equivalences follow from Lemma 9 above. Now since F hence $\overline{F}$ is weakly continuous and $\mathbb{R}^p_+ \subset C$ hence $C^* \subset \mathbb{R}^p_+$, then for any $a \in C^*$, $a \overline{F}$ is weakly continuous. On the other hand since $a F = a \overline{F}$ on $\mathcal{S}$ and $\overline{a F}$ is the unique extension of $a F$ to a weakly continuous function on $\overline{\mathcal{S}}$, we have $a \overline{F} = \overline{a F}$. Since $a F$ is weakly continuous, we deduce from Corollary 3.10 and Corollary 3.11 in [1] that $a F$ is convex iff $\overline{a F}$ hence $a \overline{F}$ is convex (in the usual sense), thus c) and d) are equivalent. □

3. SET VECTOR FUNCTION AND VECTOR OPTIMIZATION

Optimal selection of a subset of a given space does arise in several cases including electrical insulator design, optimal plasma confinement, fluid flow as well as in other economical problems. For instance as in [6], let us suppose that the cost per unit area of producing a given crop in a region $\mathcal{R}$ is c for a return u function of the total production density p which is a function of rainfall r in turn a function of longitude x and latitude y. If the area to be planted must not exceed a constant a, it is desired to choose a subregion A to optimize the profit *i.e.*

$$\max \mathrm{u}\left(\int_A p\left(r(x,\, y)\right) dx\, dy\right) - c.\,\text{measure}\,(A)$$
$$\text{subjet to measure}\,(A) \leq a,\ A \in \mathcal{A}\,(\mathcal{R})$$

where $\mathcal{A}\,(\mathcal{R})$ is an appropriate Borel structure on $\mathcal{R}$. We will deal with the multicriteria version of such a problem.

The hypothesis on $(\Omega,\, \mathcal{A},\, \mu)$, $\mathcal{S}$, C, L_μ and F still hold for the present paragraph. Further we suppose that $\mathcal{S}$ contains at least two distinct elements and that the weakly continuous F from $\mathcal{S}$ to $\mathbb{R}^p$ is C-convex.

We consider at first the primal problem

$$(\mathrm{P}_1) \quad C\text{-min}\,\{F\,(A) : A \in \mathcal{S}\}$$

the object of which is to characterize the C-minimal solutions $A^* \in \mathcal{S}$ and the C-minimal criteria values $F\,(A^*)$ s.t. there is no $A \in \mathcal{S}$ verifying $F\,(A^*) \in F\,(A) + C\backslash\{0\}$. With $E\,(\mathcal{S}) = \{F\,(A) : A \in \mathcal{S}\} = F\,(\mathcal{S})$, the set of such $F\,(A^*)$ is noted C-min $E\,(\mathcal{S})$.

11. *Remark:* Since $C \neq \{0\}$, C-min $E\,(\mathcal{S}) \subset \partial E\,(\mathcal{S})$ the relative boundary of $E\,(\mathcal{S})$ so that if $E\,(\mathcal{S})$ is open in $\mathbb{R}^p$, then C-min $E\,(\mathcal{S}) = \varnothing$. So let us suppose just for the time being that $E\,(\mathcal{S})$ is closed so that $E\,(\mathcal{S}) = \overline{E}\,(\mathcal{S}) = \overline{E\,(\mathcal{S})} = \overline{F\,(\mathcal{S})} = \overline{F}\,(\overline{\mathcal{S}})$ and $E\,(\mathcal{S})$ is a compact subset of $\mathbb{R}^p$. It follows that (Theorem 17.2 in [8]) its convex hull $\mathrm{co}\,(E\,(\mathcal{S}))$ is also compact so that (result 10.5 p. 68) in [9]) the extreme points of $\mathrm{co}\,(E\,(\mathcal{S}))$ are in $E\,(\mathcal{S})$. Now since $\mathrm{int}\, C^* = \varnothing$, it comes from these considerations that for any $a \in \mathrm{int}\, C^*$, the set $Y\,(a) = \{x_0 \in E\,(\mathcal{S}) : ax_0 = \min\,[ax : x \in E\,(\mathcal{S})]\} \neq \varnothing$ and it can easily be verified that $Y\,(a) \subset C\text{-min}\, E\,(\mathcal{S})$, so $\varnothing \neq Y\,(\mathrm{int}\, C^*) = \cup\{Y\,(a) \,:\, a \in \mathrm{int}\, C^*\} \subset C$-min $E\,(\mathcal{S})$.

Even when $E\,(\mathcal{S})$ is not closed, we do have the inclusion $Y\,(\mathrm{int}\, C^*) \subset C\text{-min}\, E\,(\mathcal{S})$ although we may have $Y\,(\mathrm{int}\, C^*) = \varnothing$. In any case the elements $A^* \in \mathcal{S}$ and $F\,(A^*)$ s.t. $F\,(A^*) \in Y\,(\mathrm{int}\, C^*)$ are said to be respectively proper C-minimal solutions and proper C-minimal criteria values for the problem (P_1). The problem of characterizing those elements could also be considered (*see* [2, 3]). With $p\,C\text{-min}\, E\,(\mathcal{S}) = Y\,(\mathrm{int}\, C^*)$, it is interesting to observe that $p\,C\text{-min}\, E\,(\mathcal{S}) = E\,(\mathcal{S}) \cap p\,C\text{-min}\, \overline{E}\,(\mathcal{S})$. For if $y_0 \in p\,C\text{-min}\, E\,(\mathcal{S})$, then $y_0 \in E\,(\mathcal{S})$ and there exists $a \subset \mathrm{int}\, C^*$ s.t.

$a_0\, y_0 \leq a_0\, y \,\forall\, y \in E(\mathcal{S})$ thus $\forall\, y \in \overline{E}(\mathcal{S})$ so that $y_0 \in E(\mathcal{S}) \cap p\, C\text{-min}\, \overline{E}(\mathcal{S})$. The converse is evident. □

The problem (P_1) seen only in the criteria value space is to characterize the elements of $C\text{-min}\, E(\mathcal{S})$ where $E(\mathcal{S}) = F(\mathcal{S})$. With $E^*(\mathcal{S}) = \{y \in \mathbb{R}^p : y \notin E(\mathcal{S}) + C\backslash\{0\}\}$, we define its dual as

$$(D_1) \quad C\text{-max}\, E^*(\mathcal{S})$$

about characterizing the C-maximal elements $y_0 \in E^*(\mathcal{S})$ s.t. there is no $y \in E^*(\mathcal{S})$ verifying $y \in y_0 + C\backslash\{0\}$, the set of such elements is C-max $E^*(\mathcal{S}) \subset \partial E^*(\mathcal{S})$ and C-max $E^*(\mathcal{S}) = \varnothing$ if $E^*(\mathcal{S})$ is open.

12. *Remarks:* a) For any $y \in E(\mathcal{S})$ and $z \in E^*(\mathcal{S})$, we evidently have $z \notin y + C\backslash\{0\}$, a kind of weak duality result.

b) If $y \in E(\mathcal{S}) \cap E^*(\mathcal{S})$ then $y \in C\text{-min}\, E(\mathcal{S}) \cap C\text{-max}\, E^*(\mathcal{S})$ (same proof as Proposition 13 in [5]).

c) C-min $E(\mathcal{S}) \subset C\text{-max}\, E^*(\mathcal{S})$ (same proof as the first part of Theorem 14 in [5]).

d) It comes from c) and the definition of $E^*(\mathcal{S})$ that if $y \in E(\mathcal{S})$ then $y \in C\text{-min}\, E(\mathcal{S})$ iff $y \in C\text{-max}\, E^*(\mathcal{S})$, in other words C-min $E(\mathcal{S}) = E(\mathcal{S}) \cap C$-max $E^*(\mathcal{S})$ which could be seen as a duality result. Further if $E(\mathcal{S}) = \overline{E}(\mathcal{S})$, since $E(\mathcal{S}) + C$ is closed and convex, then C-min $E(\mathcal{S}) = C$-max $E^*(\mathcal{S})$, the proof being the same as the one of the second part of Theorem 14 in [5]. □

13. Proposition: *We note $\overline{F}$ on $\overline{\mathcal{S}}$ also F so that*

$$E(\overline{\mathcal{S}}) = F(\overline{\mathcal{S}}) = \overline{F}(\overline{\mathcal{S}}) = \overline{F(\mathcal{S})} = \overline{E(\mathcal{S})} = \overline{E}(\mathcal{S}).$$

We set

$$E^*(\overline{\mathcal{S}}) = \{y \in \mathbb{R}^p : y \notin E(\overline{\mathcal{S}}) + C\backslash\{0\}\} = \{y \in \mathbb{R}^p : y \notin \overline{E}(\mathcal{S}) + C\backslash\{0\}\},$$

$M(\mathcal{S}) = \overline{E}(\mathcal{S}) + C$ (a closed and convex subset of $\mathbb{R}^p$),

$$T^*(\mathcal{S}) = \{y \in \mathbb{R}^p : \exists\, a \in \operatorname{int} C^*,\, ay \leq \min\,[az : z \in \overline{E}(\mathcal{S})]\}.$$

Then we have:

a) $(M(\mathcal{S}))^c \subset T^*(\mathcal{S}) \subset E^*(\overline{\mathcal{S}}) \subset \overline{(M(\mathcal{S}))^c}$ where for any set A, A^c is its complementary set.

b) $pC\text{-min}\, \overline{E}(\mathcal{S}) = C\text{-max}\, T^*(\mathcal{S}) \neq \varnothing$.

c) $pC\text{-min}\, E(\mathcal{S}) = E(\mathcal{S}) \cap p\, C\text{-min}\, \overline{E}(\mathcal{S}) = E(\mathcal{S}) \cap C\text{-max}\, T^*(\mathcal{S})$ so that the dual of the problem of characterising the elements of $p\, C\text{-min}\, E(\mathcal{S})$ is the problem of characterizing those of $E(\mathcal{S}) \cap C\text{-max}\, T^*(\mathcal{S})$. □

Proof: a) For the second inclusion, by taking $y \in T^*(\mathcal{S})$ and supposing that $y \notin E^*(\overline{\mathcal{S}})$, we get a contradiction. For the last inclusion, let $y \in E^*(\overline{\mathcal{S}})$. Then $y \notin M(\mathcal{S}) + C\backslash\{0\}$, thus $y \notin \text{int}\,(M(\mathcal{S}))$ so $y \in \overline{(M(\mathcal{S}))^c}$. It remains to show the first inclusion. Since $p\, C\text{-min}\, \overline{E}(\mathcal{S}) \neq \varnothing$ (Remark 11), there exist $y_0 \in p\, C\text{-min}\, \overline{E}(\mathcal{S})$, $a_0 \in \text{int}\, C^*$ s.t. $d(a_0) = a_0\, y_0 = \min[a_0\, x : x \in \overline{E}(\mathcal{S})]$ so that $d(a_0) = \min[a_0\, y : y \in M(\mathcal{S})]$. We may therefore state the existence of $a_0 \in \text{int}\, C^*$ s.t. $M(\mathcal{S}) \subset H^+(a_0) = \{y \in \mathbb{R}^p : ay \geq d(a_0)\}$. Now $M(\mathcal{S})$ is a closed, convex and non empty set and if for $z \in M(\mathcal{S})$ and $a \in \mathbb{R}^p$ we have $d(a) = az = \min[ay : y \in M(\mathcal{S})]$, then $y = z + c \in M(\mathcal{S})\, \forall c \in C$, thus $ay = az + ac \geq az$ so that $ac \geq 0$ hence $a \in C^*$. Consequently $M(\mathcal{S})$ is the intersection of the $H^+(a)$ with $a \in C^*$, an intersection included in the one of the $H^+(a)$ with $a \in \text{int}\, C^*$. For the reverse inclusion, if $M(\mathcal{S}) \subset H^+(a_1)$ with $a_1 \in \partial C^*$, then $\forall t \in \mathbb{R}^+\backslash\{0\}$ and $y \in M(\mathcal{S})$, $h_t(y) = (ta_1 + a_0)\, y - [td(a_1) + d(a_0)] \geq 0$ and for $z \in \text{int}\, H^-(a_1)$ with $H^-(a_1) = \{y \in \mathbb{R}^p : a_1\, y \leq d(a_1)\}$, by taking $t > 0$ large enough, we get $h_t(z) < 0$. Since $ta_1 + a_0 \in \text{int}\, C^*\, \forall t > 0$, $\exists\, t = t_0$ s.t. $\overline{a} = t_0\, a_1 + a_0 \in \text{int}\, C^*$, $M(\mathcal{S}) \subset H^+(\overline{a})$ and $z \in H^-(\overline{a})$, that is $\partial H^+(\overline{a})$ is between $M(\mathcal{S})$ and z. That implies the required reverse inclusion so that $M(\mathcal{S}) = \cap[H^+(a) : a \in \text{int}\, C^*]$, $(M(\mathcal{S}))^c = \cup[\text{int}\, H^-(a) : a \in \text{int}\, C^*] \subset \cup[H^-(a) : a \in \text{int}\, C^*] = T^*(\mathcal{S})$.

b) $p\, C\text{-min}\, \overline{E}(\mathcal{S}) = C\text{-max}\, T^*(\mathcal{S}) \neq \varnothing$. This is so because from Remark 11 above, $p\, C\text{-min}\, \overline{E}(\mathcal{S}) = \varnothing$ and if $y \in p\, C\text{-min}\, \overline{E}(\mathcal{S})$, then the same Remark 11 implies that $y \in T^*(\mathcal{S})$. Now if $y \notin C\text{-max}\, T^*(\mathcal{S})$ then $z \in y + C\backslash\{0\}$ for some $z \in T^*(\mathcal{S})$, thus there exists $a \in \text{int}\, C^*$ s.t. $az \leq ax\, \forall x \in \overline{E}(\mathcal{S})$. It follows that $ay < az \leq ax\ \forall x \in \overline{E}(\mathcal{S})$, an absurdity since $y \in \overline{E}(\mathcal{S})$. Conversely if $y \in C\text{-max}\, T^*(\mathcal{S})$, then $y \in T^*(\mathcal{S}) \subset E^*(\overline{\mathcal{S}})$, so $y \notin \overline{E}(\mathcal{S}) + C\backslash\{0\}$. On the other hand, from the first part $(M(\mathcal{S}))^c \subset T^*(\mathcal{S}) \subset \overline{(M(\mathcal{S}))^c}$. Since $y \in C\text{-max}\, T^*(\mathcal{S})$ and $(M(\mathcal{S}))^c$ is open, then $y \in \partial\, \overline{(M(\mathcal{S}))^c} = \partial M(\mathcal{S})$. Now $y \notin \overline{E}(\mathcal{S}) + C\backslash\{0\}$ implies $y \notin M(\mathcal{S}) + C\backslash\{0\}$. Consequently $y \in C\text{-min}\, M(\mathcal{S})$ and $C\text{-min}\, M(\mathcal{S}) = C\text{-min}\, \overline{E}(\mathcal{S})$ (Lemma 4.1 in [10]). As $y \in \overline{E}(\mathcal{S})$ and $y \in T^*(\mathcal{S})$, we conclude that $y \in p\, C\text{-min}\, \overline{E}(\mathcal{S})$.

c) It comes from b) above and Remark 11 that

$$p\, C\text{-min}\, E(\mathcal{S}) = E(\mathcal{S}) \cap p\, C\text{-min}\, \overline{E}(\mathcal{S}) = E(\mathcal{S}) \cap C\text{-max}\, T^*(\mathcal{S}).$$

Consequently the dual of the problem of characterizing the elements of $pC\text{-min}\,E(\mathcal{S})$ is the problem of characterizing those of $E(\mathcal{S}) \cap C\text{-max}\,\mathrm{T}^*(\mathcal{S})$. □

The solutions and values properly C-minimal to (P_1) being more desirable than those simply C-minimal (*see* [4] for example for the reasons), like in [2-3], we will, in the remaining of the paper, concentrate on the first type. On the other hand, in order to easily introduce dual variables, our interest will be focused on a special class of problems. Formally, we let

$$\mathcal{D} = \{A \in \mathcal{S} : G(A) \in -K\}$$

where $\mathcal{S}$ is as above, K is a closed convex cone in $\mathbb{R}^m$ containing $\mathbb{R}^m_+$ and is s.t. $\mathrm{int}\,K^* \neq \varnothing$ and for any sequences (y^k) and (a^k) in $\mathbb{R}^m$ verifying $\mathrm{Lim}\,\alpha^k = \alpha \in \mathbb{R}^m$ and $\alpha^k \in y^k + K$ for each k, then $\alpha \in \mathrm{Lim\,sup}\,y^k + K$ and $G = (G_1, ..., G_m)$ is a vector function from $\mathcal{S}$ to $\mathbb{R}^m$, *K-convex* and weakly continuous. We consider the primal problem

$$(P_2) \quad pC\text{-min}\,\{F(A) : A \in \mathcal{D}\}$$

consisting in characterizing the solutions $A \in \mathcal{D}$ and values $F(A)$ that are properly *C-minimal*. In the preceding notations the set of proper *C-minimal* values is $pC\text{-min}\,E(\mathcal{D})$.

We say that $\mathcal{D}$ (or G) satisfies Slater's constraint qualification or Slater's condition, if there exists $A_0 \in \mathcal{S}$ s.t. $G(A_0) \in -\mathrm{int}\,K$. In quest of a dual to (P_2) we will exploit Proposition 13, but in order to do so we must have $\overline{E}(\mathcal{D})$ C-convex, that is $M(\mathcal{D}) = \overline{E}(\mathcal{D}) + C$ convex in $\mathbb{R}^p$ and this is not necessarily the case even though $\mathcal{S}$ is convex (*see* Example 3.1 in [2]). The next two results will allow us to bypass the difficulty.

14. LEMMA: *If $\mathcal{D}$ satisfies Slater's condition, then $\mathcal{D}' = \{A \in \mathcal{S} : G(A) \in -\mathrm{int}\,K\}$ is a non empty and convex subfamily of $\mathcal{A}$.* □

Proof: $\mathcal{D}'$ is evidently non empty. Let $\alpha \in [0, 1]$, $A, B \in \mathcal{D}'$ and (Z_n) in $\mathcal{A}$ s.t. $(Z_n) \sim (\alpha, A, B)$. Since A and $B \in \mathcal{S}$ and $\mathcal{S}$ is convex, there exists a subsequence (Z_n^1) in $\mathcal{S}$. Now G being K-convex and weakly continuous, there exists a subsequence (Z_n^2) of (Z_n^1) s.t. $\alpha G(A) + (1 - \alpha) G(B) \in \mathrm{Lim}\,G(Z_n^2) + K$ and since $G(A)$ and $G(B) \in -\mathrm{int}\,K$, we have $\lim G(Z_n^2) \in -\mathrm{int}\,K$. Since $\mathrm{int}\,K$ is open, there exists a subsequence (Z_n^3) of (Z_n^2) s.t. $G(Z_n^3) \in -\mathrm{int}\,K\,\forall n$ so that $(Z_n^3) \subset \mathcal{D}'$ and consequently $\mathcal{D}'$ is convex. □

As a very slight generalization of Proposition 3.1 in [2], we have:

15. PROPOSITION: *If $\mathcal{D}$ satisfies Slater's condition, then $\overline{E}(\mathcal{D})$ is C-convex, that is $M(\mathcal{D}) = \overline{E}(\mathcal{D}) + C$ is convex in $\mathbb{R}^p$.* $\square$

Proof: It is quite similar to the one of Proposition 3.1 in [2]. We include it. By definition $\overline{E}(\mathcal{D}) = \overline{E(\mathcal{D})} = \overline{F(\mathcal{D})}$ and by the lemma $\mathcal{D}' \neq \emptyset$ and convex, thus $\overline{F(\mathcal{D}')} = \overline{F}(\overline{\mathcal{D}'})$ is C-convex according to Lemma 9. It suffices to establish that $\overline{F(\mathcal{D})} = \overline{F(\mathcal{D}')}$ or that $F(\mathcal{D}) \subset \overline{F(\mathcal{D}')}$. So let $A_0 \in \mathcal{D}'$ and $B_0 \in \mathcal{D}$. Since $\mathcal{D}$ and $\mathcal{D}' \subset \mathcal{S}$, for each $n \in \mathbb{N}^*$, there exists a sequence $(Z_{n,k})_k$ in $\mathcal{S}$ s.t. $(Z_{n,k}) \sim (1/n, A_0, B_0)$, thus there exists a subsequence $(Z^1_{n,k})_k$ s.t.

$$(1/n)\, G(A_0) + \left(1 - \frac{1}{n}\right) G(B_0) \in \operatorname*{Lim}_{k \to +\infty} G(Z^1_{n,k}) + K.$$

As $G(A_0) \in -\operatorname{int} K$ and $G(B_0) \in -K$, we have $\operatorname*{Lim}_{k \to +\infty} G(Z^1_{n,k}) \in -\operatorname{int} K$, thus there exists a subsequence $(Z^2_{n,k})_k \subset (Z^1_{n,k})$ s.t. $G(Z^2_{n,k}) \in -\operatorname{int} K\, \forall\, k$ so that $(Z^2_{n,k})_k \subset \mathcal{D}' \subset \overline{\mathcal{A}}$. Since $\overline{\mathcal{A}}$ is metrizable and I_{B_0} is a cluster point of $(Z^2_{n,k})_k$, there exists a subsequence $(Z_i) \subset (Z^2_{n,k})_{n,k}$ s.t. $\operatorname{Lim} Z_i = I_{B_0}$ weakly, so $\operatorname{Lim} F(Z_i) = F(B_0)$ and hence $F(B_0) \in \overline{F(\mathcal{D}')}$ implying that $F(\mathcal{D}) \subset \overline{F(\mathcal{D}')}$. $\square$

In the remaining of the paper, the unique weakly continuous extension of F (resp. G) from $\mathcal{S}$ to $\overline{\mathcal{S}}$, that is $\overline{F}$ (resp. $\overline{G}$), will also be noted F (resp. G). Let us also note that for any $a \in \operatorname{int} C^*$, if $\mathcal{D}$ satisfies Slater's condition, since $\overline{E}(\mathcal{D})$ is C-convex according to the preceding Proposition 15, it comes from Remark 11 that there exists $y_0 \in \overline{E}(\mathcal{D})$ s.t. $a y_0 = \min\{ay : y \in \overline{E}(\mathcal{D})\}$ and since $\overline{E}(\mathcal{D}) = \overline{E(\mathcal{D})} = \overline{F(\mathcal{D})} = F(\overline{\mathcal{D}})$, for any $y \in \overline{E}(\mathcal{D})$, there exists $h \in \overline{\mathcal{D}}$ s.t. $y = F(h)$.

16. THEOREM: *Let $\mathcal{D}$ satisfy Slater's condition, $a \in \operatorname{int} C^*$ and $y_0 \in \overline{E}(\mathcal{D})$ with $y_0 = F(h_0)$ for an $h_0 \in \overline{\mathcal{D}}$. Then $a y_0 = \min[ay : y \in \overline{E}(\mathcal{D})]$ iff there exists $u \in K^*$ s.t. $a y_0 = \inf[a F(h) + u G(h) : h \in \overline{\mathcal{S}}]$ in which case $u G(h_0) = 0$ and the infimum may be replaced by a minimum.* $\square$

Proof: 1. If there exists $u \in K^*$ s.t. $a y_0 = \inf[a F(h) + u G(h) : h \in \overline{\mathcal{S}}]$ then $a y_0 \le a F(h) + u G(h)\, \forall\, h \in \overline{\mathcal{D}}$. Since $\forall\, A \in \mathcal{D}$, $G(A) \in -K$, then $u G(A) \le 0\, \forall\, A \in \mathcal{D}$, $u G(h) \le 0\, \forall\, h \in \overline{\mathcal{D}}$ since $u G$ is weakly continuous. It follows that $a y_0 \le a F(h)\, \forall\, h \in \overline{\mathcal{D}}$ *i.e.* $a y_0 \le a y$ $\forall\, y \in F(\overline{\mathcal{D}}) = \overline{E}(\mathcal{D})$ so that $a y_0 = \min[ay : y \in \overline{E}(\mathcal{D})]$. We also have

$ay_0 \leq a\,F\,(h_0) + u\,G\,(h_0)$ so that $u\,G\,(h_0) \geq 0$ and $u\,G\,(h_0) \leq 0$, hence $u\,G\,(h_0) = 0$, $ay_0 = \min\,[ay\,:\,y \in \overline{E}\,(\mathcal{D})]$ and inf may be replaced by min.

2. Conversely, let $ay_0 = \min\,[ay\,:\,y \in \overline{E}\,(\mathcal{D})]$ and let us set $S = \{(z,\,w) \in \mathbb{R}\times\mathbb{R}^m : \exists\,h \in \overline{\mathcal{S}},\, a\,F\,(h) - ay_0 < z,\, w - G\,(h) \in \operatorname{int} K\}$ and let us show that S is convex. So let $\alpha \in [0,\,1]$, $(z_1,\,w_1)$ and $(z_2,\,w_2) \in S$. Then for $i = 1,\,2,\,\exists\,h_i \in \overline{\mathcal{S}},\; k_i \in \operatorname{int} K$ s.t. $z_i > a\,F\,(h_i) - ay_0$, $w_i - G\,(h_i) = k_i$ implying that $\alpha\,z_1 + (1-\alpha)\,z_2 > \alpha\,F\,(h_1) + (1-\alpha)\,F\,(h_2) - ay_0$, $\alpha\,w_1 + (1-\alpha)\,w_2 - [\alpha\,G\,(h_1) + (1-\alpha)\,G\,(h_2)] = \alpha\,k_1 + (1-\alpha)\,k_2$. Now $\overline{\mathcal{S}}$ is a convex subset (in the usual sense) of L^∞_μ (Remark 5) and according to Proposition 10 $a\,F$ is convex (in the usual sense) on $\overline{\mathcal{S}}$ and G is K-convex (sense of Definition 6) on $\overline{\mathcal{S}}$ so that $\alpha\,h_1 + (1-\alpha)\,h_2 \in \overline{\mathcal{S}}$. $\alpha\,z_1 + (1-\alpha)\,z_2 > a\,F\,(\alpha\,h_1 + (1-\alpha)\,h_2) - ay_0$, $\alpha\,w_1 + (1-\alpha)\,w_2 - [G\,(\alpha\,h_1 + (1-\alpha)\,h_2) + k] = \alpha\,k_1 + (1-\alpha)\,k_2$ for some $k \in K$. It follows that $(\alpha\,z_1 + (1-\alpha)\,z_2,\; \alpha\,w_1 + (1-\alpha)\,w_2) \in S$, hence S is convex.

3. Suppose that $(0,\,0) \in S$. Then there exists $h \in \overline{\mathcal{S}}$ s.t. $a\,F\,(h) - ay_0 < 0$ and $-G\,(h) \in \operatorname{int} K$. If $h = \operatorname{Lim} h_i$ weakly with $h_i \in \mathcal{S}$, then $\operatorname{Lim} G\,(h_i) = G\,(\operatorname{Lim} h_i) = G\,(h) \in -\operatorname{int} K$ and $\operatorname{Lim} a\,F\,(h_i) = a\,F\,(\operatorname{Lim} h_i) = a\,F\,(h) < ay_0$, thus for i large enough, $G\,(h_i) \in -\operatorname{int} K$ and $a\,F\,(h_i) < ay_0$, a contradiction that implies $(0,\,0) \notin S$.

Consequently, since $S = \varnothing$, there exist $\beta \in \mathbb{R}$ and $u \in \mathbb{R}^m$ s.t. $(\beta,\,u) \neq (0,\,0)$ and $\beta\,z + uw \geq 0\,\forall\,(z,\,w) \in S$ (Lemma 2 p. 47 in [7]) and since z could be arbitrarily large, we must have $\beta \geq 0$.

4. Let $h \in \overline{\mathcal{S}}$, $r_1 \in \mathbb{R}_+\backslash\{0\}$ and $k_1 \in \operatorname{int} K$ be fixed. Then for any $(r,\,k) \in \mathbb{R}_+\times K$, setting $z = a\,F\,(h) - ay_0 + r_1 + r$ and $w = G\,(h) + k_1 + k$, we have $(z,\,w) \in S$, hence $\beta\,[a\,F\,(h) - ay_0] + \beta\,r_1 + \beta\,r + u\,G\,(h) + uk_1 + uk \geq 0$. $(r,\,k)$ being arbitrary in $\mathbb{R}_+ \times K$, we must have $(\beta,\,u) \in (\mathbb{R}_+ \times K)^* = \mathbb{R}_+ \times K^*$ and so $(\beta,\,u) \in \mathbb{R}_+ \times K^*\backslash\{(0,\,0)\}$.

5. Let $\varepsilon \in \mathbb{R}_+\backslash\{0\}$ and $k \in \operatorname{int} K$. Then for any $h \in \overline{\mathcal{S}}$, setting $z = a\,F\,(h) - ay_0 + \varepsilon$ and $w = G\,(h) + \varepsilon\,k$, we have $(z,\,w) \in S$ so that $\beta\,[a\,F\,(h) - ay_0] + u\,G\,(h) \geq -\varepsilon\,(\beta + uk)$.

If $\inf\,\{\beta\,[a\,F\,(h) - ay_0] + u\,G\,(h)\,:\,h \in \overline{\mathcal{S}}\} = -\delta < 0$, we have $0 < \delta \leq \varepsilon\,(\beta + uk)$ and by taking ε small enough, we get a contradiction implying that $\beta\,[a\,F\,(h) - ay_0] + u\,G\,(h) \geq 0\;\forall\,h \in \overline{\mathcal{S}}$.

6. If $\beta = 0$, thus $u \in K^*\backslash\{0\}$. Slater's condition implies the existence of an $A_1 \in \mathcal{S}$ s.t. $G\,(A_1) \in -\operatorname{int} K$ so that $u\,G\,(A_1) \geq 0$ and $u\,G\,(A_1) < 0$.

This contradiction implies that $\beta > 0$ and we may suppose that $\beta = 1$, obtaining $a F(h) + u G(h) \geq a y_0 \ \forall h \in \overline{\mathcal{S}}$. $\square$

We will need the following remark (Remark 26 in [5]).

17. *Remark:* For any $a \in C^*\backslash\{0\}$ and $u \in K^*$, since there exists $b \in C$ s.t. $ab = 1$, taking the real $p \times m$-matrix $U \doteq (u_1 b, \ldots, u_m b)$, we get $a U = u$ and for any $v \in K$, $U v = uvb$ so that $U v \in C$. With $\mathcal{U}$ as the set of all real $p \times m$-matrices U s.t. $\forall v \in K$, $U v \in C$, that is $UK \subset C$, we just showed that for any $a \in C^*\backslash\{0\}$ and $u \in K^*$, there exists $U \in \mathcal{U}$ verifying $a U = u$, $UK \subset C$ and since $\mathbb{R}_+^p \subset C$, thus $C^* \subset \mathbb{R}_+^p$, by supposing a normalized (*i.e.* $ae = a_1 + \ldots + a_p = 1$) and taking $U = (u_1 e, \ldots, u_m e)$, we get $U v = (uv, \ldots, uv)$ noted $\ll u, v \gg \ \forall v \in K$. $\square$

The last conclusion of the following proposition is a slight improvement of Theorem 3.1 in [3].

18. Proposition: *We let $\mathcal{D}$ satisfy Slater's condition, A_0 in $\mathcal{S}$ and define the Lagrangian type function L on $\overline{\mathcal{S}} \times K^*$ by*

$$L(h, u) = F(h) + \ll u, G(h) \gg .$$

Then $F(A_0) \in pC$-min $E(\mathcal{D})$ iff there exist $a_0 \in \operatorname{int} C^*$, $u_0 \in K^*$, s.t. $a_0 F(A_0) + u_0 G(A_0) \leq a_0 F(h) + u_0 G(h) \forall h \in \overline{\mathcal{S}}$ and $u_0 G(A_0) = 0$ in which case there exists $U_0 \in \mathcal{U}$ s.t. $U_0 G(A_0) = 0$,

$$F(A_0) + U_0 G(A_0) \notin F(h) + U_0 G(h) + C\backslash\{0\} \forall h \in \overline{\mathcal{S}},$$

$$F(A_0) + \ll u_0, G(A_0) \gg \notin F(h) + \ll u_0, G(h) \gg + C\backslash\{0\} \forall h \in \overline{\mathcal{S}}$$

and

$$F(A_0) \in C\text{-min}\{L(A, u_0) : A \in \mathcal{S}\} \cap C\text{-min}\,\overline{\{L(A, u_0) : A \in \mathcal{S}\}}. \quad \square$$

Proof: We suppose that $\mathcal{D}$ satisfies Slater's condition and let $A_0 \in \mathcal{S}$. Then $F(A_0) \in pC\text{-min}\, E(\mathcal{D})$ iff $F(A_0) \in E(\mathcal{D}) \cap pC\text{-min}\, \overline{E}(\mathcal{D})$, in other words $F(A_0) \in E(\mathcal{D})$ and there exists $a_0 \in \operatorname{int} C^*$ s.t. $a_0 F(A_0) = \min[ay : y \in \overline{E}(\mathcal{D})]$. It then comes from Theorem 16 above that this is equivalent to $F(A_0) \in E(\mathcal{D})$ and the existence of $u_0 \in K^*$ s.t.

$$a_0 F(A_0) + u_0 G(A_0) \leq a_0 F(h) + u_0 G(h) \forall h \in \overline{\mathcal{S}}$$

and $u_0 G(A_0) = 0$. From the preceding Remark 17, this implies that there exists $U_0 \in \mathcal{U}$ s.t. $a_0 U_0 = u_0$ so that

$$a_0 [F(A_0) + U_0 G(A_0)] \leq a_0 [F(h) + U_0 G(h)] \forall h \in \overline{\mathcal{S}}.$$

It follows that the sets $F(A_0) + U_0 G(A_0) - C\backslash\{0\}$ and $\{F(h) + U_0 G(h) : h \in \overline{S}\}$ are strictly separated by the hyperplane $\{z \in \mathbb{R}^P : a_0 z = a_0 [F(A_0) + U_0 G(A_0)]\}$ so that $F(A_0) + U_0 G(A_0) \notin F(h) + U_0 G(h) + C\backslash\{0\} \forall h \in \overline{S}$. We now suppose a_0 normalized since $C^* \subset \mathbb{R}_+^p$. According to Remark 17 above, we may take $U_0 = (u_{01} e, ..., u_{0m} e)$ and with $\ll u_0, G(h) \gg = U_0 G(h) \forall h \in \overline{S}$, we get $F(A_0) + \ll u_0, G(A_0) \gg \notin F(h) + \ll u_0, G(h) \gg + C\backslash\{0\} \forall h \in \overline{S}$, thus $\forall h \in \mathcal{S}$ and $u_0 G(A_0) = 0$ so $U_0 G(A_0) = 0$. With $L(h, u) = F(h) + \ll u, G(h) \gg \forall h \in \overline{S}$ and $u \in K^*$, we get $F(A_0) \in C\text{-min} [L(h, u_0) : h \in \overline{S}\} \cap C\text{-min} [L(A, u_0) : A \in \mathcal{S}]$ and $u_0 G(A_0) = 0$. Because of the weak continuity of F and G on $\mathcal{S}$, hence on $\overline{S}$, we have $\{L(h, u_0) : h \in \overline{S}\} = \overline{\{L(A, u_0) : A \in \mathcal{S}\}}$. We conclude that $F(A_0) \in C\text{-min} \{L(A, u_0) : A \in \mathcal{S}\} \cap C\text{-min} \{\overline{L(A, u_0) : A \in \mathcal{S}}\}$ and $u_0 G(A_0) = 0$. □

19. *Remark:* Let $\mathcal{D}$ satisfy Slater's constraint qualification and also let us recall from Remark 11 and Proposition 13 that: $\overline{E}(\mathcal{D}) = \overline{E(\mathcal{D})} = \overline{F(\mathcal{D})} = F(\overline{\mathcal{D}}) = E(\overline{\mathcal{D}})$ ($\overline{F}$ on $\overline{\mathcal{D}}$ has been noted F), $E^*(\overline{\mathcal{D}}) = \{y \in \mathbb{R}^p : y \notin E(\overline{\mathcal{D}}) + C\backslash\{0\} = \{y \in \mathbb{R}^p : y \notin \overline{E}(\mathcal{D}) + C\backslash\{0\}\}$, $T^*(\mathcal{D}) = \{y \in \mathbb{R}^p : \exists a \in \operatorname{int} C^*, ay \leq \min [az : z \in \overline{E}(\mathcal{D})]\}$. Since $\overline{E}(\mathcal{S}) = E(\overline{\mathcal{S}})$ it comes from Proposition 13 that $p\,C\text{-min}\, E(\mathcal{D}) = E(\mathcal{D}) \cap p\,C\text{-min}\, E(\overline{\mathcal{D}}) = E(\mathcal{D}) \cap C\text{-max}\, T^*(\mathcal{D})$. Also it comes from Theorem 16 above that

$$
\begin{aligned}
T^*(\mathcal{D}) = \{y \in \mathbb{R}^p &: \exists a \in \operatorname{int} C^*, u \in K^*, \\
&ay \leq \inf [a F(h) + u G(h) : h \in \overline{S}]\} \\
= \{y \in \mathbb{R}^p &: \exists a \in \operatorname{int} C^*, u \in K^*, \\
&ay \leq a F(h) + u G(h) \forall h \in \overline{S}\}. \quad \square
\end{aligned}
$$

20. Proposition: *Let $\mathcal{D}$ satisfy Slater's constraint qualification. $T^*(\mathcal{D})$ being as in the preceding Remark* 19. *Let*

$$S^*(\mathcal{D}) = \{y \in \mathbb{R}^p : \exists U \in \mathcal{U}, y \notin F(h) + UG(h) + C\backslash\{0\} \forall h \in \overline{S}\}.$$

Then

$$C\text{-min}\, E(\overline{\mathcal{D}}) \supset C\text{-max}\, S^*(\mathcal{D}) \supset p\,C\text{-min}\, E(\overline{\mathcal{D}}) = C\text{-max}\, T^*(\mathcal{D}) \neq \varnothing$$

so that

$$
\begin{aligned}
p\,C\text{-min}\, E(\mathcal{D}) &= E(\mathcal{D}) \cap p\,C\text{-min}\, E(\overline{\mathcal{D}}) \\
&= E(\mathcal{D}) \cap C\text{-max}\, T^*(\mathcal{D}) \subset E(\mathcal{D}) \cap C\text{-max}\, S^*(\mathcal{D})
\end{aligned}
$$

with equality everywhere if $C\text{-min}\, E(\mathcal{D}) = p\, C\text{-min}\, E(\mathcal{D})$. $\square$

Proof: For any $y_0 \in T^*(\mathcal{D})$ it is easily seen (same procedure as in the proof of Proposition 18) that $y_0 \in S^*(\mathcal{D}) = \{y \in \mathbb{R}^p : \exists U \in \mathcal{U}, y \notin F(h) + UG(h) + C\backslash\{0\} \,\forall h \in \overline{\mathcal{S}}\}$, so $T^*(\mathcal{D}) \subset S^*(\mathcal{D})$. On the other hand with $\overline{D}$ in place of $\mathcal{S}$. Remarks 12 above still hold and since $E(\overline{\mathcal{D}}) = \overline{E}(\mathcal{D})$ and $E(\overline{\mathcal{D}}) + C$ is closed and convex, we get $C\text{-min}\, E(\overline{\mathcal{D}}) = C\text{-max}\, E^*(\overline{\mathcal{D}})$.

Now let $y_0 \in S^*(\mathcal{D})$. Then there exists $U_0 \in \mathcal{U}$ s.t. $y_0 \notin F(h) + U_0\, G(h) + C\backslash\{0\} \,\forall h \in \overline{\mathcal{S}}$ hence $\forall h \in \overline{\mathcal{D}}$. For any $h \in \overline{\mathcal{D}}$, either $h \in \mathcal{D}$ so that $G(h) \in -K$ or $h = \operatorname{Lim} h_i$ weakly with $h_i \in \mathcal{D}$ so $G(h_i) \in -K$ and $G(h) = G(\operatorname{Lim} h_i) = \operatorname{Lim} G(h_i) \in -K$ since K is closed and G is weakly continuous. In any case, it comes from Remark 17 that $U_0\, G(h) \in -C$ and since C is convex and pointed, we deduce that $y_0 \notin F(h) + C\backslash\{0\} \,\forall h \in \overline{\mathcal{D}}$, so $S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$, hence $T^*(\mathcal{D}) \subset S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$. It follows from Proposition 13 that $(M(\mathcal{D}))^c \subset T^*(\mathcal{D}) \subset S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}}) \subset \overline{(M(\mathcal{S}))^c}$ and $p\, C\text{-min}\, E(\overline{\mathcal{D}}) = C\text{-max}\, T^*(\mathcal{D}) \neq \varnothing$.

Further the inclusion $C\text{-max}\, S^*(\mathcal{D}) \subset C\text{-min}\, E(\overline{\mathcal{D}})$ can be shown in a similar way as the inclusion $C\text{-max}\, T^*(\mathcal{S}) \subset p\, C\text{-min}\, \overline{E}(\mathcal{S})$ in part b) of the proof of Proposition 13 since $\overline{E}(\mathcal{S}) = E(\overline{\mathcal{S}})$. To show that $p\, C\text{-min}\, E(\overline{\mathcal{D}}) \subset C\text{-max}\, S^*(\mathcal{D})$, let $y \in p\, C\text{-min}\, E(\overline{\mathcal{D}})$. Then $y \in C\text{-max}\, T^*(\mathcal{D})$ so $y \in S^*(\mathcal{D})$ since $T^*(\mathcal{D}) \subset S^*(\mathcal{D})$. If $y \notin C\text{-max}\, S^*(\mathcal{D})$ then there exists $z \in S^*(\mathcal{D})$ s.t. $z \in y + C\backslash\{0\}$. Since $z \in S^*(\mathcal{D})$, we have $z \in E^*(\overline{\mathcal{D}})$. On the other hand since $y \in p\, C\text{-min}\, E(\overline{\mathcal{D}})$, $y \in C\text{-max}\, E^*(\overline{\mathcal{D}}) = C\text{-min}\, E(\overline{\mathcal{D}})$. Consequently $y \in C\text{-max}\, E^*(\overline{\mathcal{D}})$, $z \in E^*(\overline{\mathcal{D}})$ and $z \in y + C\backslash\{0\}$. We obtain a contradiction which implies that $y \in C\text{-max}\, S^*(\mathcal{D})$. It follows that $C\text{-min}\, E(\overline{\mathcal{D}}) \supset C\text{-max}\, S^*(\mathcal{D}) \supset p\, C\text{-min}\, E(\overline{\mathcal{D}}) = C\text{-max}\, T^*(\mathcal{D}) \neq \varnothing$ and the remaining follows. $\square$

We may therefore retain as dual to (P_2) either

$$(D_2): \quad E(\mathcal{D}) \cap C\text{-max}\, T^*(\mathcal{D})$$

or

$$(D_3): \quad E(\mathcal{D}) \cap C\text{-max}\, S^*(\mathcal{D}).$$

For better informations on these duals, we have:

21. Lemma: *Let* $\mathcal{D}$ *satisfy Slater's condition.*

a) Let Φ *be the set function defined on* $\mathcal{U}$ *by*

$$\Phi(U) = \{y \in F(\mathcal{D}) : y \notin F(h) + UG(h) + C\backslash\{0\} \,\forall h \in \overline{\mathcal{S}}\}.$$

Then an equivalent formulation of (D_3) *is*

$$(D_3): \quad C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}].$$

Further if $y_0 \in C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}]$ *and* $U_0 \in \mathcal{U}$ *is s.t.* $y_0 \in \Phi(U_0)$ *then* $\Phi(U_0) \subset C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}]$ *in which case such a* U_0 *is said to be a C-maximal solution to* (D_3) *and the elements of* $\Phi(U_0)$ *are C-maximal criteria values to* (D_3).

b) Let Ψ *be the set function defined on* $(\operatorname{int} C^*) \times K^*$ *by*

$$\Psi(a, u) = \{y \in F(\mathcal{D}) : ay \leq aF(h) + uG(h) \, \forall h \in \overline{S}\}.$$

Then an equivalent formulation of (D_2) *is*

$$(D_2): \quad C\text{-max} \cup [\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*].$$

If $y_0 \in C\text{-max}\cup[\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$ *with* $a_0 \in \operatorname{int} C^*$, $u_0 \in K^*$ *s.t.* $y_0 \in \Psi(a_0, u_0)$, *then* $\Psi(a_0, u_0) \subset C\text{-max} \cup [\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$ *in which case such a* (a_0, u_0) *is a C-maximal solution to* (D_2) *and the elements of* $\Psi(a_0, u_0)$ *are C-maximal criteria values to* (D_2). $\square$

Proof: Let us at first observe that $E(\mathcal{D}) \cap C\text{-max}\, S^*(\mathcal{D}) = C\text{-max}[E(\mathcal{D}) \cap S^*(\mathcal{D})]$. For if $y_0 \in E(\mathcal{D}) \cap C\text{-max}\, S^*(\mathcal{D})$ then $y_0 \in E(\mathcal{D}) \cap S^*(\mathcal{D})$ and if there exists $y_1 \in E(\mathcal{D}) \cap S^*(\mathcal{D})$ s.t. $y_1 \in y_0 + C\backslash\{0\}$, we get a contradiction to $y_0 \in C\text{-max}\, S^*(\mathcal{D})$. Conversely, let $y_0 \in C\text{-max}[E(\mathcal{D}) \cap S^*(\mathcal{D})]$. Then $y_0 \in E(\mathcal{D}) \subset E(\overline{\mathcal{D}})$ and $y_0 \in S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$, so $y_0 \in E(\overline{\mathcal{D}}) \cap E^*(\overline{\mathcal{D}})$, and it comes from part *b*) of Remarks 12 that $y_0 \in C\text{-min}\, E(\overline{\mathcal{D}}) \cap C\text{-max}\, E^*(\overline{\mathcal{D}})$. Now $y_0 \in C\text{-max}\, E^*(\overline{\mathcal{D}})$ and $y_0 \in S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$ implies that $y_0 \in C\text{-max}\, S^*(\mathcal{D})$, thus $y_0 \in E(\mathcal{D}) \cap C\text{-max}\; S^*(\mathcal{D})$. Secondly, we have $E(\mathcal{D}) \cap S^*(\mathcal{D}) = \{y \in F(\mathcal{D}) : \exists U \in \mathcal{U}, y \notin F(h) + UG(h) + C\backslash\{0\} \, \forall h \in \overline{S}\}$. $\forall U \in \mathcal{U}$, setting $\Phi(U) = \{y \in F(\mathcal{D}) : y \notin F(h) + UG(h) + C\backslash\{0\} \, \forall h \in \overline{S}\}$, we get $E(\mathcal{D}) \cap S^*(\mathcal{D}) = \cup[\Phi(U) : U \in \mathcal{U}]$.

Thirdly, let $y_0 \in C\text{-max}[E(\mathcal{D}) \cap S^*(\mathcal{D})] = C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}]$ and $U_0 \in \mathcal{U}$ s.t. $y_0 \in \Phi(U_0)$. We show the interesting fact that $\Phi(U_0) \subset C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}]$.

Let also $y_1 \in \Phi(U_0)$. Then $y_1 \in F(\mathcal{D}) = E(\mathcal{D}) \subset E(\overline{\mathcal{D}})$ and $y_1 \in S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$, so $y_1 \in E(\overline{\mathcal{D}}) \cap E^*(\overline{\mathcal{D}})$ and again part *b*) of Remarks 12 implies that $y_1 \in C\text{-min}\, E(\overline{\mathcal{D}}) \cap C\text{-}max\, E^*(\overline{\mathcal{D}})$. It follows that $y \notin y_1 + C\backslash\{0\} \, \forall y \in E^*(\overline{\mathcal{D}})$ thus $\forall y \in S^*(\mathcal{D}) \subset E^*(\overline{\mathcal{D}})$ so that $y_1 \in E(\mathcal{D}) \cap C\text{-max}\, S^*(\mathcal{D}) = C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}\}$. Similarly, $E(\mathcal{D}) \cap C\text{-max}\, T^*(\mathcal{D}) = C\text{-max}[E(\mathcal{D}) \cap T^*(\mathcal{D})]$ and $\forall a \in \operatorname{int} C^*$,

$u \in K^*$, setting $\Psi(a, u) = \{y \in F(\mathcal{D}) : ay \leq aF(h) + uG(h) \forall h \in \overline{\mathcal{S}}\}$, we get $E(\mathcal{D}) \cap T^*(\mathcal{D}) = \cup[\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$. In addition if $y_0 \in C\text{-max} \cup [\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$ with $a_0 \in \operatorname{int} C^*$, $u_0 \in K^*$ s.t. $y_0 \in \Psi(a_0, u_0)$, then $\Psi(a_0, u_0) \subset C\text{-max} \cup [\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$. $\square$

22. THEOREM (duality): *Let $\mathcal{D}$ satisfy Slater's condition.*

a) $A_0 \in \mathcal{S}$ is a proper C-minimal solution to (P_2) iff there exists $(a_0, u_0) \in (\operatorname{int} C^*) \times K^*$ C-maximal for (D_2) and s.t. $F(A_0) \in \Psi(a_0, u_0)$ and $u_0 G(A_0) = 0$ in which case $\Psi(a_0, u_0) \subset pC\text{min}\, F(\mathcal{D}) = C\text{-max} \cup [\Psi(a, u) : a \in \operatorname{int} C^*, u \in K^*]$.

b) If $A_0 \in \mathcal{S}$ is a proper C-minimal solution to (P_2) then there exists $U_0 \in \mathcal{U}$ C-maximal for (D_3) and s.t. $F(A_0) \in \Phi(U_0)$, $U_0 G(A_0) = 0$ and $\Phi(U_0) \subset C\text{-max} \cup [\Phi(U) : U \in \mathcal{U}]$ and in the case where $C\text{-min}\, E(\mathcal{D}) = p\,C\text{-min}\, E(\mathcal{D})$ then the converse holds. $\square$

Proof: The result is a consequence of Proposition 18, Proposition 20 and Lemma 21. $\square$

23. *Remarks:* We suppose that $\mathcal{D}$ satisfies Slater's condition.

1. If $\mathcal{D} = \overline{\mathcal{D}}$ (*i.e.* $\mathcal{D}$ is weakly closed, thus weakly compact), then $\varnothing \neq p\,C\text{-min}\, E(\mathcal{D}) = C\text{-max}\, T^*(\mathcal{D}) \subset C\text{-max}\, S^*(\mathcal{D})$. It is therefore non necessary to require for any $y \in \Psi(a, u)$ (resp. $y \in \Phi(U)$) that $y \in F(\mathcal{D})$. In other words we may set:

$$\Psi(a, u) = \{y \in \mathbb{R}^p : ay \leq aF(h) + uG(h) \forall h \in \overline{\mathcal{S}}\}.$$
$$\Phi(U) = \{y \in \mathbb{R}^p : y \notin F(h) + UG(h) + C\backslash\{0\} \forall h \in \overline{\mathcal{S}}\}.$$

2. $T^*(\mathcal{D}) = \{y \in \mathbb{R}^p : \exists a \in \operatorname{int} C^*, ay \leq \min[aF(h) : h \in \overline{\mathcal{D}}]\}$. Since $C^* \subset \mathbb{R}^p_+$ we may suppose all the concerned a's normalized. From Theorem 16 and Remark 17, we have:

$$\begin{aligned} T^*(\mathcal{D}) = \{&y \in \mathbb{R}^p : \exists a \in \operatorname{int} C^*, u \in K^*, \\ &ay \leq aF(h) + uG(h) \forall h \in \overline{\mathcal{S}}\} \\ = \{&y \in \mathbb{R}^p : \exists a \in \operatorname{int} C^*, u \in K^*, \\ &ay \leq a[F(h) + (u_1 e, \ldots, u_m e) G(h)] \forall h \in \overline{\mathcal{S}}\}. \end{aligned}$$

We also have:

$$\begin{aligned} S^*(\mathcal{D}) = \{&y \in \mathbb{R}^p : \exists u \in K^*, y \notin F(h) \\ &+ \ll u, G(h) \gg + C\backslash\{0\} \forall h \in \overline{\mathcal{S}}\}, \end{aligned}$$

we recall (Remark 17) that $\ll u, G(h) \gg = (u_1 e, \ldots, u_m e) G(h)$. We may therefore define Φ on K^* rather than on $\mathcal{U}$ by setting for each $u \in K^*$, $\Phi(u) = \{y \in F(\mathcal{D}) : y \notin F(h) + \ll u, G(h) \gg + C \backslash \{0\} \, \forall h \in \overline{S}\}$ so that (D_3) becomes C-max $\cup [\Phi(u) : u \in K^*]$ and if $y_0 \in C$-max $\cup [\Phi(u) : u \in K^*]$ with $u_0 \in K^*$ s.t. $y_0 \in \Phi(u_0)$, then the elements of $\Phi(u_0)$ are all C-optimal values associated with the C-optimal solution $u_0 \in K^*$. Finally Theorem 22 could consequently be reformulated.

3. In [3], the main results that are Theorem 3.1 and the general duality Theorem 4.2 provide only necessary conditions for proper C-minimality whereas we have elsewhere provided in this paragraph necessary and sufficient conditions through a duality setting. □

4. CONCLUSION

We extended convexity to vector functions defined on convex sets of measurable sets with values in an Euclidean space ordered by a closed, convex and pointed cone. We gave some properties of such convex functions. All these notions have been used to characterize, through a duality setting, cone-optimal solutions to vector optimization problems involving set vector functions.

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