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A CONTINUOUS-TIME SEARCH MODEL WITH FINITE HORIZON (*)

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Abstract. – *A variant of the continuous-time search model of Zuckerman is studied. A fixed number of identical items are for sale over a finite time horizon. Offers of i.i.d. random sizes arrive at time instances forming a renewal process; every time the salesperson decides to wait for a new offer, a fixed amount has to be paid. Search with or without recall is considered. We derive optimal strategies (maximizing the expected gain) for both cases. In the situation with recall the optimal strategy is seen to be of the form "sell all iff the maximum offer so far is greater than or equal to some threshold value", which is a non-decreasing function of the remaining selling-time. This function is given explicitly. In the case without recall the optimal strategy is of a similar, but more complicated form.*

Keywords: Search model, continuous time, renewal process, finite horizon, optimal strategy.

Résumé. – *Nous étudions une variante du modèle d'exploration à temps continu de Zuckerman. Un nombre fixé d'objets identiques est mis en vente sur un horizon de temps fini. Les offres de tailles indépendantes identiquement distribuées arrivent au hasard à des temps formant un processus de renouvellement; chaque fois que le vendeur décide d'attendre une nouvelle offre, un montant fixé doit être payé. On considère l'exploration avec ou sans rappel. Nous trouvons la stratégie optimale (maximisant le gain moyen) dans ces deux cas. Dans le cas « avec rappel », on voit que la stratégie est de la forme « vendre tout si et seulement si l'offre maximum jusqu'à présent est supérieure ou égale à un certain seuil », seuil qui est une fonction non-décroissante du temps de vente restant à courir. Cette fonction est donnée explicitement. Dans le cas « sans rappel » la stratégie optimale a une forme semblable, mais plus compliquée.*

Mots clés : Modèle de recherche, temps continu, processus de renouvellement, horizon fini, stratégie optimale.

1. INTRODUCTION

This paper is concerned with a variant of the continuous-time search model first studied by Zuckerman [10, 11, 12]. A fixed number k of identical commodities are for sale over a prespecified finite horizon. Offers arrive at random times forming a renewal process, *i.e.* the inter-arrival times are

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independent positive random variables with a common distribution function $A(t)$. The offer sizes are also i.i.d. positive random variables; their common distribution function will be denoted by $F(x)$. Selling only takes place upon the arrival of an offer; at any such time the decision to look for a new offer costs a fixed amount of $c > 0$ monetary units, even if the next offer does not appear before the deadline. Past offers are either lost (search without recall) or can be retained (search with recall). In the second case we assume that any offer extends to all still available commodities so that it is up to the salesperson to decide how many of them he/she is willing to sell at the best current price. The objective is to find a strategy ensuring the maximum expected again.

The differences of this model to the one investigated by Zuckerman are:

(*a*) There is a finite deadline so that offers can be accepted at the latest at time $T > 0$.

(*b*) The case of several commodities is treated.

(*c*) At any time, setting out to look for a new offer causes a fixed cost $c > 0$, while in Zuckerman's model cost is proportional to time.

Apparently the first version of the problem at hand was solved by Elfving [3] and Siegmund [4] (see also Chow *et al.* [2]). They considered a Poisson arrival stream, only one commodity ($k = 1$) and a general discount function instead of costs for keeping the offer stream going. This model was generalized to the case $k > 1$ by Stadje [6]. Zuckerman [10] studied the search problem with Poisson arrivals and time-proportional observation costs over an infinite time horizon. A related optimal pricing model with a finite deadline is discussed in Stadje [7].

In two subsequent papers Zuckerman [11, 12] dealt with NBU and NBUE inter-arrival distributions. He was primarily interested in finding conditions to ensure the optimality of some control-limit policy, *i.e.* a strategy for which the first offer exceeding a critical value is selected. Recently, Boshuizen and Gouweleeuw [1] extended Zuckerman's original model to the case of arbitrary (absolutely continuous) renewal arrival streams. They derived non-explicit expressions for the optimal stopping rules and showed that it can be optimal to stop strictly in between two arrivals. Further generalizations can be found in Stadje [8, 9]. In both papers the offer size distribution is allowed to be time-dependent. In [9] the arrival stream is assumed to be the superposition of several renewal processes (with the distribution of the offer sizes also depending on which process the corresponding arrival time belongs to), while in [8] it is a point process whose intensity at any time t depends on t and the number of offers received up to t.

For simplicity we assume throughout that A and F are continuous and that $0 < m = \int_0^\infty x\,dF(x) < \infty$.

2. SEARCH WITH RECALL

In this Section we suppose that at any arrival time of an offer the salesperson is allowed to sell an arbitrary subset of his/her remaining commodities at the highest price offered so far. In this case the problem of selling k commodities can be reduced to that of selling only one, since the selling decisions regarding any of the commodities will be the same. If we denote by $V_k(t, x)$ the maximum expected gain achievable from k commodities for sale over a time horizon of t time units, given that the current maximal offer is equal to x, then we have $V_k(t, x) = k\,V_1(t, x)$. Thus it suffices to consider $V(t, x) = V_1(t, x)$.

If $k = 1$, a strategy can be described as a function $\delta : (0, \infty) \times (0, \infty) \to \{0, 1\}$, where $\delta(t, x) = 1\,(0)$ means that the commodity is (not) sold if the maximum previous offer size is x and there are still t more time units to go.

The dynamic programming equation for $V(t, x)$ reads as follows:

$$\begin{aligned} V(t, x) = \max\Bigg[x,\, -c + x(1 - A(t)) \\ + \int_0^t \bigg(F(x)\,V(t - s, x) \\ + \int_x^\infty V(t - s, u)\,dF(u)\bigg)\,dA(s)\Bigg]. \quad (2.1) \end{aligned}$$

To see (2.1), note that either the item is sold at the price of x monetary units or a new offer is looked for, causing a cost of c monetary units. In the later case the item is sold at the price x after t time units if no offer has appeared before. Otherwise, the next offer arrives after time $s \le t$. Then with probability $F(x)$ it does not exceed x, leaving us with the residual expected gain $V(t - s, x)$ or the new offer is greater than x, say equal to $u > x$ (which happens with probability $dF(u)$), in which case the residual expected gain is given by $V(t - s, u)$. Summing over all these possibilities yields (2.1).

In order to find $V(t, x)$ and an optimal strategy, we introduce the set X_T of all measurable functions $f(t, x)$ on $[0, T] \times [0, \infty)$ satisfying

$$\|f\| = \sup \left\{ \frac{|f(t, x)|}{\max(1, x)} \,|x \geq 0,\ 0 \leq t \leq T \right\} < \infty. \tag{2.2}$$

Endowed with the norm defined in (2.2), X_T becomes a Banach space. For a strategy δ we define the operator $I_\delta : X_T \to X_T$ by

$$(I_\delta f)(t, x) = \begin{cases} x, & \text{if } \delta(t, x) = 1 \\ -c + x(1 - A(t)) & \\ +\displaystyle\int_0^t \Big[F(x) f(t-s, x) & \\ \qquad + \displaystyle\int_x^\infty f(t-s, u)\, dF(u) \Big] dA(s), & \text{if } \delta(t, x) = 0 \end{cases}$$

where $f \in X_T$. Further we need the operator $I : X_T \to X_T$ given by

$$(I f)(t, x) = \max \Big[x,\ -c + x(1 - A(t)) + \int_0^t \Big[F(x) f(t-s, x) + \int_x^\infty f(t-s, u)\, dF(u) \Big] dA(s) \Big].$$

Obviously we have $IV = V$. Let us show that V, considered as an element of X_T, is uniquely determined by this relation and that it can be approximated uniformly by iterating I.

THEOREM 1: *The value function V is the only fixed point of I and $\lim_{n \to \infty} \|I^n f - V\| = 0$ for every $f \in X_T$.*

Proof: Banach's fixed point theorem cannot be applied directy because I is not necessarily a contraction operator on X_T. Let $f, g \in X_T$ and fix $t, x \geq 0$. Let δ_0 be a strategy for which $(I_{\delta_0} f)(t, x) = (If)(t, x)$. If $\delta_0(t, x) = 1$ then $(I f)(t, x) - (I g)(t, x) \leq 0$, because $(I g)(t, x) \geq x$. Setting

$$\beta(x) = \max(x, 1), \qquad \beta_0 = \int_0^\infty \beta(x)\, dF(x)$$

we can now conclude that

$$\begin{aligned}(I\,f)(t,\,x)-(I\,g)(t,\,x) &\leq \max\,[0,\,(I_{\delta_0}\,f)(t,\,x)-(I_{\delta_0}\,g)(t,\,x)]\\ &\leq \Bigg|\int_0^t\Bigg[F(x)\,(f(t-s,\,x)-g(t-s,\,x))\\ &\qquad+\int_x^\infty (f(t-s,\,u)-g(t-s,\,u))\,dF(u)\Bigg]\,dA(s)\Bigg|\\ &\leq \|f-g\|\int_0^t\Bigg[F(x)\,\beta(x)+\int_x^\infty \beta(u)\,dF(u)\Bigg]\,dA(s)\\ &\leq \|f-g\|\,(\beta(x)+\beta_0)\,A(t).\end{aligned} \tag{2.3}$$

Interchanging f and g, we see that

$$\begin{aligned}|(I\,f)(t,\,x)-(I\,g)(t,\,x)| &\leq \|f-g\|\,(\beta(x)+\beta_0)\,A(t)\\ \|I\,f-I\,g\| &\leq \|f-g\|\,(1+\beta_0)\,A(T).\end{aligned} \tag{2.4}$$

By (2.4), it follows as in (2.3) that

$$\begin{aligned}&(I^2\,f)(t,\,x)-(I^2\,g)(t,\,x)\\ &\leq\Bigg|\int_0^t\Bigg[F(x)\,[(I\,f)(t-s,\,x)-(I\,g)(t-s,\,x)]\\ &\quad+\int_x^\infty [(I\,f)(t-s,\,u)-(I\,g)(t-s,\,u)]\,dF(u)\Bigg]\,dA(s)\Bigg|\\ &\leq \|f-g\|\int_0^t\Bigg[F(x)\;(\beta(x)+\beta_0)\,A(t-s)\\ &\qquad\qquad+\int_x^\infty (\beta(u)+\beta_0)\,A(t-s)\,dF(u)\Bigg]\,dA(s)\\ &\leq \|f-g\|\,(\beta(x)+2\,\beta_0)\,A_2(t),\end{aligned}$$

where $A_n(\cdot)$ denotes the n fold convolution of A with itself. Carrying out a straightforward complete induction (and using symmetry) we obtain

$$|(I^n\,f)(t,\,x)-(I^n\,g)(t,\,x)| \leq \|f-g\|\,(\beta(x)+n\,\beta_0)\,A_n(T) \tag{2.5}$$

for all $n \in \mathbb{N}$. (2.5) entails

$$\|(I^n f) - (I^n g)\| \leq (1 + n\,\beta_0)\, A_n(T)\, \|f - g\|.$$

Choose $\varepsilon > 0$ and $\delta > 0$ such that $1 - A(\varepsilon) \geq \delta$. It is easy to see that

$$A_n(T) \leq (1 - \delta^{[T/\varepsilon]+1})^{[n/([T/\varepsilon]+1)]} \leq C\, q^n$$

for some $C > 0$ and $q \in (0, 1)$, where $[\cdot]$ here denotes the integer part function (Stein's lemma; *see* Siegmund [5; p. 12]). Indeed, if $Z_1, Z_2, \ldots$ are i.i.d. random variables with distribution function A and $m = [T/\varepsilon] + 1$, then obviously,

$$\begin{aligned} A_n(T) &= P(Z_1 + \ldots + Z_n \leq T) \\ &\leq P\left(\bigcap_{i=1}^{[n/m]} \{Z_{(i-1)\,m+1} + \ldots + Z_{im} \leq T\}\right) \\ &= P(Z_1 + \ldots + Z_m \leq T)^{[n/m]} \\ &\leq (1 - P(Z_1 > \varepsilon)^m)^{[n/m]}. \end{aligned}$$

In particular, $\|(I^n f) - (I^n g)\| \leq \alpha\, \|f - g\|$ for some $\alpha < 1$ if n is sufficiently large, say $n \geq n_0$. Thus I^{n_0} is a contraction on X_T and has exactly one fixed point. The relation $I f = f$ implies $I^{n_0} f = f$, and consequently, $f = V$. Further we have

$$\|I^{j n_0} f - V\| \leq \alpha^j\, \|f - V\|/(1 - \alpha) \to 0, \quad \text{as } j \to \infty,$$

so that for every $f \in X_T$

$$\|I^n f - V\| \leq \max_{0 \leq l < n_0} \|I^{[n/n_0]\, n_0} (I^l f) - V\| \to 0, \quad \text{as } n \to \infty.$$

The Theorem is proved.

Note that

$$V(t, x) = x + \max[0, \Delta(t, x)], \tag{2.6}$$

where

$$\Delta(t, x) = -c - x\, A(t) + \int_0^t \left[F(x)\, V(t - s, x) + \int_x^\infty V(t - s, u)\, dF(u)\right] dA(u).$$

THEOREM 2: *The function* $\Delta(t, x)$ *is non-decreasing in* t *and non-increasing in* x.

Proof: Since $t \mapsto V(t, x)$ is obviously non-decreasing, so is $t \mapsto \Delta(t, x)$.

Let $V^{(0)}(t, x) = x$ and $V^{(n)}(t, x) = (I^n V^{(0)})(t, x)$, $n \geq 1$. It is easily seen that $(I f)(t, x)$ is non-decreasing in x if $f(t, x)$ has this property. Thus, all $V^{(n)}(t, x)$ are non-decreasing in x. $V^{(n)}(t, x)$ is the maximum expected gain if at most n future offers are allowed, given that the initial offer size is x and the horizon is t. Therefore $n \mapsto V^{(n)}(t, x)$ is non-decreasing. By Theorem 1,

$$\sup_{n \geq 1} V^{(n)}(t, x) = V(t, x). \tag{2.7}$$

Let

$$\Delta^{(n)}(t, x) = -c - x A(t) + \int_0^t \Big[F(x) V^{(n-1)}(t-s, x) + \int_x^\infty V^{(n-1)}(t-s, u)\, dF(u) \Big] dA(s), \quad n \geq 1. \tag{2.8}$$

By (2.7), (2.8) and monotone convergence, it follows that $\Delta^{(n)}(t, x) \to \triangle(t, x)$, as $n \to \infty$. Thus it suffices to prove that every function $\Delta^{(n)}$ is non-increasing in x. This can be shown by induction.

For $n = 1$ we have

$$\Delta^{(1)}(t, x) = -c + A(t) \int_x^\infty (u - x)\, dF(u), \tag{2.9}$$

and the right-hand side of (2.9) is clearly non-increasing in x. For the induction step from n to $n + 1$ we note that

$$V^{(n)}(t, x) - x = \max[\Delta^{(n)}(t, x), 0].$$

We can now write

$$\begin{aligned}\Delta^{(n+1)}(t, x) &= -c + \int_0^t \Big[\int_x^\infty (V^{(n)}(t-s, u) - V^{(n)}(t-s, x))\, dF(u) + V^{(n)}(t-s, x) - x \Big] dA(s) \\ &= -c + \int_0^t \Big[\int_x^\infty (V^{(n)}(t-s, u) - V^{(n)}(t-s, x))\, dF(u) + \max[\Delta^{(n)}(t-s, x), 0] \Big] dA(s). \end{aligned} \tag{2.10}$$

The two summands in the integral over s on the right-hand side of (2.10) are non-increasing in x. To see this, note that the inner integral $\int_x^\infty$ is equal to

$$E\left([V^{(n)}(t-s,\,U) - V^{(n)}(t-s,\,x)]^+\right),$$

where U is a random variable distributed according to F, and that $x \mapsto V^{(n)}(t-s,\,x)$ is non-decreasing. Further, the function $x \mapsto \max[\Delta^{(n)}(t-s,\,x),\,0]$ is non-decreasing by the induction hypothesis. This yields the assertion for $\Delta^{(n+1)}$. The Theorem is proved.

Theorem 2 implies that there is a non-decreasing function $h(t)$ such that $\Delta(t,\,x) \le 0$ iff $x \ge h(t)$. Since $V(t,\,x) = x$ iff $\Delta(t,\,x) \le 0$ [*see* (2.6)], the strategy δ^* defined by

$$\delta^*(t,\,x) = 1 \quad \text{iff } x \ge h(t)$$

is optimal. The function $h(t)$ can be given explicitly as follows. Let $R(\alpha) = \int_0^\infty (u-\alpha)^+\,dF(u)$, $\alpha \in \mathbb{R}$. Note that R is strictly decreasing on the interval $\{\alpha | R(\alpha) > 0\}$ and continuous everywhere. One has $(0,\,m] \subset R(\mathbb{R}) \subset [0,\,m]$, and $0 \in R(\mathbb{R})$ iff F is concentrated on a bounded interval.

THEOREM 3: *The function* $h(t)$ *is given by*

$$h(t) = \begin{cases} \alpha(t), & \text{if } A(t) > 0, \quad c \le m\,A(t) \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = \alpha(t)$ *is the unique solution of the equation* $R(\alpha) = c/A(t)$.

Proof: If $A(t) = 0$, there will be no offer within the next t time units so that $h(t) = 0$. Thus let $A(t) > 0$. Clearly, $V(t,\,x) = x$ for $x \ge h(t)$. Since h is non-decreasing, it follows that $V(t-s,\,u) = u$ for $s \in [0,\,t]$ and $u \ge h(t)$. From the definition of $\Delta(t,\,x)$ we can conclude that for every $x \ge h(t)$

$$\begin{aligned}\Delta(t,\,x) &= -c - x\,A(t) + \int_0^t \left[F(x)\,x + \int_x^\infty u\,dF(u)\right] dA(s) \\ &= -c + A(t)\,R(x).\end{aligned} \tag{2.11}$$

If $c \le m\, A(t)$, the right-hand side of (2.11) is non-positive iff $x \ge \alpha(t)$; if $c > m\, A(t)$, it is negative for all $x \ge 0$. This proves the Theorem.

The optimal strategy has the following myopic property: An offer is accepted if it is at least as large as the expected value of the reward obtained by stopping at the next offer, if there is any, or at the end of the horizon if there is no more offer.

3. SEARCH WITHOUT RECALL

In this Section every offer is valid for only one item and cannot be retained after being rejected. We assign the value zero to items which are not sold before the deadline. Once an offer has arrived, there are three possible decisions: (*a*) sell one item and look for a new offer, (*b*) sell one item and stop the process, (*c*) reject the offer and look for the next one. For $k = 1$ one will clearly never decide for option (a).

Let $W(k, t, x)$ be the maximum expected gain, given that k commodities are for sale over a time horizon t and an offer of size x has just arrived. The dynamic programming equation for $W(k, t, x)$ is given by

$$\begin{aligned} W(k, t, x) &= \max \Big[x - c + \int_0^t \int_0^\infty W(k-1, t-s, u)\, dF(u)\, dA(s), \\ &\qquad x,\ -c + \int_0^t \int_0^\infty W(k, t-s, u)\, dF(u)\, dA(s) \Big], \quad k \ge 1 \\ W(0, t, x) &= 0. \end{aligned} \tag{3.1}$$

A strategy is now a function $\delta : \{0, 1, \ldots, k\} \times (0, \infty) \times [0, \infty) \to \{0, 1, 2\}$, where $\delta(j, t, x) = 0$ (or 1 or 2) means that in state (j, t, x) decision (*a*) [or (*b*) or (*c*)] is made. Proceeding along the same lines as in Section 2, we introduce the Banach space Y_T or all measurable functions $f : \{0, 1, \ldots, k\} \times [0, T] \times [0, \infty) \to \mathbb{R}$ satisfying $f(0, t, x) \equiv 0$ and

$$\|f\| = \sup \left\{ \frac{|f(j, t, x)|}{\max(x, 1)} \,\middle|\, j \in \{1, \ldots, k\},\ 0 \le t \le T,\ x \ge 0 \right\} < \infty, \tag{3.2}$$

where Y_T is of course endowed with the norm defined by (3.2). The operators $J_\delta : Y_T \to Y_T$ (for a given strategy δ) and $J : Y_T \to Y_T$ are given by

$$(J_\delta f)(j, t, x) = \begin{cases} x - c + \int_0^t \int_0^\infty f(j-1, t-s, u) \, dF(u)\, dA(s), & \text{if } \delta(j, t, x) = 0 \\ x, & \text{if } \delta(j, t, x) = 1 \\ -c + \int_0^t \int_x^\infty f(j, t-s, u) \, dF(u)\, dA(s), & \text{if } \delta(j, t, x) = 2 \end{cases}$$

$$(J f)(j, t, x) = \max \left[x - c + \int_0^t f(j-1, t-s, u)\, dF(u)\, A(s), \; x, \; -c + \int_0^t \int_0^\infty f(j, t-s, u)\, dF(u)\, dA(s) \right].$$

THEOREM 4: *J^{n_0} is a contraction operator on Y_T for some $n_0 \in \mathbb{N}$. The value function W is its only fixed point. For every $f \in Y_T$, $\|J^n f - W\|$ tends to zero exponentially fast, as $n \to \infty$.*

Proof: We only have to show the contraction property of J^{n_0} for some $n_0 \in \mathbb{N}$, the other assertions then being obvious. Let $f, g \in Y_T$, $(j, t, x) \in \{1, \ldots, k\} \times (0, \infty) \times [0, \infty)$ and let δ_0 be a strategy satisfying $(J_{\delta_0} f)(j, t, x) = (J f)(j, t, x)$. Further recall the constant $\beta_0 = \int_0^\infty \max(u, 1)\, dF(u)$. As in the proof of Theorem 1 we obtain

$$\begin{aligned} (J f)(j, t, x) - (J g)(j, t, x) &= (J_{\delta_0} f)(j, t, x) - (J g)(j, t, x) \\ &\le \max \left[\left| \int_0^t \int_0^\infty (f(j-1, t-s, u) - g(j-1, t-s, u))\, dF(u)\, dA(s) \right|, \right. \\ &\qquad \left. \left| \int_0^t \int_0^\infty (f(j, t-s, u) - g(j, t-s, u))\, dF(u)\, dA(s) \right| \right] \\ &\le \|f - g\|\, A(t)\, \beta_0. \end{aligned}$$

It follows that

$$\|J f - J g\| = \sup \left\{ \frac{|(J f)(j, t, x) - (J g)(j, t, x)|}{\max(x, 1)} \,\middle|\, \begin{array}{c} j \in \{1, \ldots, k\} \\ 0 \leq t \leq T, \, x \geq 0 \end{array} \right\}$$
$$\leq \|f - g\| \, \beta_0 \, A(T).$$

By induction, it is seen that

$$\|J^n f - J^n g\| \leq A_n(T) \, \beta_0 \, \|f - g\|, \qquad n \in \mathbb{N}.$$

Since $A_n(T) \to 0$, as $n \to \infty$, J^n is a contraction operator for n sufficiently large.

Now let $A^{-1}(x) = \sup\{s \geq 0 | A(s) \leq x\}$, $x \geq 0$, and define the function $g(j, t)$ by

$$g(j, t) = \int_0^t \int_0^\infty [W(j, t - s, u) - W(j - 1, t - s, u)] \, dF(u) \, dA(s).$$

THEOREM 5: *The following strategy is optimal*:

$$\delta(j, t, x) = \left\{ \begin{array}{ll} 0, & \text{if } x \geq g(j, t), \, t \geq A^{-1}(c/m) \\ 1, & \text{if } t < A^{-1}(c/m) \\ 2, & \text{if } x < g(j, t), \; t \geq A^{-1}(c/m) \end{array} \right\} \tag{3.3}$$

Proof: From the definition of W it is obvious that $j \mapsto W(j, t, x)$ is non-decreasing. Thus, in state (j, t, x) it is optimal to sell one item and then stop the process iff

$$-c + \int_0^t \int_0^\infty W(j, t - s, u) \, dF(u) \, dA(s) \leq 0. \tag{3.4}$$

Let us show that (3.4) is equivalent to $t \leq A^{-1}(c/m)$ (which in turn is tantamount to $A(t) \leq c/m$). As $W(j, t, u) \geq u$, we have

$$\int_0^t \int_0^\infty W(j, t - s, u) \, dF(u) \, dA(s) \geq m \, A(t).$$

Hence, (3.4) does not hold if $A(t) > c/m$.

Now assume that $A(t) \leq c/m$. Let $W^{(n)}(j, t, x)$ be the maximum expected gain achievable when starting at (j, t, x) under the condition that at most n future offers are permitted. Clearly, $W^{(n)}(j, t, x)$ in non-decreasing in n and satisfies

$$\left.\begin{aligned} W^{(n)}(j, t, x) &= \max \left[x - c + \int_0^t \int_0^\infty W^{(n-1)}(j-1, t-s, u)\right. \\ &\qquad \left. dF(u)\, dA(s),\ x,\ -c + \int_0^t \int_0^\infty dF(u)\, dA(s)\right], \\ &\qquad n \geq 1, \quad j \geq 1 \\ W^{(n)}(0, t, x) &= 0 \\ W^{(0)}(j, t, x) &= x, \quad j \geq 1. \end{aligned}\right\} \tag{3.5}$$

By Theorem 3,

$$W^{(n)}(j, t, x) \nearrow W(j, t, x), \quad \text{as } n \to \infty. \tag{3.6}$$

We prove by induction on n that $A(t) \leq c/m$ implies that

$$-c + \int_0^t \int_0^\infty W^{(n)}(j, t-s, u)\, dF(u)\, dA(s) \leq 0, \qquad n \in \mathbb{Z}_+. \tag{3.7}$$

Using the monotone convergence theorem, we conclude (3.4) from (3.6) and (3.7). (3.7) is trivial for $j = 0$. So let $j \geq 1$. Then

$$\int_0^t \int_0^\infty W^{(0)}(j, t-s, u)\, dF(u)\, dA(s) = m\, A(t),$$

so that the assertion holds for $n = 0$. Now suppose (3.7) is true for $n - 1$ instead of n for some $n \in \mathbb{N}$. Then (3.5) yields $W^{(n)}(j, t-s, u) = u$ for all $u \geq 0$ and $s \leq t$. Therefore,

$$\int_0^t \int_0^\infty W^{(n)}(j, t-s, u)\, dF(u)\, dA(s) = m\, A(t) \leq c,$$

and the induction is complete.

It remains to consider the case $t \geq A^{-1}(c/m)$. Then the decision to sell and stop is not taken, and it follows from (3.1) that it is optimal to sell one item and look for further offers iff

$$x - c + \int_0^t \int_0^\infty W(k-1,\, t-s,\, u)\, dF(u)\, dA(s) \\ \geq -c + \int_0^t \int_0^\infty W(k,\, t-s,\, u)\, dF(u)\, dA(s).$$

It is now easily seen that the strategy given by (3.3) is optimal.

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