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ON A CONVOLUTION OPERATION OBTAINED BY ADDING LEVEL SETS: CLASSICAL AND NEW RESULTS (*)

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Abstract. – *The purpose of this paper is studying the convolution* $x \in X \mapsto \operatorname{Inf}\{f(x-v) \vee g(v) : v \in X\}$ *of two functions f and g. Researchers working in different branches of mathematics have already encountered this operation, but a systematic study of it has not been yet undertaken. We will review most of the known properties of this operation, and establish several new results.*

Keywords: Level sum, Lipschitzian approximation, Fenchel conjugate, subdifferential.

Résumé. – *On étudie la convolution* $x \in X \mapsto \operatorname{Inf}\{f(x-v) \vee g(v) : v \in X\}$ *de deux fonctions f et g. Cette opération se rencontre dans diverses branches des mathématiques, mais son étude systématique n'a pas été entreprise. On rappelle ici la plupart des propriétés connues de cette opération et nous établissons plusieurs nouveaux résultats.*

Mots clés : Somme en niveaux, approximation lipchitzienne, conjugaison de Fenchel, sous-différentiel.

1. INTRODUCTION

Let f and g be two real valued functions defined over a (real) linear space X. Convoluting f and g amounts to splitting the variable $x \in X$ in the form $x = (x - v) + v$, and then mixing the pairs $\{(f(x-v), g(v)) : v \in X\}$. Various ways of performing this mixture gives rise to different concepts of convolution. For instance, integrating all the products $\{f(x-v)\, g(v) : v \in X\}$ yields the usual convolution appearing in functional analysis, and taking the infimum of all the sums $\{f(x-v) + g(v) : v \in X\}$ yields the inf-convolution operation appearing in convex analysis. Besides these two well known examples, one can also

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consider the convolution

$$x \in X \mapsto [f \Delta g](x) := \operatorname*{Inf}_{v \in X} \{ f(x-v) \vee g(v) \} \tag{1.1}$$

and its symmetric version

$$x \in X \mapsto [f \nabla g](x) := \operatorname*{Sup}_{v \in X} \{ f(x-v) \wedge g(v) \}. \tag{1.2}$$

Here the symbols $\vee$ and $\wedge$ have their usual meaning, *i.e.*,

$$\left.\begin{aligned} a \vee b &:= \text{maximum of } a \in \mathbb{R} \text{ and } b \in \mathbb{R}, \\ a \wedge b &:= \text{minimum of } a \in \mathbb{R} \text{ and } b \in \mathbb{R}. \end{aligned}\right\} \tag{1.3}$$

If the functions f and g have values on the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, definitions (1.1)-(1.2) remain valid with the obvious rule

$$\left.\begin{aligned} a \vee (+\infty) &= (+\infty) \vee a = +\infty \text{ for all } a \in \overline{\mathbb{R}}, \\ a \wedge (-\infty) &= (-\infty) \wedge a = -\infty \text{ for all } a \in \overline{\mathbb{R}}. \end{aligned}\right\} \tag{1.4}$$

Several important applications justify the study of the operations (1.1) and (1.2). This point is illustrated with the help of the following examples.

Example 1.1 (on the algebra of fuzzy sets): Let the universe of discourse X be a linear space, and let $\mu_A : X \to [0, 1]$ denote the membership function of a fuzzy set A in X. Thus, $\mu_A(x) = 0$ means that x does not belong to A, $\mu_A(x) = 1$ means that x belongs to A, while $0 < \mu_A(x) < 1$ means that x partially belongs to A. In other words, $\mu_A(x)$ measures the grade of membership of x in A. The definitions of the basic operations of the algebra of fuzzy sets are usually given in terms of the respective membership functions. For instance, the sum $A \oplus B$ of the fuzzy sets A and B is given by

$$\mu_{A \oplus B}(x) = \operatorname*{Sup}_{v \in X} \{ \mu_A(x-v) \wedge \mu_B(v) \} \quad \text{for all } x \in X.$$

This corresponds, of course, to the operation introduced in (1.2). The operation (1.1) appears if one passes to the complements. The complement $\tilde{C}$ of a fuzzy set C in X is defined by

$$\mu_{\tilde{C}}(x) := 1 - \mu_C(x) \quad \text{for all } x \in X.$$

Thus, $\mu_{\tilde{C}}(x)$ measures the grade of nonmembership of x with respect to C. A straightforward calculus shows that

$$\mu_{\widetilde{A \oplus B}}(x) = \operatorname*{Inf}_{v \in X} \{ \mu_{\tilde{A}}(x-v) \vee \mu_{\tilde{B}}(v) \} \quad \text{for all } x \in X.$$

For more material on this example, the reader may consult Dubois and Prade [8] or Fedrizzi [10].

Example 1.2 (on the distance function): On a normed linear space $(X, \|\cdot\|)$, the distance of a point $x \in X$ to a nonempty subset $C \subset X$ is defined as the number

$$d_C(x) = \operatorname*{Inf}_{v \in C} \|x - v\|.$$

The distance function $d_C : X \to \mathbb{R}$ is just a particular case of a function obtained by performing a convolution like in (1.1). Indeed, if one introduces the indicator function

$$x \in X \mapsto \Psi_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

of the set C, then one can write

$$d_C(x) = [\|\cdot\| \Delta \Psi_C](x) \quad \text{for all } x \in X.$$

Example 1.3 (on cost minimization): Suppose two factories have to join their effort in order to produce a given vector $x \in \mathbb{R}^n_+$ of commodities. If the first factory produces $x_1 \in \mathbb{R}^n_+$, then its corresponding cost is $C_1(x_1)$. Similarly, $C_2(x_2)$ represents the cost for the second factory when its production level is $x_2 \in \mathbb{R}^n_+$. Which is the best way of splitting $x = x_1 + x_2$ if one desires to render the maximum cost $C_1(x_1) \vee C_2(x_2)$ as small as possible? Answering to this question amounts to solving the minimization problem

$$\operatorname*{Minimize}_{x_1, x_2 \in \mathbb{R}^n_+} \{ C_1(x_1) \vee C_2(x_2) : x_1 + x_2 = x \}. \tag{1.5}$$

Up to a minor modification, this corresponds to the convolution operation mentioned in (1.1). Indeed, the optimal value of problem (1.5) is precisely $[\overline{C}_1 \Delta \overline{C}_2](x)$, where

$$\overline{C}_i(x_i) = \begin{cases} C_i(x_i) & \text{if } x_i \in \mathbb{R}^n_+ \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for} \quad i = 1, 2.$$

Example 1.4 (on utility maximization): Suppose two consumers wish to share a given vector $x \in \mathbb{R}^n_+$ of commodities. Let $U_1(x_1)$ represents the utility of the first consumer derived from the consumption of $x_1 \in \mathbb{R}^n_+$. The term $U_2(x_2)$ is interpreted in a similar way. Which is the best way of splitting

$x = x_1 + x_2$ if one desires to render the minimum utility $U_1(x_1) \wedge U_2(x_2)$ as large as possible? This time one has to solve

$$\underset{x_1, x_2 \in \mathbb{R}^n_+}{\text{Maximize}} \{ U_1(x_1) \wedge U_2(x_2) \,:\, x_1 + x_2 = x \}. \tag{1.6}$$

The optimal value of problem (1.6) is just $[\overline{U}_1 \nabla \overline{U}_2](x)$, where

$$\overline{U}_i(x_i) = \begin{cases} U_i(x_i) & \text{if } x_i \in \mathbb{R}^n_+ \\ -\infty & \text{otherwise} \end{cases} \qquad \text{for} \quad i = 1, 2.$$

We thank J.-P. Crouzeix (Clermont-Ferrand) for providing us with this example.

Example 1.5 (a reliability problem): The present example is taken from a recent paper by M. Kolonko [14]. Consider a replacement scheme for a system of two components. Let $U_i(x_i)$ denote the lifetime of the i-th component under the "load" $x_i \in \mathbb{R}_+$. E.g., consider an air filter that is worn out after time $U_i(x_i)$ if the dust concentration is x_i. Replacement of components may be either sequential or "parallel". In the later case both components are put into work at the same time, the load $x \in \mathbb{R}_+$ is shared as $x = x_1 + x_2$ between the partners, and the joint lifetime is given by $U_1(x_1) \wedge U_2(x_2)$. Note that here the system fails as soon as one of its components fails. E.g. a pair of filters adjusted to cope with concentration x_1 and x_2 respectively can no longer clear the full amount x if one of the filters fails. The question here is how to split the load x in order to render the lifetime of the system as large as possible.

There are cases in which the operation ∇ plays a more natural role than its symmetric version Δ. For passing from the former operation to the latter, it suffices to apply the identity

$$f \nabla g = -[(-f) \Delta (-g)].$$

So, for the sake of simplicity in our exposition, we will focuss our attention only toward the operation Δ.

Besides having the specific interpretations pointed out in the previous examples, the operation Δ serves as general tool of analysis for handling various types of theoretical questions. For instance, under the name of "quasi-inf-convolution", it has been used by Volle [28, 29, 30] in connection with the approximation, regularization, and variational convergence of functions. Additional results concerning the operation Δ appear in Rockafellar [21, p. 40], Hassouni [11, p. 12], Penot and Volle [20, p. 215], Abdulfattah and

Soueycatt [1, Section 1.4], Seeger [24], and Elqortobi [9] among others. A special mention is deserved by the important contribution of Kusraev and Kutateladze [15].

2. GENERAL PROPERTIES OF THE CONVOLUTION Δ

2.1. The operation Δ interpreted as level set addition

From now on, the function $f\Delta g$ will be referred to as the *level sum* of f and g. This terminology has the disadvantage of hidding the convolutive nature of $f\Delta g$, but it underlines the fact that this new function can be obtained by adding the level sets of f and g. A more precise statement is recorded next. In what follows, the notation

$$\{h < \alpha\} := \{x \in X : h(x) < \alpha\}$$

stands for the strict (lower) level set of $h : X \to \overline{\mathbb{R}}$ at the level $\alpha \in \mathbb{R}$. Addition of sets in X is always understood in the sense of Minkowski, that is to say,

$$A + B := \{a + b : a \in A,\, b \in B\},$$

with the convention $A + \emptyset = \emptyset + A = \emptyset$.

PROPOSITION 2.1 (Rockafellar [21, p. 40]): *Let f and g be in $\overline{\mathbb{R}}^X$. Then, for all $\alpha \in \mathbb{R}$, one has*

$$\{f\Delta g < \alpha\} = \{f < \alpha\} + \{g < \alpha\}. \tag{2.1}$$

The proof of Proposition 2.1 is straightforward. What this result says is that the function $f\Delta g$ is entirely defined by the sum of the level sets $\{f < \alpha\}$ and $\{g < \alpha\}$. Indeed, it says that

$$[f\Delta g](x) = \operatorname{Inf}\{\alpha \in \mathbb{R} : x \in \{f < \alpha\} + \{g < \alpha\}\} \quad \text{for all } x \in X.$$

As an immediate consequence of Proposition 2.1, one gets

$$\operatorname{dom}(f\Delta g) = \operatorname{dom} f + \operatorname{dom} g,$$

where

$$\operatorname{dom} h := \{x \in X : h(x) < +\infty\}$$

stands for the effective domain of $h \in \overline{\mathbb{R}}^X$. An equality like (2.1) does not hold in general for level sets of the form

$$\{ h \leq \alpha \} := \{ x \in X : h(x) \leq \alpha \}.$$

However, if $f \Delta g$ is *exact* (in the sense that, for each $x \in X$, the infimum in (1.1) is attained), then one can write

$$\{ f \Delta g \leq \alpha \} = \{ f \leq \alpha \} + \{ g \leq \alpha \}$$

for all $\alpha \in \mathbb{R}$.

2.2. The role of epigraphs

As mentioned before, level sets play an important role in the analysis of the operation Δ. We point out, however, that epigraphs have also their word to say. Recall that the strict epigraph of a function $h \in \overline{\mathbb{R}}^X$ is defined by

$$\operatorname{epi}_s h := \{ (x, \alpha) \in X \times \mathbb{R} : h(x) < \alpha \}.$$

Since this set lies in the product space $X \times \mathbb{R}$, it can be viewed as the graph of a multivalued mapping from X into $\mathbb{R}$. Indeed, one has

$$\operatorname{epi}_s h = \operatorname{Graph} E_h := \{ (x, \alpha) \in X \times \mathbb{R} : \alpha \in E_h(x) \},$$

where $E_h : X \rightrightarrows R$ is the mapping given by

$$\operatorname{dom} E_h = \{ x \in X : h(x) < +\infty \},$$

$$E_h(x) :=]\, h(x), +\infty[\quad \text{for all } x \in \operatorname{dom} E_h.$$

Notice that strict level sets of h correspond to the values of the inverse mapping $E_h^{-1} : \mathbb{R} \rightrightarrows X$. More precisely,

$$E_h^{-1}(\alpha) := \{ x \in X : \alpha \in E_h(x) \} = \{ h < \alpha \} \quad \text{for all } \alpha \in \mathbb{R}.$$

To fully appreciate next result, recall that two mappings, say $E_f : X \rightrightarrows \mathbb{R}$ and $E_g : X \rightrightarrows \mathbb{R}$, can be added either in series

$$(E_f + E_g)(x) := E_f(x) + E_g(x) \quad \text{for all } x \in X,$$

or in parallel

$$(E_f \,\|\, E_g)(x) := (E_f^{-1} + E_g^{-1})^{-1}(x) \quad \text{for all } x \in X.$$

A detailed discussion on these operations can be found in Passty [19]. Without further ado we state:

PROPOSITION 2.2: *Let f and g be in $\overline{\mathbb{R}}^X$. Then, the strict* epigraph *mapping $E_{f\Delta g} : X \rightrightarrows \mathbb{R}$ associated to $f\Delta g$, coincides with the parallel sum* of the strict epigraph *mappings $E_f : X \rightrightarrows \mathbb{R}$ and $E_g : X \rightrightarrows \mathbb{R}$. In short,*

$$E_{f\Delta g} = E_f \parallel E_g. \tag{2.2}$$

Proof: Equality (2.2) amounts to saying that

$$E_{f\Delta g}^{-1}(\alpha) = E_f^{-1}(\alpha) + E_g^{-1}(\alpha) \quad \text{for all } \alpha \in \mathbb{R}.$$

But this is just another way to express formula (2.1) stated in Proposition 2.1. ■

2.3. On the minimization of $f\Delta g$

Next we state a couple of results relating the minimization of the level sum $f\Delta g$ and the separate minimization of the component functions f and g. As customary, the notation

$$\varepsilon\text{-argmin}\, h := \{x \in X : h(x) \leq \varepsilon + \text{Inf}\, h\}$$

refers to the set of points which minimize the function $h \in \overline{\mathbb{R}}^X$ within a tolerance $\varepsilon \in \mathbb{R}_+$ or, in short, the set of ε-minima of h. The above notation will be used only in the case in which h is minorized from below. When $\varepsilon = 0$, we write simply

$$\text{argmin}\, h = \{x \in X : h(x) \leq \text{Inf}\, h\}.$$

The infimal value

$$\text{Inf}\, h := \text{Inf}_{x \in X}\, h(x)$$

is, of course, understood as an extended real number.

PROPOSITION 2.3: *Let f and g be in $\overline{\mathbb{R}}^X$. Then,*

$$\text{Inf}\,(f\Delta g) = (\text{Inf}\, f) \vee (\text{Inf}\, g). \tag{2.3}$$

Proof: Formula (2.3) is a consequence of the following chain of equalities:

$$\begin{aligned}\text{Inf}_{x \in X}\,(f\Delta g)(x) &= \text{Inf}_{x \in X} \text{Inf}_{\substack{u, v \in X \\ u+v=x}} \{f(u) \vee g(v)\} \\ &= \text{Inf}_{u, v \in X} \{f(u) \vee g(v)\} = [\text{Inf}_{u \in X}\, f(u)] \vee [\text{Inf}_{v \in X}\, g(v)].\end{aligned}$$

The last equality is a particular case of the general identity

$$\operatorname*{Inf}_{i \in I, j \in J} \{ a_i \vee b_j \} = [\operatorname*{Inf}_{i \in I} a_i] \vee [\operatorname*{Inf}_{j \in J} b_j],$$

where the $a_i{}'s$ and $b_j{}'s$ are extended real numbers, and the set of indices I and J are possibly infinite. ■

It has been observed by Abdulfattah and Soueycatt [1] that if u and v are ε-minima of f and g, respectively, then $u + v$ is an ε-minimum of $f \Delta g$. In other words,

$$\varepsilon\text{-argmin}\, f + \varepsilon\text{-argmin}\, g \subset \varepsilon\text{-argmin}\,(f \Delta g) \quad \text{for all } \varepsilon \in \mathbb{R}_+.$$

It is not difficult to construct an example which shows that the above inclusion may be strict. An exact estimate of the set ε-argmin $(f \Delta g)$ is given in next proposition. The notation γ_+ refers to the positive part of the number $\gamma \in \mathbb{R}$, *i.e.*, $\gamma_+ = \operatorname{Max}\{0, \gamma\}$. Recall that a function $h : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if dom h is nonempty.

PROPOSITION 2.4: *Let f, g : $X \to \mathbb{R} \cup \{+\infty\}$ be two proper functions which are minorized from below, that is to say, assume that the numbers $\alpha = \operatorname{Inf} f$ and $\beta = \operatorname{Inf} g$ are finite. Then, for all $\varepsilon \in \mathbb{R}_+$, one has*

$$\begin{aligned} \varepsilon\text{-argmin}\,(f \Delta g) = \bigcap_{\delta > \varepsilon} [(\delta + (\beta - \alpha)_+)&\text{-argmin}\, f \\ &+ (\delta + (\alpha - \beta)_+)\text{-argmin}\, g]. \end{aligned} \tag{2.4}$$

Proof: According to Proposition 2.3 one has

$$\varepsilon\text{-argmin}\,(f \Delta g) = \{ f \Delta g \leq \gamma + \varepsilon \}, \tag{2.5}$$

with $\gamma = \alpha \vee \beta$. A general result concerning level sets allows us to write

$$\{ f \Delta g \leq \gamma + \varepsilon \} = \bigcap_{\delta > \varepsilon} \{ f \Delta g < \gamma + \delta \}. \tag{2.6}$$

By combining (2.5), (2.6) and Proposition 2.1, one obtains

$$\varepsilon\text{-argmin}\,(f \Delta g) = \bigcap_{\delta > \varepsilon} [\{ f < \gamma + \delta \} + \{ g < \gamma + \delta \}].$$

A simple calculus shows that the above equality can also be written in the form

$$\varepsilon\text{-argmin}\,(f\Delta g) = \bigcap_{\delta>\varepsilon} [\{ f \leq \gamma + \delta \} + \{ g \leq \gamma + \delta \}].$$

Since $\gamma = \alpha + (\beta - \alpha)_+ = \beta + (\alpha - \beta)_+$, one gets finally

$$\varepsilon\text{-argmin}\,(f\Delta g) = \bigcap_{\delta>\varepsilon} [\{ f \leq \alpha + (\beta - \alpha)_+ + \delta \} + \{ g \leq \beta + (\alpha - \beta)_+ + \delta \}].$$

This is precisely what we wanted to demonstrate. ■

We end this section by mentioning two important cases in which the expression for ε-argmin $(f\Delta g)$ reduces to a very simple form.

COROLLARY 2.1: *Let* $f,\ g\ :\ X \to \mathbb{R} \cup \{ +\infty \}$ *be two proper functions such that* $\operatorname{Inf} f = \operatorname{Inf} g \in \mathbb{R}$. *Then, for all* $\varepsilon \in \mathbb{R}_+$, *one has*

$$\varepsilon\text{-argmin}\,(f\Delta g) = \bigcap_{\delta>\varepsilon} [\delta\text{-argmin}\, f + \delta\text{-argmin}\, g].$$

COROLLARY 2.2: *Let* $f,\ g\ :\ X \to \mathbb{R} \cup \{ +\infty \}$ *be two proper functions such that the numbers* $\alpha = \operatorname{Inf} f$ *and* $\beta = \operatorname{Inf} g$ *are finite. Suppose that for each* $x \in X$, *the infimum*

$$(f\Delta g)(x) = \operatorname*{Inf}_{v\in X}\ \{ f(x - v) \vee g(v) \}$$

is attained. Then, for all $\varepsilon \in \mathbb{R}_+$, *one has*

$$\varepsilon\text{-argmin}\,(f\Delta g) = (\varepsilon + (\beta - \alpha)_+)\text{-argmin}\, f + (\varepsilon + (\alpha - \beta)_+)\text{-argmin}\, g.$$

3. LIPSCHITZIAN APPROXIMATION VIA LEVEL ADDITION

In this section the linear space X is supposed to be equipped with a norm denoted by $\|\,.\,\|$. Recall that a function $g\ :\ X \to \mathbb{R}$ is said to be *Lipschitzian* with Lipschitz constant $L \in \mathbb{R}_+$ if

$$|\, g(x) - g(y) \,| \leq L \,\|\, x - y \,\| \quad \text{for all } x,\ y \quad \text{in } X.$$

In many areas, including optimization theory and nonlinear analysis, it is important to construct a Lipschitzian approximation of a given function. In this respect, the operation of level addition has a fruitful role to play. To convince the reader of this fact, we start by establishing the following basic result.

PROPOSITION 3.1: *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If $g : X \to \mathbb{R}$ is minorized from below and Lipschitzian with Lipschitz constant $L \in \mathbb{R}_+$, then so is the level sum $f \Delta g : X \to \mathbb{R}$.*

Proof: First of all, write $f \Delta g$ in the form

$$(f \Delta g)(x) = \operatorname{Inf} \{ h_v(x) : v \in \operatorname{dom} f \},$$

with

$$h_v(x) = f(v) \vee g(x - v).$$

For each $v \in \operatorname{dom} f$, the function $h_v : X \to \mathbb{R}$ is Lipschitzian with Lipschitz constant L. Indeed, for all x and y in X, one can write

$$\begin{aligned} | h_v(x) - h_v(y) | &= | f(v) \vee g(x - v) - f(v) \vee g(y - v) | \\ &\leq | g(x - v) - g(y - v) | \\ &\leq L \| x - y \|. \end{aligned}$$

To prove that $f \Delta g : X \to \mathbb{R}$ is Lipschitzian with Lipschitz constant L, we write first

$$h_v(y) - L \| x - y \| \leq h_v(x) \leq h_v(y) + L \| x - y \|,$$

and then we take the infimum with respect to $v \in \operatorname{dom} f$. One gets in this way

$$(f \Delta g)(y) - L \| x - y \| \leq (f \Delta g)(x) \leq (f \Delta g)(y) + L \| x - y \|,$$

i.e.,

$$| (f \Delta g)(x) - (f \Delta g)(y) | \leq L \| x - y \|. \quad \blacksquare$$

Roughly speaking, what Proposition 3.1 says is that the level sum $f \Delta g$ inherits the Lipschitzian property of g, no matter how bad is the function f.

Example 3.1: Take, for instance, $g = \|\,.\,\|$. Then, for any proper function $f : X \to \mathbb{R} \cup \{+\infty\}$, the level sum

$$x \in X \mapsto [f \Delta \|\,.\,\|](x) = \operatorname*{Inf}_{v \in X} \{ f(v) \vee \| x - v \| \}$$

is Lipschitzian with Lipschitz constant $L = 1$.

In the above example, there is no reason to believe that $f \Delta \|\,.\,\|$ is a "good" approximation of f. In fact, $f \Delta \|\,.\,\|$ contains only a very rough information on the function f itself. To take care of the quality of the approximation, we incorporate a parameter $r > 0$, and choose a function g of the form

$$g(x) = r \|x\| \quad \text{for all } x \in X.$$

PROPOSITION 3.2: *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then, for each $r > 0$, the level sum*

$$x \in X \mapsto f^{[r]}(x) = \operatorname*{Inf}_{v \in X} \{ f(v) \vee r \| x - v \| \} \tag{3.1}$$

is Lipschitzian with Lipschitz constant r. Moreover, if f is nonnegative, then the upper envelope of the family $\{ f^{[r]} : r > 0 \}$ coincides with the lower-semicontinuous hull cl f *of f*, i.e.,

$$(\operatorname{cl} f)(x) = \operatorname*{Sup}_{r > 0} f^{[r]}(x) \quad \text{for all } x \in X. \tag{3.2}$$

Proof: The first part follows directly from Proposition 3.1, and the fact that $r \|\,.\,\|$ is Lipschitzian with Lipschitz constant r. Suppose now that f is nonnegative. For all $x \in X$ and $r > 0$, one has

$$f^{[r]}(x) = \operatorname*{Inf}_{v \in X} \{ f(v) \vee r \| x - v \| \} \le f(x) \vee 0 = f(x),$$

and therefore $f^{[r]}(x) \le (\operatorname{cl} f)(x)$. This implies that

$$g(x) := \operatorname*{Sup}_{r > 0} f^{[r]}(x) \le (\operatorname{cl} f)(x) \quad \text{for all } x \in X.$$

To prove the reverse inequality, we proceed as follows. For a given $x \in X$, take any $\lambda \in \mathbb{R}$ such that $\lambda < (\operatorname{cl} f)(x)$. We need to show that $g(x) \ge \lambda$. From the very definition of

$$(\operatorname{cl} f)(x) = \operatorname*{Sup}_{\alpha > 0} \operatorname*{Inf}_{\|v - x\| < \alpha} f(v),$$

we know there exists a real number $\alpha > 0$ satisfying

$$\lambda \leq \operatorname*{Inf}_{\|v-x\|<\alpha} f(v),$$

and, *a fortiori*,

$$\operatorname*{Inf}_{\|v-x\|<\alpha} \{ f(v) \vee r \| v - x \| \} \geq \lambda. \tag{3.3}$$

One has also

$$\operatorname*{Inf}_{\|v-x\|\geq\alpha} \{ f(v) \vee r \| v - x \| \} \geq r\alpha. \tag{3.4}$$

By combinining (3.3) and (3.4), one gets

$$f^{[r]}(x) = \operatorname*{Inf}_{v \in X} \{ f(v) \vee r \| v - x \| \} \geq \operatorname{Min} \{ \lambda, r\alpha \}.$$

If one takes r sufficiently large, namely $r \geq \alpha^{-1}\lambda$, then the above inequality reduces to $f^{[r]}(x) \geq \lambda$. To complete the proof it suffices to observe that $g(x) \geq f^{[r]}(x)$. ■

It is important to observe that $f^{[r]}(x)$ is nondecreasing as a function of the parameter $r > 0$. What Proposition 3.2 says then is that $f^{[r]}(x)$ converges monotonically upwards to the level $(\operatorname{cl} f)(x)$, as the parameter r tends to $+\infty$. A similar type of result is known for the Moreau-Yosida approximation (*cf.* Attouch [2])

$$x \in X \mapsto f_r(x) := \operatorname*{Inf}_{v \in X} \left\{ f(v) + \frac{r}{2} \| v - x \|^2 \right\}$$

and for the Baire-Wijsman approximation (*cf.* Martinez-Legaz [17], Corollary 3.6)

$$x \in X \mapsto f^r(x) := \operatorname*{Inf}_{v \in X} \{ f(v) + r \| v - x \| \}.$$

The function $f^{[r]}$ not only serves as Lipschitzian approximation of f, but also preserves the infimal value and the local minima of f. This fact is recorded in next proposition, where we use the symbol

$$B(x_0, \delta) = \{ x \in X : \| x - x_0 \| < \delta \}$$

for denoting the open ball centered at $x_0 \in X$ and with radius $\delta > 0$.

PROPOSITION 3.3: *Let f be a nonnegative function over X and let $r > 0$. Then,*

$$\operatorname{Inf} f^{[r]} = \operatorname{Inf} f \tag{3.5}$$

Moreover, for $f(x_0) \in]0, +\infty[$, one has the equivalence

$$f^{[r]}(x_0) = f(x_0) \quad \Leftrightarrow \quad x_0 \text{ minimizes } f \text{ over } B(x_0, r^{-1} f(x_0)). \tag{3.6}$$

Proof: Equality (3.5) is a particular case of (2.3). To prove the equivalence (3.6), recall that one has always the inequality $f^{[r]}(x_0) \leq f(x_0)$. The opposite inequality amounts to saying that

$$f(x_0) \leq f(v) \vee r \| v - x_0 \| \quad \text{for all } v \in X,$$

or, what is equivalent,

$$f(x_0) \leq f(v) \quad \text{whenever } r \| v - x_0 \| < f(x_0).$$

In other words, $f^{[r]}(x_0) = f(x_0)$ if and only if $f(v) \geq f(x_0)$ for all $v \in B(x_0, r^{-1} f(x_0))$. ■

4. FENCHEL CONJUGATE, SUBDIFFERENTIAL, AND APPROXIMATE SUBDIFFERENTIAL OF THE LEVEL SUM OF TWO CONVEX FUNCTIONS

The aim of this section is deriving formulas for computing the Fenchel conjugate, the subdifferential, and the approximate subdifferential of the level sum $f \Delta g$ of two convex functions f and g. An appropriate mathematical setting for dealing with this issue is that of a couple (X, X^*) of locally convex topological linear spaces in duality by means of a bilinear form $\langle . , . \rangle : X^* \times X \mapsto \mathbb{R}$ (*see* [5], p. 48). So, X and X^* are supplied with topologies compatible with this duality [5], p. 67, so that each one can be identified with the space of continuous linear functionals on the other.

4.1. Fenchel conjugate of a level sum

The Fourier transformate of the classical convolution of two functions is the product of their corresponding Fourier transformates. This result is a consequence of Fubini's theorem allowing to exchange the order of two iterated integrals. Similarly, in the context of convex analysis, the Fenchel conjugate of the inf-convolution of two functions coincides with the sum of their corresponding Fenchel conjugates. The later result is obtained by exchanging the order of two iterated maximizations. The situation is more involved when it comes to evaluate the Fenchel conjugate of a level sum.

Recall that the Fenchel conjugate of $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is the function φ^* given by

$$\varphi^*(y) := \operatorname*{Sup}_{x \in X} \{\langle y, x\rangle - \varphi(x)\} \quad \text{for all } y \in Y.$$

In the next proposition we give a formula for computing the Fenchel conjugate of the level sum of two proper convex functions on X. In what follows, the symbol Λ stands for the elementary simplex

$$\Lambda := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0, \lambda_2 \geq 0\},$$

and $\varphi^* \lambda$ denotes the right multiplication of φ^* by the finite scalar $\lambda \geq 0$, that is to say,

$$(\varphi^* \lambda)(y) := \begin{cases} \lambda \varphi^*(\lambda^{-1} y) & \text{if } \lambda > 0, \\ \operatorname{Sup}\{\langle y, x\rangle : x \in \operatorname{dom}\varphi\} & \text{if } \lambda = 0. \end{cases}$$

PROPOSITION 4.1: *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper convex functions. Then, the Fenchel conjugate of $f \Delta g$ is given by*

$$(f \Delta g)^*(y) = \operatorname*{Inf}_{(\lambda_1, \lambda_2) \in \Lambda} (f^* \lambda_1 + g^* \lambda_2)(y) \quad \text{for all } y \in X^*. \tag{4.1}$$

Proof: First of all, observe that $f \Delta g$ can be expressed in the form

$$(f \Delta g)(x) = \operatorname*{Inf}_{(u, v) \in M} \operatorname*{Sup}_{(\lambda_1, \lambda_2) \in \Lambda} (\lambda_1 f(u) + \lambda_2 g(v)\},$$

where

$$M := \{(u, v) \in \operatorname{dom} f \times \operatorname{dom} g : u + v = x\}.$$

By plugging this expression into the definition of $(f \Delta g)^*(y)$, one gets

$$(f \Delta g)^*(y) = \operatorname*{Sup}_{x \in X} \{\langle y, x\rangle - \operatorname*{Inf}_{(u, v) \in M} \operatorname*{Sup}_{(\lambda_1, \lambda_2) \in \Lambda} \{\lambda_1 f(u) + \lambda_2 g(v)\}\}.$$

A simple calculus yields

$$\begin{aligned}(f\Delta g)^*(y) &= \operatorname*{Sup}_{x\in X}\ \operatorname*{Sup}_{(u,v)\in M}\ \operatorname*{Inf}_{(\lambda_1,\lambda_2)\in\Lambda}\ \{\langle y, x\rangle - \lambda_1 f(u) - \lambda_2 g(v)\}\\ &= \operatorname*{Sup}_{\substack{u\in\operatorname{dom} f\\ v\in\operatorname{dom} g}}\ \operatorname*{Inf}_{(\lambda_1,\lambda_2)\in\Lambda}\ \{\langle y, u\rangle - \lambda_1 f(u) + \langle y, v\rangle - \lambda_2 g(v)\}.\end{aligned}$$

We apply now a minimax theorem stated in Sion [26], Theorem 4.2. After exchanging the order of the supremum and the infimum in the above line, one obtains

$$\begin{aligned}(f\Delta g)^*(y) = \operatorname*{Inf}_{(\lambda_1,\lambda_2)\in\Lambda}\ \Big[&\operatorname*{Sup}_{u\in\operatorname{dom} f}\ \{\langle y, u\rangle - \lambda_1 f(u)\}\\ &+ \operatorname*{Sup}_{v\in\operatorname{dom} g}\ \{\langle y, v\rangle - \lambda_2 g(v)\}\Big].\end{aligned}$$

To complete the proof it suffices to observe that

$$\operatorname*{Sup}_{u\in\operatorname{dom} f}\ \{\langle y, u\rangle - \lambda_1 f(u)\} = (f^*\lambda_1)(y),$$

and similarly

$$\operatorname*{Sup}_{v\in\operatorname{dom} g}\ \{\langle y, v\rangle - \lambda_2 g(v)\} = (g^*\lambda_2)(y). \quad \blacksquare$$

We point out that formula (4.1) appears already in Attouch [3] and Zalinescu [31]. Notice that formula (4.1) can also be written in the form

$$(f\Delta g)^*(y) = \operatorname*{Inf}_{(\lambda_1,\lambda_2)\in\Lambda}\ \{(\lambda_1 f)^*(y) + (\lambda_2 g)^*(y)\}, \tag{4.2}$$

provided the left multiplication of $\varphi \in \Gamma_0(X)$ by the scalar 0 is understood in the sense

$$(0\varphi)(x) = \begin{cases} 0 & \text{if } \ x \in \operatorname{dom}\varphi,\\ +\infty & \text{if } \ x \notin \operatorname{dom}\varphi.\end{cases}$$

Of course, no confusion arises in the interpretation of (4.2) if both functions f and g are finite everywhere.

We close this paragraph by mentioning an interesting application of Proposition 4.1. In what follows we use the notation

$$\begin{aligned}\Gamma_0(X) := \{\varphi : X \to \mathbb{R}\cup\{+\infty\} :\\ \varphi \text{ is proper convex lower-semicontinuous}\}.\end{aligned}$$

COROLLARY 4.1: *Let $(X, \|\,.\,\|)$ be a normed space, and let f and g be two nonnegative functions in $\Gamma_0(X)$. Then, the following statements are equivalent:*

(a) there exists $r > 0$ such that $f^{[r]} = g^{[r]}$;

(b) for all $r > 0$, one has $f^{[r]} = g^{[r]}$;

(c) $f = g$.

Proof: It suffices to prove that $(a) \Rightarrow (c)$. Suppose

$$f \Delta r \|\,.\,\| = g \Delta r \|\,.\,\|$$

for some $r > 0$. The norm $\|\,.\,\|$ can be written as support function of the dual unit ball $B_* = \{ y \in X^* : \|y\|_* \leq 1 \}$, *i.e.*,

$$\|x\| = \Psi^*_{B_*}(x) := \operatorname*{Sup}_{y \in B_*} \langle y, x \rangle.$$

Thus, $r \|\,.\,\|$ is the support function of rB_*. According to Proposition 4.1, the Fenchel conjugate of the level sum

$$f \Delta r \|\,.\,\| = f \Delta \Psi^*_{rB_*}$$

is given by

$$(f \Delta r \|\,.\,\|)^*(y) = \operatorname*{Inf}_{(\lambda_1, \lambda_2) \in \Lambda} \{ (f^* \lambda_1)(y) + \Psi_{\lambda_2 r B_*}(y) \} \quad \text{for all } y \in X^*,$$

where

$$\Psi_{\lambda_2 r B_*}(y) = \begin{cases} 0 & \text{if } y \in \lambda_2 r B_*, \\ +\infty & \text{if } y \notin \lambda_2 r B_*. \end{cases}$$

By taking into account that f is nonnegative, after a short calculus one gets

$$(f \Delta r \|\,.\,\|)^*(y) = \begin{cases} \left(1 - \frac{1}{r}\|y\|_*\right) f^*\left(\left(1 - \frac{1}{r}\|y\|_*\right)^{-1} y\right) & \text{if } \|y\|_* < r \\ \operatorname{Sup}\{ \langle y, x \rangle : x \in \operatorname{dom} f \} & \text{if } \|y\|_* = r \\ +\infty & \text{if } \|y\|_* > r. \end{cases}$$

A similar formula holds of course for the Fenchel conjugate of $g\Delta r \|\,.\,\|$. Since

$$(f\Delta r \|\,.\,\|)^* (y) = (g\Delta r \|\,.\,\|)^* (y) \quad \text{for all } y \in X^*,$$

one can write in particular

$$f^* \left(\left(1 - \frac{1}{r} \| y \|_* \right)^{-1} y \right) = g^* \left(\left(1 - \frac{1}{r} \| y \|_* \right)^{-1} y \right) \qquad \text{for} \quad \| y \|_* < r.$$

But any element z in X^* can be written in the form

$$z = \left(1 - \frac{1}{r} \| y \|_* \right)^{-1} y \qquad \text{with} \quad \| y \|_* < r.$$

To see this, just take

$$y = \left(1 + \frac{1}{r} \| z \|_* \right)^{-1} z.$$

Hence, $f^* (z) = g^* (z)$ for all $z \in X^*$. This yields finally the desired equality $f = g$. ■

It is worth mentioning that a result like Corollary 4.1 does not apply to the case of the Baire-Wijsman approximation. Indeed, it is possible to construct two functions $f, g \in \Gamma_0 (X)$ which are different, but such that their corresponding Baire-Wijsman approximates f^r and g^r coincide for some parameter $r > 0$.

4.2. The subdifferential of a level sum

Given a convex function $\varphi : X \to \mathbb{R} \cup \{ +\infty \}$, the subdifferential of φ at the point $x_0 \in \operatorname{dom} \varphi$ is defined by

$$\partial \varphi (x_0) = \{ y \in X^* : \varphi (x) \geq \varphi (x_0) + \langle y, x - x_0 \rangle \text{ for all } x \in X \}.$$

This set reflects the first-order behaviour of the function φ around x_0. A detailed discussion on the properties of this set can be found in any standard text on convex analysis.

Next proposition provides a formula which serves to compute the subdifferential $\partial (f \Delta g) (x_0)$ of the level sum of the convex functions f

and g. For the sake of simplicity, we suppose that there exists an element $v_0 \in X$ at which the infimum

$$(f \Delta g)(x_0) = \operatorname*{Inf}_{v \in X} \{ f(x_0 - v) \vee g(v) \}$$

is attained. For convenience, we use the symbol

$$A \# B := \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} \{ \lambda_1 A \cap \lambda_2 B \} \tag{4.3}$$

for denoting the *inverse sum* of the compact sets $A \subset X^*$ and $B \subset X^*$; *cf.* [21], p. 21.

PROPOSITION 4.2: *Let f and g be two functions in $\Gamma_0(X)$. Let $f \Delta g$ be finite at $x_0 \in X$, and let $v_0 \in X$ be a point satisfying*

$$(f \Delta g)(x_0) = f(x_0 - v_0) \vee g(v_0).$$

Suppose f and g are continuous at $x_0 - v_0$ and v_0, respectively. Then,

$$\partial(f \Delta g)(x_0) = \left. \begin{matrix} \partial f(x_0 - v_0) \# \partial g(v_0) & \text{if } f(x_0 - v_0) = g(v_0), \\ \{0\} & \text{if } f(x_0 - v_0) \neq g(v_0). \end{matrix} \right\} \tag{4.4}$$

Proof: We write $f \Delta g$ in the form

$$(f \Delta g)(x) = \operatorname*{Inf}_{v \in X} (H_1 \vee H_2)(x, v),$$

with

$$H_1(x, v) = f(x - v) \qquad \text{and} \qquad H_2(x, v) = g(v).$$

By applying Rockafellar's rule [22], Theorem 2.4 on the subdifferential of a marginal function, one obtains

$$\partial(f \Delta g)(x_0) = \{ y \in X^* : (y, 0) \in \partial(H_1 \vee H_2)(x_0, v_0) \}.$$

Evaluating $\partial(H_1 \vee H_2)(x_0, v_0)$ requires to distinguish between two cases. Consider first the case in which $f(x_0 - v_0) = g(v_0)$. According to Valadier [27], it is possible to write

$$\partial(H_1 \vee H_2)(x_0, v_0) = \text{cl conv}[\partial H_1(x_0, v_0) \cup \partial H_2(x_0, v_0)], \tag{4.5}$$

where "cl" and "conv" refer to the closure and convex hull operation, respectively. The continuity hypothesis on f and g makes the closure operation in (4.5) unnecessary, and allows us to write

$$\partial H_1(x_0, v_0) = \{(y_1, -y_1) : y_1 \in \partial f(x_0 - v_0)\},$$
$$\partial H_2(x_0, v_0) = \{(0, y_2) : y_2 \in \partial g(v_0)\}.$$

Thus, $(y, 0) \in \partial(H_1 \vee H_2)(x_0, v_0)$ if and only if, there exists $(\lambda_1, \lambda_2) \in \Lambda$ such that

$$(y, 0) = \lambda_1(y_1, -y_1) + \lambda_2(0, y_2), \tag{4.6}$$

with $y_1 \in \partial f(x_0 - v_0)$ and $y_2 \in \partial g(v_0)$. By writing (4.6) in the form

$$y = \lambda_1 y_1 = \lambda_2 y_2,$$

one sees that $(y, 0) \in \partial(H_1 \vee H_2)(x_0, v_0)$ if and only if

$$y \in \partial f(x_0 - v_0) \# \partial g(v_0) := \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} \{\lambda_1 \partial f(x_0 - v_0) \cap \lambda_2 \partial g(v_0)\}.$$

Consider now the case $f(x_0 - v_0) \neq g(v_0)$. Take, for instance, $f(x_0 - v_0) > g(v_0)$. At the point (x_0, v_0), the functions $H_1 \vee H_2$ and H_1 have the same subdifferential. Thus,

$$\partial(H_1 \vee H_2)(x_0, v_0) = \{(y_1, -y_1) : y_1 \in \partial f(x_0 - v_0)\}.$$

Hence, $(y, 0) \in \partial(H_1 \vee H_2)(x_0, v_0)$ if and only if $y = 0$. The case $f(x_0 - v_0) < g(v_0)$ is treated in the same way. ■

Remark 4.1: Under the same assumptions of Proposition 4.2, it can be shown that the condition $\operatorname{Inf} f = \operatorname{Inf} g$ implies that $f(x_0 - v_0) = g(v_0)$. This can be shown by combining Propositions 4.2 and 2.3.

4.3. Approximate subdifferential of a level sum

This paragraph is more technical than the previous one, and is addressed to the reader which is familiar with the following variant

$$\partial_\varepsilon \varphi(x_0) = \{y \in X^* : \varphi(x) \geq \varphi(x_0) + \langle y, x - x_0 \rangle - \varepsilon \text{ for all } x \in X\} \tag{4.7}$$

of the subdifferential $\partial \varphi(x_0)$. The set $\partial_\varepsilon \varphi(x_0)$ is referred to as the *approximate subdifferential* of φ at x_0. There are several reasons which justify introducing the parameter $\varepsilon > 0$ in the definition (4.7). In Hiriart-Urruty and Seeger [13], and Seeger [23, 25], it is explained how the set $\partial_\varepsilon \varphi(x_0)$ can be used to obtain higher-order information on the behaviour of φ around x_0. Approximate subdifferentials have many other uses in convex optimization: design of algorithms [8, 16, 32], characterization of approximate optimal solutions [12, 18], and formulation of variational principles [4, 6], are just a few examples.

In next proposition we derive a formula for computing the approximate subdifferential $\partial_\varepsilon (f \Delta g)(x_0)$ of the level sum of two convex functions f and g.

PROPOSITION 4.3: *Let f and g be two functions in $\Gamma_0(X)$. Let $f\Delta g$ be finite at $x_0 \in X$, and let $v_0 \in X$ be a point satisfying*

$$(f \Delta g)(x_0) = f(x_0 - v_0) \vee g(v_0).$$

Then, for all $\varepsilon \geq 0$, one can write

$$\partial_\varepsilon (f \Delta g)(x_0) = \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} \bigcup_{(\alpha_1, \alpha_2)} \{ \partial_{\alpha_1} (\lambda_1 f)(x_0 - v_0) \cap \partial_{\alpha_2} (\lambda_2 g)(v_0) \}, \tag{4.8}$$

where the inner union is taken with respect to all pairs $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ such that

$$\alpha_1 + \alpha_2 = \varepsilon + \lambda_1 f(x_0 - v_0) + \lambda_2 g(v_0) - (f \Delta g)(x_0). \tag{4.9}$$

Proof: The approximate subdifferential of the function $f\Delta g$ can be characterized in terms of its Fenchel-conjugate $(f \Delta g)^*$, namely

$$\partial_\varepsilon (f \Delta g)(x_0) = \{ y \in X^* : (f \Delta g)^*(y) + (f \Delta g)(x_0) - \langle y, x_0 \rangle \leq \varepsilon \}. \tag{4.10}$$

Now we take advantage of Proposition 4.1, that is to say, we use the formula (4.2).

The infimum in (4.2) is attained because Λ is a compact set and $(\lambda_1, \lambda_2) \in \Lambda \mapsto (\lambda_1 f)^* (y) + (\lambda_2 g)^* (y)$ is a lower-semicontinuous function. Plugging (4.2) into (4.10), one gets

$$\partial_\varepsilon (f \Delta g)(x_0) = \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} A_{\lambda_1, \lambda_2},$$

where

$$A_{\lambda_1, \lambda_2} := \{ y \in X^* : (\lambda_1 f)^* (y) + (\lambda_2 g)^* (y) + (f \Delta g)(x_0) - \langle y, x_0 \rangle \leq \varepsilon \}.$$

After a short calculus one sees that $y \in A_{\lambda_1, \lambda_2}$ if and only if

$$\begin{aligned} &[(\lambda_1 f)^* (y) + (\lambda_1 f)(x_0 - v_0) - \langle y, x_0 - v_0 \rangle] \\ &\quad + [(\lambda_2 g)^* (y) + (\lambda_2 g)(v_0) - \langle y, v_0 \rangle] \leq \varepsilon + \delta, \end{aligned}$$

where

$$\delta = \lambda_1 f (x_0 - v_0) + \lambda_2 g (v_0) - (f \Delta g)(x_0).$$

Now observe that both expressions between brackets are nonnegative. Thus, $y \in A_{\lambda_1, \lambda_2}$ if and only if there are two coefficients $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, with $\alpha_1 + \alpha_2 = \varepsilon + \delta$, such that

$$\left. \begin{aligned} &(\lambda_1 f)^* (y) + (\lambda_1 f)(x_0 - v_0) - \langle y, x_0 - v_0 \rangle \leq \alpha_1, \\ &(\lambda_2 g)^* (y) + (\lambda_2 g)(v_0) - \langle y, v_0 \rangle \leq \alpha_2. \end{aligned} \right\} \tag{4.11}$$

The inequalities in (4.11) are equivalent to the conditions

$$y \in \partial_{\alpha_1} (\lambda_1 f)(x_0 - v_0) \quad \text{and} \quad y \in \partial_{\alpha_2} (\lambda_2 g)(v_0),$$

respectively. This means that

$$A_{\lambda_1, \lambda_2} = \bigcup_{(\alpha_1, \alpha_2)} \partial_{\alpha_1} (\lambda_1 f)(x_0 - v_0) \cap \partial_{\alpha_2} (\lambda_2 g)(v_0),$$

where the union is taken with respect to the pairs $(\alpha_1, \alpha_2) \in \mathbb{R}_+^2$ satisfying (4.9). This completes the proof of the proposition. ■

Usually the infimum in the definition

$$(f \Delta g)(x_0) = \operatorname*{Inf}_{v \in X} \{ f(x_0 - v) \vee g(v) \}$$

is attained at some point $v_0 \in X$ such that

$$(f \Delta g)(x_0) = f(x_0 - v_0) = g(v_0).$$

In such a case, formula (4.8) reduces to the expression

$$\partial_\varepsilon (f \Delta g)(x_0) = \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} \ \bigcup_{\substack{\alpha_1 + \alpha_2 = \varepsilon \\ \alpha_1 \geq 0,\, \alpha_2 \geq 0}} \{ \partial_{\alpha_1} (\lambda_1 f)(x_0 - v_0) \cap \partial_{\alpha_2} (\lambda_2 g)(v_0) \}.$$

Observe that the particular choice $\varepsilon = 0$ yields an alternative expression for the set $\partial (f \Delta g)(x_0)$, namely

$$\partial (f \Delta g)(x_0) = \bigcup_{(\lambda_1, \lambda_2) \in \Lambda} \{ \partial (\lambda_1 f)(x_0 - v_0) \cap \partial (\lambda_2 g)(v_0) \}. \quad (4.12)$$

In constrast with the expression (4.4) given in Proposition 4.2, the formula (4.12) applies even without the continuity assumption made on the functions f and g.

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