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PROPERTIES OF ORDINARY AND WEIGHTED SUMS OF ORDER p USED FOR DISTANCE ESTIMATION (*)

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Abstract. – *Sums of order p are encountered in numerous practical applications; an example is the popular l_p norm found in continuous location theory. This paper attempts to generalize various properties of ordinary and weighted sums of order p which are of theoretical and practical interest. We expand the conditions for which Jensen's inequality holds. In addition, an important convexity result is generalized and an open question posed by Beckenbach is resolved.*

Keywords: Weighted sums, order p, distance estimation.

Résumé. – *On rencontre des sommes de l'ordre p dans de nombreuses applications pratiques; c'est ainsi que l'on trouve la norme populaire l_p dans la théorie de la localisation continue. Dans cet article, les auteurs essaient de généraliser les diverses propriétés des sommes ordinaires et des sommes pondérées de l'ordre p qui présentent un intérêt théorique et pratique. Ils développent les conditions pour lesquelles l'inégalité de Jensen est valable. De plus, ils généralisent un résultat de convexité important et ils résolvent une question en suspens posée par Beckenbach.*

Mots clés : Sommes pondérées, ordre p, estimation, distance.

1. INTRODUCTION

Distance predicting functions are used in many different applications. Ginsberg and Hansen [5] utilized a distance predicting function (p.d.f.) to check the accuracy of actual travel distance data. Westwood [19] incorporated a p.d.f. into a distribution model. Kolesar, Walker and Hausner [9] incorporated a p.d.f. into a response-time model for emergency vehicles such as fire engines. Eilon *et al.* [4] utilized a p.d.f. to compute depot-to-customer distances in locational analysis studies. Kleindorfer *et al.* [8] discuss the use of a p.d.f. in routing problems, and Klein [7] uses p.d.f.s for

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construction of Voronoi diagrams. The use of p.d.f.s in location-allocation models is given by Love *et al.* [13]. Two of the most widely used truck scheduling software packages in North America, Truckstops [16] and Roadnet [15] incorporate p.d.f.s since they are much more efficient and comprehensive to use in practice than attempting to assemble large files of distance data.

Love and Morris [10, 11, 12] applied several distance norms including the weighted l_p (kl_p) norm to Germany and several regions of the United States. Let $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$ be any two points in the plane. The kl_p norm is given by

$$kl_p(x, y) = k\,[|x_1 - y_1|^p + |x_2 - y_2|^p]^{1/p}, \qquad k > 0, \quad p > 0.$$

Love and Morris found that the kl_p norm was relatively easy to fit to a geographical region and it has excellent predictive properties. Ward and Wendell [17, 18] have introduced the concept of utilizing block norms as distance predictors. A new study by Love and Walker [14] shows that although marginally better results can be obtained by using a block norm with eight or more parameters than by using the kl_p norm, the computation cost of fitting the block norms can be prohibitive. Conversely, the original studies by Love and Morris [10, 11] show that the kl_p norm usually gives much superior results compared to other simpler norms such as the weighted Euclidean or weighted rectangular norms.

In the present paper we introduce a generalized kl_p norm in the form of a weighted sum of order p. This is in effect adding a single parameter to the kl_p norm since one of the two weights in the sum of order p function replaces the k in the kl_p norm. Properties of the weighted sum as a function of the parameter p are derived which can be utilized when fitting the norm to actual geographical data.

2. PRELIMINARY RESULTS

A weighted sum of order p is defined as follow (e.g., section 2.10 of [6]):

$$T(y;\, b,\, p) = \left[\sum_{i=1}^{K} b_i\, y_i^p\right]^{1/p}, \tag{1}$$

where

$$y = (y_1, \ldots, y_K)^T, \qquad y_i > 0, \quad i = 1, \ldots, K,$$
$$b = (b_1, \ldots, b_K)^T, \qquad b_i > 0, \quad i = 1, \ldots, K,$$

and $p \neq 0$.

The vector b and the scalar p can be considered as a set of parameter values. If all the weights $b_i = 1$, then T becomes the ordinary sum of order p which is well-known in the literature (e.g., section 1-16 of [2]). Note that the function $T(y;\, b,\, p)$ has the form of a generalized l_p distance given by

$$l_{bp}(x) = \left[\sum_{i=1}^{K} b_i \, |x_i|^p\right]^{1/p}, \tag{2}$$

where $x = (x_1, \ldots, x_K)^T \in R^K$, p is generally assumed to be greater than zero, and l_{bp} estimates the distance between any two points $y,\, z \in R^K$ such that $x = y - z$. The weights b_i can be used to represent non-symmetric costs along the axis directions in a location model. (A comprehensive treatment of continuous location models is given in [13].) We are interested here in studying the behaviour of the sum T as a function of its parameter p.

Consider first the asymptotic behaviour of T. Letting $y_m = \min_i (y_i)$ and $y_M = \max_i (y_i)$, we obtain

$$\begin{aligned}
\lim_{p \to +\infty} \{T(y;\, b,\, p)\} &= \lim_{p \to +\infty} \left\{\left[\sum_{i=1}^{K} b_i \, y_i^p\right]^{1/p}\right\} \\
&= y_M \lim_{p \to +\infty} \left\{\left[\sum_{i=1}^{K} b_i \left(\frac{y_i}{y_M}\right)^p\right]^{1/p}\right\} \\
&= y_M,
\end{aligned} \tag{3}$$

and similarly,

$$\begin{aligned}
\lim_{p \to -\infty} \{T(y;\, b,\, p)\} &= y_m \lim_{p \to -\infty} \left\{\left[\sum_{i=1}^{K} b_i \left(\frac{y_m}{y_i}\right)^{-p}\right]^{1/p}\right\} \\
&= y_m.
\end{aligned} \tag{4}$$

Thus, the function T approaches the same horizontal asymptotes irrespective of the positive weights b_i, $i = 1, \ldots, K$.

Without loss in generality, let us assume that all the y_i's have distinct values; that is, $y_i \neq y_j$, $i \neq j$, for all $i, j \in \{1, .., K\}$. (If this is not the case, common terms can be added together and K adjusted accordingly.) Denoting the weights associated with y_m and y_M by b_m and b_M respectively, it is clear from (3) and (4) that for $K \geq 2$,

$$\lim_{p \to +\infty} T = \begin{cases} y_M^+, & \text{if} \quad b_M \geq 1, \\ y_M^-, & \text{if} \quad 0 < b_M < 1, \end{cases} \tag{5}$$

and

$$\lim_{p \to -\infty} T = \begin{cases} y_m^-, & \text{if} \quad b_m \geq 1, \\ y_m^+, & \text{if} \quad 0 < b_m < 1. \end{cases} \tag{6}$$

Thus, the direction of approach from above or below the horizontal asymptotes y_M, y_m depends on the magnitude of the corresponding weights b_M, b_m.

We now examine the behaviour of T near $p = 0$. Letting

$$\beta = \sum_{i=1}^{K} b_i, \tag{7}$$

it is readily seen that for $\beta > 1$,

$$\lim_{p \to 0^+} T = +\infty, \qquad \lim_{p \to 0^-} T = 0; \tag{8}$$

while for $\beta < 1$,

$$\lim_{p \to 0^+} T = 0, \qquad \lim_{p \to 0^-} T = +\infty. \tag{9}$$

When $\beta = 1$, $T(y;\, b,\, p)$ becomes the mean value function, for which the following well-known result applies:

$$\lim_{p \to 0} T = \prod_{i=1}^{K} y_i^{b_i}. \tag{10}$$

It follows from (8), (9) and (10) that T is continuous at $p = 0$ if, and only if, $\beta = 1$.

We now calculate the first and second-order partial derivatives of $\ln T$ with respect to p. Letting

$$a_i = \frac{b_i}{\beta}, \qquad i = 1, \ldots, K, \tag{11}$$

equation (1) can be rewritten as

$$T(y;\, b,\, p) = \beta^{1/p} \left[\sum_{i=1}^{K} a_i\, y_i^p \right]^{1/p}, \tag{12}$$

where $a_i > 0$, $i = 1, \ldots, K$, and

$$\sum_{i=1}^{K} a_i = 1.$$

Then

$$\ln T = \frac{1}{p} \ln \beta + \frac{1}{p} \ln \left(\sum_{i=1}^{K} a_i\, y_i^p \right). \tag{13}$$

a) First derivative

$$\begin{aligned} \frac{\partial}{\partial p} \ln T &= -\frac{1}{p^2} \ln \beta - \frac{1}{p^2} \ln \left(\sum_{i=1}^{K} a_i\, y_i^p \right) + \frac{1}{p} \cdot \frac{1}{\sum_{i=1}^{K} a_i\, y_i^p} \sum_{i=1}^{K} a_i\, y_i^p \ln y_i \\ &= \frac{1}{p^2 \sum_{i=1}^{K} a_i\, y_i^p} \left[\sum_{i=1}^{K} a_i\, y_i^p \ln \left(\frac{y_i^p}{\beta \sum_{j=1}^{K} a_j\, y_j^p} \right) \right], \qquad p \neq 0. \end{aligned} \tag{14}$$

Since

$$\frac{\partial}{\partial p} \ln T = \frac{1}{T} \frac{\partial T}{\partial p},$$

we immediately get

$$\frac{\partial T}{\partial p}=\frac{\beta^{1/p}}{p^2}\left[\sum_{i=1}^{K} a_i\, y_i^p\right]^{(1-p)/p}\left[\sum_{i=1}^{K} a_i\, y_i^p \ln\left(\frac{y_i^p}{\beta\sum_{j=1}^{K} a_j\, y_j^p}\right)\right], \tag{15}$$

$$p \neq 0.$$

It is interesting to note that for $\beta > 1$,

$$\begin{aligned}\lim_{p\to 0^-}\frac{\partial T}{\partial p} &= -\ln\beta\,.\lim_{p\to 0^-}\left[\frac{\beta^{1/p}}{p^2}\left(\sum_{i=1}^{K} a_i\, y_i^p\right)^{1/p}\right]\\ &= -\ln\beta\,.\prod_{i=1}^{K} y_i^{a_i}\,.\lim_{p\to 0^-}\left[\frac{\beta^{1/p}}{p^2}\right] \qquad \text{[equation (10)]}\\ &= 0^-.\end{aligned} \tag{16}$$

Meanwhile, for $\beta < 1$, we obtain in similar fashion the following result.

$$\lim_{p\to 0^+}\frac{\partial T}{\partial p}=0^+. \tag{17}$$

b) Second derivative

In the following summations $i,\ j \in \{1, \ldots, K\}$ is understood, but omitted to simplify the notation.

$$\begin{aligned}\frac{\partial^2}{\partial p^2}\ln T =& \frac{2}{p^3}\ln\beta+\frac{2}{p^3}\ln\Big(\sum_i a_i\, y_i^p\Big)-\frac{2}{p^2\sum_i a_i\, y_i^p}\sum_i a_i\, y_i^p \ln y_i\\ &-\frac{1}{p}\cdot\frac{1}{\big(\sum_i a_i\, y_i^p\big)^2}\Big(\sum_i a_i\, y_i^p \ln y_i\Big)^2\\ &+\frac{1}{p\sum_i a_i\, y_i^p}\cdot\sum_i a_i\, y_i^p\,(\ln y_i)^2.\end{aligned}$$

After some re-arranging this reduces to

$$\frac{\partial^2}{\partial p^2} \ln T = \frac{1}{p^3 \sigma^2} \left[2 \sigma \sum_i a_i y_i^p \ln \left(\frac{\beta \sigma}{y_i^p} \right) + p^2 \sum_{j<i} \sum a_i a_j y_i^p y_j^p (\ln y_i - \ln y_j)^2 \right], \qquad p \neq 0, \quad (18)$$

where

$$\sigma = \sum_{i=1}^{K} a_i y_i^p. \quad (19)$$

3. PROPERTIES

Setting all the $b_i = 1$, we obtain the ordinary sum of order p, denoted as follows:

$$S(y;\, p) = \left(\sum_{i=1}^{K} y_i^p \right)^{1/p} \quad (20)$$

This sum satisfies the well-known relation,

$$S(y;\, p_2) < S(y;\, p_1), \quad 0 < p_1 < p_2, \quad K \geq 2, \quad (21)$$

which is usually referred to as Jensen's inequality. (For two different proofs of (21), *see* Theorem 19 in [6], and the proof by Beckenbach [1].) Beckenbach also shows that $S(y;\, p)$ is convex in p for $p > 0$. His proof utilizes techniques from convex analysis. Using the results of the preceding section, we now obtain a general condition whereby Jensen's inequality and the convexity result of Beckenbach [1] are extended to the weighted sum.

THEOREM 1: *Consider the function* $\mathrm{T}(y;\, b,\, p)$ *defined in* (1), *with given (constant) vectors y and b. Assume without loss in generality that* $y_M = \max_i (y_i)$ *occurs for a unique* $M \in \{1, \ldots, K\}$; i.e., *there are no ties. (If this is not the case, add the coefficients* (b_i) *of the ties to form one term.) Then for* $K \geq 2$, $T(y;\, b,\, p)$ *is a decreasing function of p for* $0 < p < +\infty$, *if, and only if,* $b_M \geq 1$, *where* b_M *is the coefficient of* y_M. *Furthermore, if* $b_M \geq 1$, T *is also a strictly convex function in p over this interval.*

Proof:

(i) (If) Since $p > 0$, $K \geq 2$ and $b_M \geq 1$, it follows that

$$\frac{y_i^p}{\beta\sigma} = \frac{y_i^p}{\beta \sum_{j=1}^{K} a_j\, y_j^p} = \frac{y_i^p}{\sum_{j=1}^{K} b_j\, y_j^p} < \frac{y_i^p}{b_M\, y_M^p} \leq 1, \qquad i = 1, \ldots, K. \tag{22}$$

From equations (14) and (18), we see that

$$\frac{\partial \ln T}{\partial p} < 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial p^2} \ln T > 0.$$

Hence, $\ln T$ is decreasing and strictly convex in p for $0 < p < +\infty$. It immediately follows that T is a decreasing function of p in this interval. Furthermore,

$$\frac{\partial^2 T}{\partial p^2} = T\, \frac{\partial^2}{\partial p^2} \ln T + \frac{1}{T} \left(\frac{\partial T}{\partial p} \right)^2; \tag{23}$$

so that

$$\frac{\partial^2 T}{\partial p^2} \geq T\, \frac{\partial^2}{\partial p^2} \ln T > 0.$$

Thus, T is also strictly convex in p, for $0 < p < +\infty$. We conclude that $b_M \geq 1$ is a sufficient condition for T to be a decreasing strictly convex function of $p \in (0, +\infty)$.

(ii) (Only if) That $b_M \geq 1$ is a necessary condition for T to be decreasing in p immediately follows from the asymptotic behaviour of T as $p \to +\infty$, shown in (5). If $b_M < 1$, T approaches y_M asymptotically from below, and hence is increasing and concave for sufficiently large p.

COROLLARY 1: *$T(y;\, b,\, p)$ with $K \geq 2$ is a decreasing function of $p > 0$ for given weights b and all (positive) y, if, and only if, $b_i \geq 1$, $i = 1, \ldots, K$. Furthermore, T is also strictly convex in p under these conditions.*

Proof: Consider any y such that the y_i's are not all equal. Clearly, $b_M \geq 1$ if all the $b_i \geq 1$. By Theorem 1, we know that T is decreasing and strictly convex in $p > 0$. Now consider any y such that all the y_i's are equal. Then, $T = \beta^{1/p}\, y_1$, where $\beta = \Sigma_{i=1}^{K} b_i > 1$ if all the $b_i \geq 1$. It is readily shown that T is once again decreasing and strictly convex in $p > 0$. Thus, $b_i \geq 1$,

$i = 1, \dots, K$, is a sufficient condition. That this is also a necessary condition is readily seen by contradiction. Suppose $b_r < 1$, for some $r \in \{1, \dots, K\}$. Construct a vector y such that $y_i \neq y_j$, $i \neq j$, $\forall i, j \in \{1, \dots, K\}$, and $y_r = \max_i (y_i)$. By the theorem, we know that T is not a decreasing function of $p \in (0, +\infty)$, for this y.

The shape of T as a function of p becomes more complex when the criteria on the weights are changed, as shown in the following result.

PROPERTY 1: *Consider a vector of weights b, such that $\beta = \Sigma_i b_i > 1$, and $b_r < 1$ for at least one $r \in \{1, \dots, K\}$. Then, for any given y there exists a $\delta > 0$ such that T is decreasing and strictly convex in $p \in (0, \delta)$. However, if $y_r = \max_i (y_i)$, and there are no ties, then T is increasing and strictly concave for sufficiently large positive p.*

Proof: Follows immediately from the limit $p \to 0^+$ in (8) and the limit $p \to +\infty$ in (5).

Note that the function T described in the preceding result is neither increasing or decreasing in p nor convex or concave in p over the entire interval $0 < p < +\infty$, and that at least one inflection point exists in this interval. This is illustrated in Figure 1.

The fact that T is a decreasing function of p in the interval $(0, +\infty)$ for all y if, and only if, $b_i \geq 1$, $i = 1, \dots, K$ (Corollary 1), has been recognized previously (Theorem 23, [6]). However, their proof is different than ours and does not show the important result that T is convex in p under these conditions. The third and final case to consider for the weights b is where $\beta = \Sigma_i b_i \leq 1$. In the same theorem, the above authors prove that T is non-decreasing in p over the interval $(0, +\infty)$ for all y if, and only if, this condition holds. Thus, the following property can be given without proof.

PROPERTY 2: *A necessary and sufficient condition to have*

$$T(y;\, b, p_1) \leq T(y;\, b, p_2), \qquad 0 < p_1 < p_2,$$

for given weights b and all y, is that $\beta \leq 1$. Furthermore, there is strict inequality unless all the y_i are equal and $\beta = 1$.

An analogous result holds for negative values of p. In this case, rewrite

$$T(y;\, b,\, p) = \left[\sum_{i=1}^{K} b_i\, y_i^p\right]^{1/p}$$

$$= 1 \Bigg/ \left[\sum_{i=1}^{K} b_i \left(\frac{1}{y_i}\right)^{|p|}\right]^{1/|p|}$$

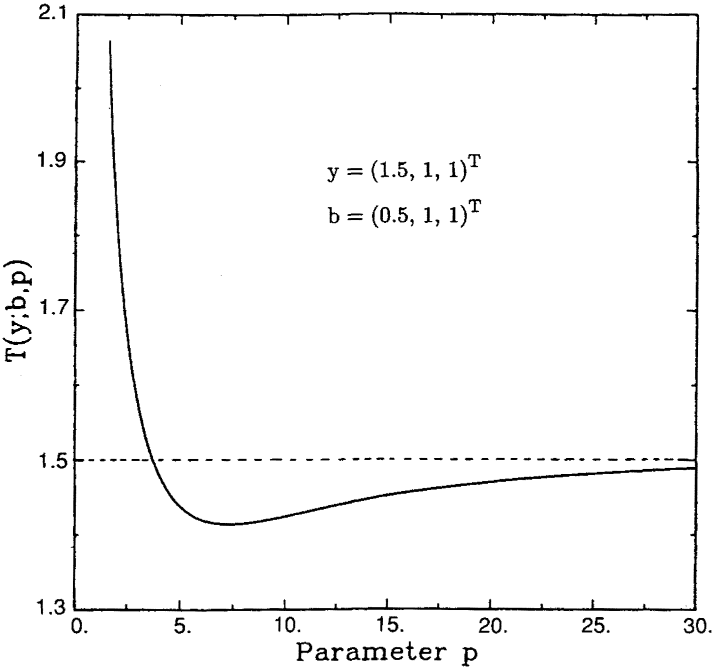

Figure 1. – General shape of T under conditions of Property 1.

It follows from Property 2 that

$$T(y;\, b,\, p_1) \geq T(y;\, b,\, p_2), \qquad p_2 < p_1 < 0,$$

for given weights b and all y, if and only if $\beta \leq 1$.

Use of negative p when the weighted sum T is a distance function in location models does not appear to have a physical interpretation. However, there may be other situations where $p < 0$ might be considered. In any case, we would like to take full advantage of our lengthy calculations of derivatives. This questionable motivation leads to the following results for $p < 0$.

THEOREM 2: *Consider the function* $T(y;\, b,\, p)$ *defined in* (1), *with given (constant) vectors* y *and* b. *Assume without loss in generality that* $y_m = \min_i(y_i)$ *occurs for a unique* $m \in \{1, \dots, K\}$; i. e., *there are no ties. (If this is not the case, add the coefficients* (b_i) *of the ties to form one term.) Then for* $K \geq 2$, $T(y;\, b,\, p)$ *is a decreasing function of* p *in the interval* $(-\infty, 0)$, *if, and only if,* $b_m \geq 1$, *where* b_m *is the coefficient of* y_m. *Furthermore, if* $b_m \geq 1$, $\ln T$ *is also a strictly concave function of* p *over this interval.*

Proof:

(i) (If). Since $p < 0$, $K \geq 2$, and $b_m \geq 1$, it follows that

$$\frac{y_i^p}{\sum_{j=1}^{K} b_j\, y_j^p} < \frac{y_i^p}{b_m\, y_m^p} = \frac{1}{b_m}\left(\frac{y_m}{y_i}\right)^{|p|} \leq 1, \qquad i = 1, \dots, K. \tag{24}$$

Returning to equations (14) and (18), we can readily show that

$$\frac{\partial \ln T}{\partial p} < 0 \qquad \text{and} \qquad \frac{\partial^2 \ln T}{\partial p^2} < 0, \qquad p < 0.$$

Hence, $\ln T$ is decreasing and strictly concave in p in the interval $(-\infty, 0)$. It immediately follows that T is a decreasing function of p in this interval. We conclude that $b_m \geq 1$ is a sufficient condition for T to be decreasing in p and $\ln T$ strictly concave in p, for $-\infty < p < 0$.

(ii) (Only if). That $b_m \geq 1$ is a necessary condition for T to be decreasing in p immediately follows from the asymptotic behaviour of T as $p \to -\infty$, shown in (6). If $b_m < 1$, T approaches y_m asymptotically from above; so that T (or $\ln T$) is increasing and convex for sufficiently large negative values of p.

COROLLARY 2: $T(y;\, b,\, p)$ *with* $K \geq 2$ *is a decreasing function of* p *in the interval* $(-\infty, 0)$ *for given weights* b *and all (positive)* y, *if, and only if,* $b_i \geq 1$, $i = 1, \dots, K$. *Furthermore,* $\ln T$ *is strictly concave in* p *under these conditions.*

Proof: Consider any y such that the y_i's are not all equal. Clearly, $b_m \geq 1$ if all the $b_i \geq 1$. By Theorem 2, we know that T is decreasing and $\ln T$ is strictly concave in $p \in (-\infty, 0)$. Now consider any y such that all the y_i's are equal. Then, $T = \beta^{1/p}\, y_1$, where $\beta = \Sigma_i\, b_i > 1$, if all the b_i's ≥ 1. It is readily shown once again that T is decreasing and $\ln T$ is strictly concave in $p \in (-\infty, 0)$. We conclude that $b_i \geq 1, i = 1, \ldots, K$, is a sufficient condition. That this is also a necessary condition is readily seen by contradiction, similar to the procedure in Corollary 1.

An open question posed by Beckenbach [1] concerns a lower bound on the number of inflection points of $S(y; p)$ as a function of p in the interval $(-\infty, 0)$. This question is resolved below for the more general weighted sum.

THEOREM 3: *The function $T(y; b, p)$ with $\beta > 1$ has at least one inflection point, and hence is neither convex nor concave in p, in the interval $-\infty < p < 0$.*

Proof: From relations (6) and (8), it follows that T cannot be convex in p over the entire interval $(-\infty, 0)$. However, from (16) we see that a $\delta > 0$ exists such that T is convex in p in the interval $(-\delta, 0)$. Hence, we conclude that at least one inflection point exists.

It immediately follows that $S(y; p)$ with $K \geq 2$ has at least one inflection point, and hence is neither convex nor concave in p, in the interval $-\infty < p < 0$. As a final comment we note that Theorem 3 applies when $\beta < 1$ and $p \in (0, +\infty)$. The proof follows in a similar fashion from relations (5), (9) and (17).

4. CONCLUSIONS

This paper investigates properties of the weighted sum of order p as a function of the parameter p. The weighted sum is a generalized form of the ordinary sum of order p, which is alternatively known in location theory as the popular l_p distance function or l_p norm when $p \geq 1$.

The well-known result that the ordinary sum is a non-increasing, convex function of p in the interval $(0, +\infty)$ has been useful in deriving efficient algorithms to find the best-fitting parameter values of the weighted l_p norm for estimating distances in a given geographical region (*see* Brimberg and Love [3]). General conditions have been presented here for extending the monotonicity and convexity properties to the weighted sum. This sum can be interpreted as a generalized l_p distance which allows non-symmetric travel in the axial directions through the use of unequal weights on the coordinates. The new form should provide greater accuracy for estimating travel distances on a transportation network. However, there is a computational cost for the gain in accuracy, since additional parameters must be fitted to the network. For example, the weighted l_p norm in the plane has two unknown parameters, whereas the generalized form discussed here will have three. The theoretical results presented in this paper are being applied by the authors to extend the existing fitting algorithms to the general weighted l_p norm. In addition, we plan to do empirical testing of the new metric.

This paper also resolves an open question in the literature concerning the existence of inflection points for sums of order p (*see* Beckenbach [1]). Since the conditions under which these inflection points occur are typically not encountered in practical applications of the sum of order p as a distance function, these particular results are currently of theoretical interest only.

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