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THE GENERALIZED WEBER PROBLEM WITH EXPECTED DISTANCES (*)

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Abstract. – *In this paper we study a generalized Weber problem where both demand locations and the facility to be located may be regions and are assumed to be distributed according to some probability measures inside each region. A general notation is proposed to describe these location problems and several properties are proved which enable the resolution of the problem using existing algorithms. In some special cases the gradient of the objective function is evaluated, leading to the resolution of a wide range of problems even when the exact expression for the objective function is unknown. This methodology is applied to some cases.*

Keywords: Location theory, average distances, regional facilities.

Résumé. – *Dans cet article on étudie un problème de Weber généralisé où la demande et l'origine qu'on doit localiser peuvent être des régions et on assume qu'elles sont distribuées selon quelque mesure de probabilité dans chaque région. On propose une notation générale pour décrire ces problèmes de localisation et on prouve quelques propriétés qui vont nous permettre la résolution du problème en utilisant des algorithmes qui existent déjà. Dans quelques cas spéciaux on évalue le gradient de la fonction objective et on arrive à la résolution d'un vaste ensemble de problèmes même quand l'expression exacte de la fonction objective n'est pas connue. On applique cette méthodologie dans quelques cas.*

Mots clés : Localisation, distances moyennes, services sur régions.

1. INTRODUCTION

The continuous single-facility location problem in its many variations and generalizations has been widely studied in the literature of Location Theory [11, 20]. The objective of the problem is to place a facility somewhere on a region in some optimal manner in order to serve the existing demand.

Although most existing papers suppose that the demand is concentrated at a discrete set of points and the facility has negligible size, (thus assumed to be a point), there are cases where the demand fits better when it is

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considered to be distributed over a region rather than concentrated at points. In addition, the assumption that the facility has also an area leads to more realistic models, which can be applied, for example, to the location of public facilities such as parks or industrial areas to serve different communities.

Different motivations may be made for the problem to be studied. For example, existing facility or demands may be points, although each point is a random variable with a probability measure over an area. This is the case of location problems under conditions of uncertainty [6, 9, 14]. Secondly the facilities may have areas instead of points, as suggested for instance in [16]. A third interpretation is that the problem deals with a very large number of demand and facility points clustered in some neighborhoods, thus being more appropriate to model it as an area-demand and area-facility problem rather than a conglomerate of many points [7].

Traditionally the Weber problem [10, 11] has been addressed in its point version. However, since the early seventies one can find the regional approach in different papers. Love [17] presents a computational procedure for locating optimally a facility when the demand is distributed in a union of rectangles. Drezner [5] and Juel [14] discuss the Weber problem with demands uniformly distributed on circular regions. Drezner and Wesoloswky [8] develop an iterative procedure for solving problems with regional demand when distances are measured by means of l_p norms ($p > 1$). Drezner [7] considers the Weber problem where both demand and the facility are assumed to be uniformly distributed on circular shapes. Aly and Marucheck [1, 18] deal with the problem of demand uniformly distributed over rectangular regions with the Manhattan norm. Recently Koshizuka and Kurita [16] have studied the Weber problem with uniform demand on circular areas using approximate formulas for average distances.

As each author develops a particular solution-method which does not work directly for other problems, it is desirable to develop a unified approach, this is the aim of the paper. If one could obtain explicit functional forms for the expected distance between regions then these problems could be solved by standard optimization procedures. However, observe that the expected distance between two regions is given by

$$\int \int \|x - y\| \, dP(x) \, dQ(y)$$

where P and Q are probability measures in $\mathbb{R}^2$ and even in the most simple case of uniform distributions, the evaluation of these formulae, when possible, is very time-consuming [16].

In the literature there exist other approaches which avoid the exact calculation of expected distances [1, 4, 7, 18] but are only valid for particular norms and shapes.

Now we describe the contents of the paper. In Section 2, we present the model to be studied and we introduce a notation which is inspired in the standard in Queuing and Scheduling Theory. In Section 3 we state some properties extending well-known results for the classical Weber problem which are the basis of an algorithm which does not need the evaluation of expected distances to solve the general problem. Finally we apply this methodology to two different cases in Sections 4 and 5.

2. FORMULATION AND NOTATION

The problem we study consists of placing a facility, whose shape F is fixed (but not its location), to serve a demand D. For instance, if one wants to locate a circle of radius R, then F is given by the circle centered at the origin with radius R, and the problem is where the circle should be located, which is equivalent to locating its center.

The only constraints imposed on F is compactness. We also assume that both the facility and the demands are distributed following independent random variables inside their regions. The distances between points are measured by a gauge γ, *see* [19, 21].

We must note that the unique decision variable is the translation vector x that moves F to the facility region $x + F = \{x + f : f \in F\}$.

Hence, the general location model we consider, hereafter called the Generalized Weber problem, is formulated as follows

$$\min_{x \in \mathbb{R}^2} \int_F \int_D \gamma(x + f - d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \tag{1}$$

where

$x = (x_1, x_2)$, $d = (d_1, d_2) \in D$, $f = (f_1, f_2) \in F$

γ = a gauge in $\mathbb{R}^2$

D = the set of demand points

F = the shape of the facility

P = probability measure over D

Q = probability measure over F

It is useful to define the following functions:

$$\psi : \quad x \mapsto \psi(x) = \int_D \gamma(x - d)\, \mathrm{d}P(d)$$

$$\nu_F : \quad x \mapsto \nu_F(x) = \int_F \psi(x+f)\,\mathrm{d}Q(f)$$

$\psi(x)$ represents the expected distance (measured by the gauge γ) between x and the set D of demand points and $\nu_F(x)$ represents the expected distance from the facility in $x+F$ to the region D.

The formulation of Problem 1 includes as particular instances different problems proposed in the literature. For instance, for $F = \{(0, 0)\}$, a finite set $D = \{a_1, \ldots, a_n\}$ and probability measures

$$Q(x) = \begin{cases} 1 & \text{if } x = (0, 0) \\ 0 & \text{otherwise} \end{cases}, \qquad P(x) = \begin{cases} w_i & \text{if } x = a_i \quad i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

where $\sum_{i=1}^{n} w_i = 1$, we obtain the classical Weber problem [11]

$$\min_{x \in \mathbb{R}^2} \sum_{i=1}^{n} w_i\, \gamma(x - a_i) \tag{2}$$

For $F = \{(0, 0)\}$, $D = \bigcup_{i=1}^{n} R_i$, where each R_i is a rectangle, $\gamma(\cdot) = \|\cdot\|_2$, Q is the degenerate measure over (0, 0) and P is a mixture of uniform measures over each R_i, we obtain the location problem

$$\min_{x \in \mathbb{R}^2} \sum_{i=1}^{n} w_i \int_{R_i} \|x - b\|_2\, \mathrm{d}P(b)$$

addressed in [17].

Finally, for $\gamma(\cdot) = \|\cdot\|_2$ and uniform distributions over the sets F and D one obtains (*see* [7])

$$\min_{x \in \mathbb{R}^2} \frac{1}{\mu(F)\,\mu(D)} \int_F \int_D \|x + f - a\|_2\, \mathrm{d}a\, \mathrm{d}f$$

where μ is the Lebesgue measure in $\mathbb{R}^2$.

We have now developed enough terminology to introduce the notation we propose to classify these location problems. The notation discussed in this section is inspired in Kendall's notation [15] for Queuing Theory. Each of these location problems is described by six characteristics:

$$1/2/3/4/5/6$$

The first and the second characteristics describe the type of the probability measure over the facility F and the demand set D. The following standard abbreviations are used:

D = deterministic, which corresponds to assume that the facility or the demand are points.

U = uniform measure

MU = mixture of uniform measures

G = general probability measure

Any necessary parameter has to be added.

The third characteristic is the number of facilities to be located.

The fourth characteristic specifies the type of facility to be located. The following standard abbreviations are used:

x = point facility

F = regional facility with shape F.

The fifth characteristic specifies the shape of the demand set D. When D is the union of more simple sets D_i, $i \in I$, one can use the following convention $D = \bigcup_{i \in I} D_i$.

The sixth characteristic describes the gauge used to measure the distances. The standard abbreviations used in Location Theory may appear.

As an illustration, $D/G/2/x/R/\|\cdot\|_2$ represents a problem with two point facilities with demand set R, a general probability measure over D and distances measured by means of the Euclidean norm.

3. PROPERTIES

In this section we discuss some properties of the objective function ν_F of the Generalized Weber problem, which show some similarities and differences that exist between Problem 1 and its point version, Problem 2. First of all, we assume that the facility F and the demand D are compact sets, which is not restrictive from a practical viewpoint. In what follows we assume it without explicit mention.

As these results do not require other assumptions on F than compactness, they also hold for ψ, which corresponds to $\nu_{\{(0,0)\}}$.

PROPERTY 3.1: *If the probability measures P and Q are absolutely continuous then*

$$\nu_F(x) > 0, \qquad \forall x \in \mathbb{R}^2$$

The proof follows from the definition of ν_F.

PROPERTY 3.2: *There exists a constant k such that $\nu_F(x) \leq \gamma(x) + k$, $\forall x \in \mathbb{R}^2$, hence the function ν_F is finite everywhere.*

Proof: By definition,

$$\nu_F(x) = \int_F \int_D \gamma(x + f - d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f)$$

Let $\alpha = \max_{d \in D} \gamma(-d)$ and $\beta = \max_{f \in F} \gamma(f)$. Observe that α and β are finite because D and F are, by assumption, compact sets.

Then,

$$\begin{aligned}\int_D \gamma(x + f - d)\, \mathrm{d}P(d) &\leq \int_D (\gamma(x) + \gamma(f) + \gamma(-d))\, \mathrm{d}P(d) \\ &\leq \int_D (\gamma(x) + \beta + \alpha)\, \mathrm{d}P(d) = \gamma(x) + \beta + \alpha.\end{aligned}$$

Hence,

$$\nu_F(x) \leq \int_F (\gamma(x) + \beta + \alpha)\, \mathrm{d}Q(b) = \gamma(x) + \beta + \alpha$$

Taking $k = \beta + \alpha$, the result holds. □

PROPERTY 3.3: *ν_F is a proper convex function.*

Proof: First of all ν_F is finite everywhere. Beside for any x, $y \in \mathbb{R}^2$, $\lambda \in [0, 1]$, one has

$$\begin{aligned}&\nu_F(\lambda x + (1 - \lambda) y) \\ &\quad = \int\int \gamma(\lambda x + (1 - \lambda) y + f - d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\ &\quad = \int\int \gamma(\lambda(x + f - d) + (1 - \lambda)(y + f - d))\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\ &\quad \leq \int\int (\lambda\gamma(x + f - d) + (1 - \lambda)\gamma(y + f - d))\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\ &\quad = \lambda\nu_F(x) + (1 - \lambda)\nu_F(y)\end{aligned}$$

Hence ν_F is a proper convex function. □

As a consequence of property above one has

PROPERTY 3.4: *The function ν_F is continuous in $\mathbb{R}^2$.*

PROPERTY 3.5: *The function ν_F has compact lower level sets.*

Proof: For all $R > 0$ let $L(R) = \{x \in \mathbb{R}^2 : \nu_F(x) \leq R\}$. By Property 3.4, $L(R)$ is a closed set. It is only necessary to show that $L(R)$ is also bounded. Indeed, for any $x \in L(R)$, one has

$$\begin{aligned} R \geq \nu_F(x) &= \int_F \int_D \gamma(x + f - d)\, \mathrm{d}P(d)\, dQ(f) \\ &\geq \int_F \int_D (\gamma(x) - \gamma(-f) - \gamma(d))\, dP(d)\, dQ(f) \\ &\geq \gamma(x) - \max_{f \in F} \gamma(-f) - \max_{d \in D} \gamma(d) \end{aligned}$$

Hence, $L(R) \subset \{x : \gamma(x) \leq R + \max_{f \in F} \gamma(-f) + \max_{d \in D} \gamma(d)\}$, which is bounded. □

Properties above lead us to prove the existence of a solution for Problem 1. This is stated in the following theorem.

THEOREM 3.1: *The set of optimal solutions to Problem* 1 *is convex, compact and not empty.*

Proof: By Property 3.5 the set M of optimal solutions to Problem 1 is compact and not empty. Besides, as ν_F is convex M is also convex. □

A gauge γ is said to be strict if the boundary of its unit ball does not contain nondegenerate segments.

THEOREM 3.2: *If γ is a strict gauge and P is absolutely continuous then Problem* 1 *has a unique optimal solution.*

Proof: It suffices to show that ν_F is a strictly convex function. Let $x \neq y \in \mathbb{R}^2$, $0 < \lambda < 1$. For all $f \in F$ define the set

$$\begin{aligned} Z(f) = \{d \in D :\ & 0,\ \lambda(x + f - d), \\ & \text{and } (1 - \lambda)(y + f - d) \text{ are not collinear}\}, \end{aligned}$$

which is a set with probability $P(Z(f)) = 1$. Indeed, the complement $Z(f)^c$ of $Z(f)$ is the intersection of D with the line passing through $x + f$ and $y + f$, thus $Z(f)^c$ has zero Lebesgue measure, and by assumption, P is absolutely continuous. Hence $P(Z(f)) = 1$.

As γ is a strict gauge, for any $d \in Z(f)$ one has:

$$\gamma(\lambda(x+f-d)+(1-\lambda)\ (y+f-d)) < \lambda\gamma(x+f-d)+(1-\lambda)\gamma(y+f-d)$$

Hence,

$$\begin{aligned}
&\nu_F(\lambda x + (1-\lambda) y) \\
&\quad = \int \int \gamma(\lambda(x+f-d) + (1-\lambda)(y+f-d))\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\
&\quad = \int \int_{Z(f)} \gamma(\lambda(x+f-d) + (1-\lambda)(y+f-d))\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\
&\quad < \int \int_{Z(f)} (\lambda\gamma(x+f-d) + (1-\lambda)\gamma(y+f-d))\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\
&\quad = \lambda \int \int_{Z(f)} \gamma(x+f-d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\
&\qquad + (1-\lambda) \int \int_{Z(f)} \gamma(y+f-d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f) \\
&\quad = \lambda\, \nu_F(x) + (1-\lambda)\, \nu_F(y)
\end{aligned}$$

Hence ν_F is strictly convex, thus Problem 1 has exactly one optimal solution. □

The result above is not true for general gauges. As a simple counterexample, consider the problem $D/U/1/x/[0,\,1]^2 \cup [2,\,3]^2/l_1$. It can be shown that the set of optimal solutions to this problem consists of the square $[1,\,2]^2$.

These results lead to the characterization of a dominant set for the Problem 1, which extends the localization property given by Wendell and Hurter [24] for Problem 2.

Denote by conv (A) the convex hull of the set A and by $A - B$ the set $\{a - b\, :\, a \in A,\, b \in B\}$.

THEOREM 3.3: *If γ is a norm then* conv $(D - F)$ *contains at least an optimal solution to Problem* 1.

Proof: Let x^* be an optimal solution to (1). For each $d \in D$, $f \in F$, define:

$$R_{d-f}(x^*) = \{x : \gamma(x+f-d) \leq \gamma(x^* + f - d)\}$$

We are in position to show that conv $(D-F) \cap \bigcap_{z \in D-F} R_z(x^*) \neq \emptyset$. Indeed, if conv $(D-F) \cap \bigcap_{z \in D-F} R_z(x^*) = \emptyset$, by Helly's theorem there would exist

$z_1, z_2, z_3 \in D - F$ such that $\operatorname{conv}(D - F) \cap \bigcap_{i=1,2,3} R_{z_i}(x^*) = \emptyset$. Then

$$\operatorname{conv}(\{z_i : i = 1, 2, 3\}) \cap \bigcap_{i=1,2,3} R_{z_i}(x^*) = \emptyset$$

which contradicts Theorem 3 in [24]. Then

$$\operatorname{conv}(D - F) \cap \bigcap_{z \in D-F} R_z(x^*) \neq \emptyset.$$

Take

$$x' \in \operatorname{conv}(D - F) \cap \bigcap_{z \in D-F} R_z(x^*).$$

By construction, $\gamma(x' + f - d) \leq \gamma(x^* + f - d)$ for all $f \in F$, $d \in D$, hence

$$\int \gamma(x' + f - d)\, \mathrm{d}P(d) \leq \int \gamma(x^* + f - d)\, \mathrm{d}P(d), \qquad \forall f \in F$$

thus $\nu_F(x') \leq \nu_F(x^*)$.

As x^* was by assumption an optimal solution to (1), so is x' and this proves the theorem. □

An interesting question concerning Location Problems is to develop optimality conditions. Their usefulness, for deriving iterative resolution procedures, is well known [19].

Since ν_F is a convex function, the primal optimality condition at a point x is $0 \in \partial \nu_F(x)$. We shall prove that when P is absolutely continuous it is possible to obtain a simpler condition because of the properties of function ν_F.

Applying Theorem 1 in [13] and Corollary 25.5.1 in [21] one obtains the following

THEOREM 3.4: *If P is absolutely continuous, then ν_F is continuously differentiable everywhere and*

$$\nabla \nu_F(x) = \int\int \nabla \gamma(x + f - d)\, \mathrm{d}P(d)\, \mathrm{d}Q(f).$$

To sum up, we have shown that (1) is an unconstrained convex problem which always has an optimal solution. Furthermore, when P is absolutely continuous, the objective function ν_F is differentiable, and its gradient is given in Theorem 3.4. Hence, a number of existing algorithms could be used. Nevertheless, as ν_F is given by an integral, even its evaluation may be a hard task.

It is sometimes easier to evaluate $\nabla \nu_F$ rather than ν_F; for these cases, algorithms that solve the problem avoiding the evaluation of ν_F, like the Steepest Descent method with fixed step [2] or the Ellipsoid Method with central cut [3, 12] seem to be more appropriate. We discuss these ideas in the next two sections.

4. THE $D/G/1/x/D/\gamma_{\text{Polyhedral}}$ MODEL

In this section we address the problem of locating a point facility when the demand D has an absolutely continuous probability measure P and distances are measured by a polyhedral gauge γ, *i.e.*, a gauge whose unit ball is a polyhedron, (*see* [19, 23] for further details).

Then the problem can be formulated as

$$\min_{x \in \mathbb{R}^2} \psi(x) = \int_D \gamma(x - d)\, \mathrm{d}P(d) \tag{3}$$

In order to solve Problem 3 we first obtain a tractable expression for the gradient. Let v^i, $i = 1, \ldots, n$ be the extreme points of the unit ball of the gauge dual to γ. It is well known that $\gamma(x) = \max_{i=1, \ldots, n} v^i x'$. For $i = 1, \ldots, n$, let $Q_i = \{x \in \mathbb{R}^2 : \gamma(x) = v^i x'\}$.

THEOREM 4.1: *Let P be absolutely continuous and γ a polyhedral gauge. Then*

$$\nabla \psi(x) = \sum_{i=1}^{n} P(x - Q_i)\, v^i$$

Proof: By Theorem 3.4 one has

$$\nabla \psi(x) = \int \nabla \gamma(x - d)\, \mathrm{d}P(d)$$

As $\bigcup_{i=1}^{n} Q_i = \mathbb{R}^2$, $\operatorname{int}(Q_i) \cap \operatorname{int}(Q_j) = \emptyset$, $\forall i \neq j$ and $P(Q_i) = P(\operatorname{int}(Q_i))$ it follows that

$$\begin{aligned} \nabla \psi(x) &= \sum_{i=1}^{n} \int_{x-\operatorname{int}(Q_i)} \nabla \gamma(x-d)\, \mathrm{d}P(d) \\ &= \sum_{i=1}^{n} \int_{x-\operatorname{int}(Q_i)} v^i\, \mathrm{d}P(d) \\ &= \sum_{i=1}^{n} \int_{x-\operatorname{int}(Q_i)} \mathrm{d}P(d)\, v^i \\ &= \sum_{i=1}^{n} P(x - \operatorname{int}(Q_i))\, v^i \end{aligned}$$

This completes the proof. □

The theorem above leads to simple optimality conditions in the polyhedral case.

COROLLARY 4.1: *$x^* \in \mathbb{R}^2$ is an optimal solution to Problem* 3 *iff* $\sum_{i=1}^{n} P(x^* - Q_i)\, v^i = 0.$

Some examples that show the usefulness of this result are now given.

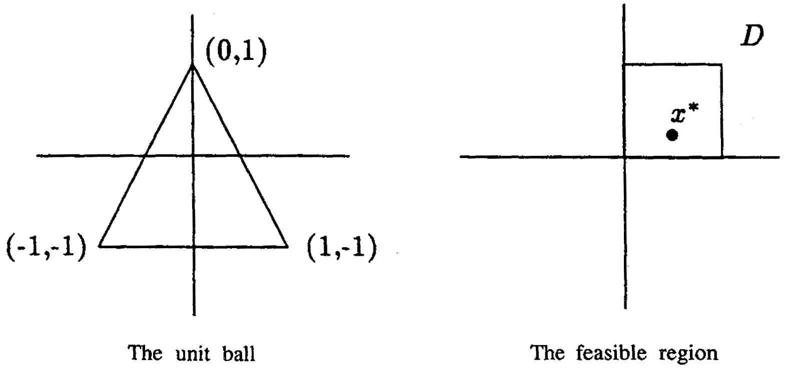

Figure 1. – Example 4.1.

Example 4.1: Consider the asymmetric polyhedral gauge γ_T whose unit ball is the triangle with vertices $(-1, -1)$, $(1, -1)$, $(0, 1)$. It is easy to show that $\gamma_T(x) = \max_{1 \leq i \leq 3} v^i x'$, with $v^1 = (2, 1)$, $v^2 = (-2, 1)$, $v^3 = (0, -1)$.

Consider the square $D = [0, 1]^2$ and let P be the uniform measure over D. In this example the corollary above can be applied to obtain an optimal solution. Indeed x^* is an optimal solution iff

$$\begin{aligned} 2P(x^* - Q_1) - 2P(x^* - Q_2) \qquad\qquad\qquad &= 0 \\ P(x^* - Q_1) + P(x^* - Q_2) - P(x^* - Q_3) &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} P(x^* - Q_1) &= 1/4 \\ P(x^* - Q_2) &= 1/4 \\ P(x^* - Q_3) &= 1/2 \end{aligned}$$

This system of equations is easily solved giving $x^* = (1/2, 1/4)$.

Example 4.2: Let $\|\cdot\|_1$ be the l_1-norm,

$$\|x\|_1 = |x_1| + |x_2| = \max_{1 \leq i \leq 4} v^i x'$$

where $v^1 = (1, 1)$, $v^2 = (-1, 1)$, $v^3 = (-1, -1)$, $v^4 = (1, -1)$.

The optimality condition above states that x^* is an optimal solution iff

$$\begin{aligned} P(x^* - Q_1) - P(x^* - Q_2) - P(x^* - Q_3) + P(x^* - Q_4) &= 0 \\ P(x^* - Q_1) + P(x^* - Q_2) - P(x^* - Q_3) - P(x^* - Q_4) &= 0 \end{aligned}$$

which occurs iff x^* is the 2-dimensional median. With the demand set of Example 4.1 the optimal solution is $x^* = (1/2, 1/2)$.

The rest of this section is devoted to illustrate how Theorem 4.1 can be applied in order to evaluate the gradient of the objective function for Problem 3, enabling the use of an optimization scheme to solve the problem when demands are distributed uniformly on circles and distances are given by the $l_{1,\infty}$ norm [22]. Remark that the same methodology with minor changes can be applied to any polyhedral gauge.

The problem to be solved has the following formulation

$$\min_{x \in \mathbb{R}^2} \sum_{i=1}^{n} \lambda_i \int_{C_i} \|x - f\|_{1,\infty} \, dP_i(f)$$

where each P_i represents the uniform measure over the circle C_i.

Thanks to the linearity of the gradients and integrals and as by definition $\|x\|_{1,\infty} = \alpha_1 \sqrt{2} \|x\|_\infty + \alpha_2 \|x\|_1$, it follows that calculating the gradient of the objective function above reduces to evaluating the gradient of the expected distances to a circle in $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

We first derive an expression for the gradient, hereafter called $\nabla_\infty(x)$, of the expected l_∞ distance from an arbitrary point x to the circle C centered at (0, 0) with radius r.

In this situation the sets Q_i are given by

$$\begin{aligned}
Q_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0, \ -x_1 + x_2 \geq 0\} \\
Q_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 0, \ -x_1 + x_2 \geq 0\} \\
Q_3 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 0, \ -x_1 + x_2 \leq 0\} \\
Q_4 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 0, \ -x_1 + x_2 \leq 0\}
\end{aligned}$$

and then

$$\begin{aligned}
\nabla_\infty(x) = &P(x - Q_1)(0, 1) + P(x - Q_2)(-1, 0) \\
&+ P(x - Q_3)(0, -1) + P(x - Q_4)(1, 0)
\end{aligned}$$

Define for each i, $p_i(x) = P(x - Q_i)$. Thanks to the symmetry of the circle and the unit ball of the norm, the calculation of $p_i(x)$ is reduced to the case $x \in Q_1$. Indeed, for example, let $x \in Q_2$, then

$$\begin{aligned}
p_1(x) &= P(\{(c_1, c_2) \in C : -x_1 + c_1 + x_2 - c_2 \geq 0, \\
&\qquad x_1 - c_1 + x_2 - c_2 \geq 0\}) \\
&= P(\{(c_1, c_2) \in C : x_1 - c_1 - x_2 + c_2 \leq 0, \\
&\qquad -x_1 + c_1 - x_2 + c_2 \leq 0\}) \\
&= P((x_2, -x_1) - Q_4) \\
&= p_4(x_2, -x_1)
\end{aligned}$$

With similar arguments, one can obtain the following table of equivalences

	$x \in Q_1$	$x \in Q_2$	$x \in Q_3$	$x \in Q_4$
$p_1(x)$	$p_1(x_1, x_2)$	$p_4(x_2, -x_1)$	$p_3(-x_1, -x_2)$	$p_2(-x_2, x_1)$
$p_2(x)$	$p_2(x_1, x_2)$	$p_1(x_2, -x_1)$	$p_4(-x_1, -x_2)$	$p_3(-x_2, x_1)$
$p_3(x)$	$p_3(x_1, x_2)$	$p_2(x_2, -x_1)$	$p_1(-x_1, -x_2)$	$p_4(-x_2, x_1)$
$p_4(x)$	$p_4(x_1, x_2)$	$p_3(x_2, -x_1)$	$p_2(-x_1, -x_2)$	$p_1(-x_2, x_1)$

Now, we show how to calculate $p_i(x)$ when $x \in Q_1$. In order to do that we decompose Q_1 into 5 zones, Z_1 to Z_5, defined as follows

$$\begin{aligned}
Z_1 &= \{(x_1, x_2) : -x_1 + x_2 \geq \sqrt{2}\, r,\ x_1 + x_2 \geq \sqrt{2}\, r\} \\
Z_2 &= \{(x_1, x_2) : -x_1 + x_2 < \sqrt{2}\, r,\ x_1 + x_2 \geq \sqrt{2}\, r\} \\
Z_3 &= \{(x_1, x_2) : -x_1 + x_2 \geq \sqrt{2}\, r,\ x_1 + x_2 < \sqrt{2}\, r\} \\
Z_4 &= \{(x_1, x_2) : -x_1 + x_2 < \sqrt{2}\, r,\ x_1 + x_2 < \sqrt{2}\, r,\ \|x\|_2 \geq r\} \\
Z_5 &= \{(x_1, x_2) : -x_1 + x_2 < \sqrt{2}\, r,\ x_1 + x_2 < \sqrt{2}\, r,\ \|x\|_2 < r\}
\end{aligned}$$

Within each zone the area of $\{x - Q_i\} \cap C$ is easily calculated using the formula

$$r^2\, a\cos\left(\frac{h}{r}\right) - hr\sqrt{1 - \left(\frac{h}{r}\right)^2}$$

which gives the area of the region S of Figure 2, where h is the distance from the origin to the line l.

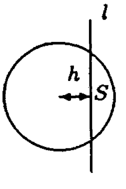

Figure 2. – Circular segment S.

In order to give a compact expression for p_i we introduce the function

$$\rho(t) = \frac{1}{\pi}\left(a\cos(t) - t\sqrt{1 - t^2}\right)$$

Then within each zone, the probabilities p_i are easily shown to be given by

• **Zone** Z_1

$$\begin{aligned}
p_1(x) &= 1 \\
p_2(x) &= 0 \\
p_3(x) &= 0 \\
p_4(x) &= 0
\end{aligned}$$

• **Zone** Z_2

$$\begin{aligned} p_1(x) &= 1 - p_4(x) \\ p_2(x) &= 0 \\ p_3(x) &= 0 \\ p_4(x) &= \rho\left(\frac{|x_1 - x_2|}{\sqrt{2}\,r}\right) \end{aligned}$$

• **Zone** Z_3

$$\begin{aligned} p_1(x) &= 1 - p_2(x) \\ p_2(x) &= \rho\left(\frac{|x_1 + x_2|}{\sqrt{2}\,r}\right) \\ p_3(x) &= 0 \\ p_4(x) &= 0 \end{aligned}$$

• **Zone** Z_4

$$\begin{aligned} p_1(x) &= 1 - p_2(x) - p_4(x) \\ p_2(x) &= \rho\left(\frac{|x_1 + x_2|}{\sqrt{2}\,r}\right) \\ p_3(x) &= 0 \\ p_4(x) &= \rho\left(\frac{|x_1 - x_2|}{\sqrt{2}\,r}\right) \end{aligned}$$

• **Zone** Z_5

$$\begin{aligned} p_1(x) &= 1 - p_2(x) - p_3(x) - p_4(x) \\ p_2(x) &= \rho\left(\frac{|x_1 + x_2|}{\sqrt{2}\,r}\right) - p_3(x) \\ p_3(x) &= \frac{|x_1 - x_2|\,|x_1 + x_2|}{2\,r^2\,\pi} \\ &\quad + \frac{1}{2}\left(\rho\left(\frac{|x_1 - x_2|}{2\sqrt{2}}\right) + \rho\left(\frac{|x_1 + x_2|}{2\sqrt{2}}\right)\right) - \frac{1}{4} \\ p_4(x) &= \rho\left(\frac{|x_1 - x_2|}{\sqrt{2}\,r}\right) - p_3(x) \end{aligned}$$

This concludes the description of the five subcases and leads, as shown above, to the determination of the p_i, $i=1, 2, 3, 4$ in the whole plane. Hence we can determine the gradient of the expected distance with the norm l_∞.

In order to obtain the gradient of the expected l_1 distance, the required probabilities can be evaluated after rotating an angle $\pi/4$ the demand set.

Now as an application consider the following location situation: some fire vigilance posts are placed on a forest. One can assume that regions covered by these posts are circles. Let us suppose known the expected fire alarm rate λ_i associated with the i-th circle in the period of interest (alarms per hour). A fire station minimizing the expected distance to the fire must be located. The use of the $l_{1,\infty}$ norm in this case might be applicable due to the existence of fire-break, which limit movements to the travel directions of this norm.

To give a numerical example we use the gradient method [2] to solve the problem with all the λ_i $i = 1, \ldots, n$ equal to 1 and coordinate posts and radii given in Table I.

TABLE I
Demand regions

Demand region	Radius	a_1	a_2
C_1	3	8	10
C_2	1	12	3
C_3	1.5	13	6
C_4	2	2	6
C_5	1	1	1

The following form of the norm is used

$$\|x\|_{1,\infty} = \sqrt{2}\,(1-\mu)\,\|x\|_\infty + \mu\,\|x\|_1$$

and the problem is solved for the different values of μ from 0 to 1 with stepsize 0.01. For each μ, the algorithm was stopped when the norm of the gradient was less than 10^{-3} and the difference between the coordinates of two successive solutions was less than 10^{-4}.

A curve fitting the solutions obtained is represented on Figure 3.

5. THE $U/MU/1/\mathbf{R}/\bigcup_{1\le i\le n} R_i/l_1$ MODEL

The aim of this section is to show how the methodology developed in section above can be applied to solve a location problem when both the facility and demand have an area. We obtain formulae for the gradient

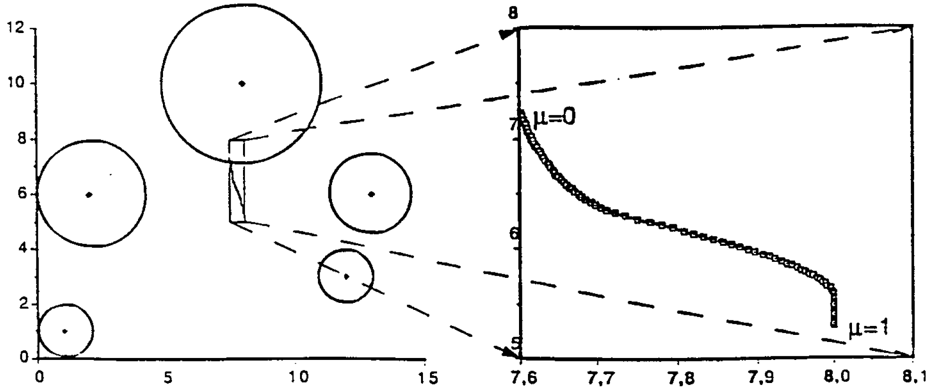

Figure 3. – Fitting solutions.

of the expected l_1-distance between two rectangles with parallel sides to the coordinate axes. Although in this case the problem could be solved exactly by others methods, the strategy proposed here can even be applied to the calculation of the gradient of ν_F, for any other polyhedral gauge and polyhedral regions.

The model dealt with is the Generalized Weber Problem with several rectangular-area demands, one rectangular-area facility, mixture of uniform distributions over the rectangles and distances measured by the l_1-norm.

In order to obtain the gradient of the objective function we consider, without loss of generality, a square S centered at the origin with side length 2 and a rectangle R centered at the point (x_0, y_0), with side lengths $2a$ and $2b$ respectively. Note that it is always possible to reduce the calculations to the case where $x_0 \geq 0$, $y_0 \geq 0$.

Let $\psi_R(u, v)$ and $\nabla \psi_R(u, v)$ denote respectively the expected l_1-distance from a point $P = (u, v)$ inside S to R and its gradient. As in Section 4, let $Q_i = \{x \in \mathbb{R}^2 : \|x\|_1 = v^i x'\}$, $i = 1, \ldots, 4$, where $v^1 = (1, 1)$, $v^2 = (-1, 1)$, $v^3 = (-1, -1)$, $v^4 = (1, -1)$. We first obtain the expressions for $\nabla \psi_R(u, v)$. Four different cases must be considered.

Case 1. The rectangle R is contained in the set $P + Q_1$.

$$\nabla \psi_R(u, v) = (-1, -1)$$

Case 2. The rectangle R intersects the sets $P + Q_1$, $P + Q_4$ and is contained in $P + (Q_1 \cup Q_4)$.

$$\nabla \psi_R(u, v) = \left(-1, \frac{v - y_0}{b}\right)$$

Case 3. The rectangle R intersects the sets $P + Q_1$, $P + Q_2$ and is contained in $P + (Q_1 \cup Q_2)$.

$$\nabla \psi_R(u, v) = \left(\frac{u - x_0}{a}, -1\right)$$

Case 4. The rectangle R contains the point P.

$$\nabla \psi_R(u, v) = \left(\frac{u - x_0}{a}, \frac{v - y_0}{b}\right)$$

The proofs directly follow from Theorem 4.1.

Our attention now turns to the computation of $\nabla \nu_R(0)$, the gradient of the expected l_1 distance from S to R. Four cases must be distinguished. It is noted that by the symmetry of the shapes and by means of rotations with $\pm\pi/2$ and $\pm\pi$ angles any other case can be reduced to one of those.

Let $V = (x_0 - a, y_0 - b)$, the lower, left vertex of R. The following cases have to be considered:

Case A.1. The square S is inside the set $V + Q_3$

$$\nabla \nu_R(0) = (-1, -1)$$

Proof: All the points inside S (*see* Figure 4) verify the Case 1 above. Thus, by direct integration one obtains the result. □

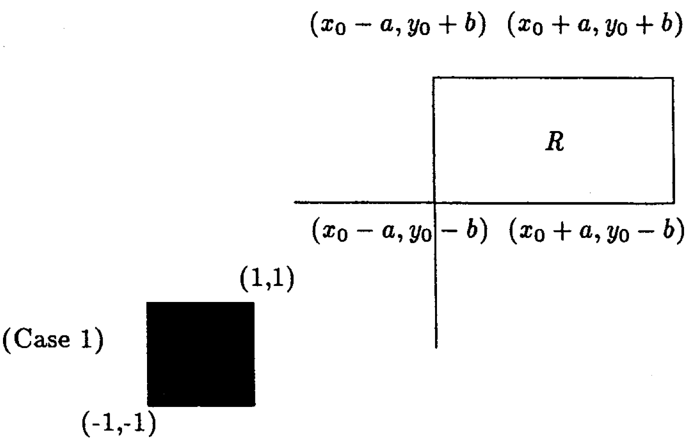

Figure 4. – Case A.1.

Case A.2. The square S intersects the sets $V + Q_2$, $V + Q_3$ and is contained in $V + (Q_2 \cup Q_3)$

$$\nabla \nu_R (0) = \left(-1, \; \frac{(y_0 - b)^2 - 2\,(y_0 + b) + 1}{4\,b}\right)$$

Proof: In this case (*see* Figure 5) the points inside S belong to either Case 1 or 2 above. The points inside the rectangle $[-1, 1] \times [-1, y_0 - b]$ are in Case 1. The points inside the rectangle $[-1, 1] \times [y_0 - b, 1]$ are in Case 2. Thus $\nabla \nu_R (0)$ is given by

$$\nabla \nu_R (0) = \frac{1}{4}\left(\int_{-1}^{1}\int_{-1}^{y_0 - b} (-1, \, -1)\, dv\, du + \int_{-1}^{1}\int_{y_0 - b}^{1} \left(-1, \; \frac{v - y_0}{b}\right) dv\, du\right)$$

Integrating the expression above the result follows. □

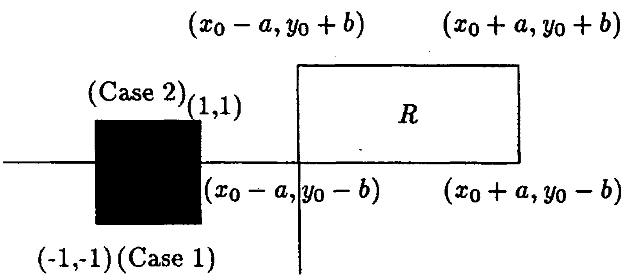

Figure 5. – Case A.2.

Case A.3. The square S intersects the sets $V + Q_3$, $V + Q_4$ and is contained in $V + (Q_3 \cup Q_4)$

$$\nabla \nu_R (0) = \left(\frac{(x_0 - a)^2 - 2\,(x_0 + a) + 1}{4\,a}, \; -1\right)$$

Proof: In this case (*see* Figure 6) the points inside S belong to one of the Cases 1 or 3. The points inside the rectangle $[-1, x_0 - a] \times [-1, 1]$ are in Case 1. The points inside the rectangle $[x_0 - a, 1] \times [-1, 1]$ are in Case 3. Hence

$$\nabla \nu_R (0) = \frac{1}{4}\left(\int_{-1}^{x_0 - a}\int_{-1}^{1} (-1, \, -1)\, dv\, du + \int_{x_0 - a}^{1}\int_{-1}^{1} \left(\frac{u - x_0}{a}, \; -1\right) dv\, du\right)$$

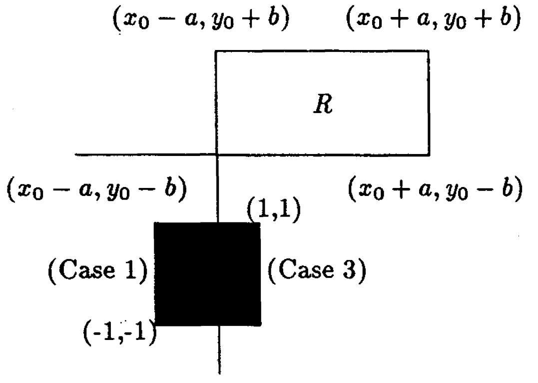

Figure 6. – Case A.3.

Integrating the expression above the result follows. $\square$

Case A.4. The square S contains the point $V = (x_0 - a,\ y_0 - b)$:

$$\nabla \nu_R(0) = \left(\frac{(x_0 - a)^2 - 2(x_0 + a) + 1}{4a}, \frac{(y_0 - b)^2 - 2(y_0 + b) + 1}{4b} \right)$$

Proof: In this case (*see* Figure 7) there exist points inside S belong to all the four cases. The points inside the rectangle $[-1,\ x_0 - a] \times [-1,\ y_0 - b]$ are in Case 1. The points inside the rectangle $[-1,\ x_0 - a] \times [y_0 - b,\ 1]$ are in Case 2. The points inside the rectangle $[x_0 - a,\ 1] \times [-1,\ y_0 - b]$ are in Case 3. Finally the points inside the rectangle $[x_0 - a,\ 1] \times [y_0 - b,\ 1]$

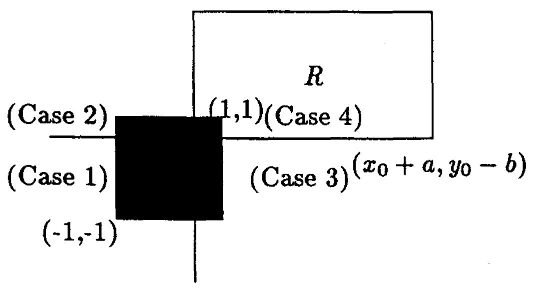

Figure 7. – Case A.4.

are in Case 4. Hence

$$\begin{aligned}\nabla \nu_R(0) &= \frac{1}{4}\left(\int_{-1}^{x_0-a}\int_{-1}^{y_0-b}(-1,\,-1)\,dv\,du\right.\\ &\quad+\int_{-1}^{x_0-a}\int_{y_0-b}^{1}\left(-1,\,\frac{v-y_0}{b}\right)dv\,du\\ &\quad+\int_{x_0-a}^{1}\int_{-1}^{y_0-b}\left(\frac{u-x_0}{a},\,-1\right)dv\,du\\ &\quad\left.+\int_{x_0-a}^{1}\int_{y_0-b}^{1}\left(\frac{u-x_0}{a},\,\frac{v-y_0}{b}\right)dv\,du\right)\end{aligned}$$

Integrating the expression above the result follows. □

It may be remarked that, once one has an expression for the gradient, it can be integrated into an optimization scheme similar to the one used in the section above and the problem can be solved.

6. CONCLUDING REMARKS

In this paper we have addressed the Generalized Weber Problem with Expected Distances, which includes as particular cases most Weber problems studied in the literature. We develop a methodology to solve the general problem which avoids the difficult calculation of the expected distance between two regions, because it only needs the calculation of the gradient, which is in general simpler.

We propose an easy method to compute the gradient, which reduces its computation to the evaluation of some simple probabilities, when the distances are measures by a polyhedral gauge.

Some properties of the Classical Weber Problem are extended to the Generalized Problem. Localization results, stability and geometrical characterization of optimal solutions, or even the determination of the optimal shape for the facility when the area is fixed are open questions which are now under study.

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