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STABILITY OF NASH EQUILIBRIA IN LOCATIONAL GAMES (*)

by J. BHADURY [1] and H. A. EISELT [1]

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Abstract. – *Consider a locational game on a network in which two competing facilities charge fixed, but not necessarily equal, prices and the decision variables are their respective locations. Rather than deciding in a given situation whether or not an equilibrium exists, we devise a stability index that measures the stability or instability of a given situation. In other words, given that an equilibrium exists, our index indicates how much external effort (or subsidy) is required to destroy that equilibrium; if equilibria do not exist, the index shows how much external effort (or tax) is needed to "generate" an equilibrium. Computational evidence for randomly generated problems is presented.*

Keywords: Competitive location, Nash equilibria, stability.

Résumé. – *Nous considérons un jeu de localisation dans un réseau avec deux établissements compétitifs, des prix fixes, mais pas nécessairement égaux. Les variables de décision sont les localisations des établissements. Au lieu de déterminer s'il y a un équilibre dans une certaine situation, nous construirons un indice de stabilité qui mesure la stabilité ou l'instabilité d'un arrangement de localisations. En d'autres termes, si un équilibre existe, notre indice indique combien d'effort extérieur (ou de subvention) est nécessaire pour détruire cet équilibre; si l'équilibre n'existe pas, l'indice indique combien d'effort extérieur (ou d'impôt) est nécessaire pour « générer » un équilibre. Nous présentons des résultats sur une série des problèmes générés de façon aléatoire.*

Mots clés : Problèmes de localisation compétitive, indice de stabilité, résultats numériques.

1. INTRODUCTION

The model that forms the basis of this paper has its roots in the work of Hotelling (1929). In his model two dupolists locate on a linear market, *i.e.* a line segment with potential customers evenly distributed along the market. The products offered by the competitors are homogeneous, and customers make their purchases from the cheapest source. The decision variables available to each of the two competitors are price and location. Hotelling concluded that if both duopolists were to charge equal prices and locate at the center of the market, a Nash equilibrium would be reached,

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i.e. a situation in which neither competitor can unilaterally improve its profit by changing its price or location. Even though d'Aspremont *et al.* (1979) showed that Hotelling's argument was flawed, his analysis has become the basis of a plethora of competitive spatial models put forward by researchers over the last sixty years.

Most subsequent authors have focused on existence and uniqueness of equilibria in Hotelling models. For some introductory surveys, the reader is referred to Graitson (1982), Greenhut *et al.* (1987), Friesz *et al.* (1988), Hakimi (1990), Gabszewicz and Thisse (1986 and 1991), Eiselt *et al.* (1993), and Labbé *et al.* (1993). The discussion of equilibria is the focusses almost exclusively on the yes – no dichotomy: either an equilibrium exists or it does not. Clearly, life is more subtle than that (and so is the theory of location games!). For instance, in physics we distinguish between stable (or self-restoring), neutral, and unstable equilibria. A similar approach is taken in this paper, where we replace the "*equilibrium – no equilibrium*" dichotomy by a continuum that is separated by a "stability line", a line that separates equilibria from non-equilibria. Loosely speaking, points on the continuum express the degree of stability (instability) in case an equilibrium does (or does not) exist by the distance of a given situation from the stability line. The introduction of stability in the discussion of competitive situations is not entirely new, though. Schofield (1978) studied stability in the context of dynamic games, more recently Kohlberg and Mertens (1986) and Wilson (1992) have employed the notion of stability in general two-person games. Stability of equilibria is also a topic that is well known and studied in political science and voting theory, *see*, e.g., Kramer (1977) and Tovey (1991 and 1993). The discussion in this paper will focus exclusively on the stability of locational games.

The remainder of this paper is organized as follows. In the second section we introduce our basic model and develop the stability index. In the third section we discuss the case in with equilibria exist, in the fourth section we investigate the case of disequilibria, and in the fifth section we present some results of computational experiments. We conclude by offering some thoughts on possible extensions of our results.

2. THE BASIC MODEL

Even though our arguments can be applied to competitive location problems with any number of competitors in arbitrary spaces, we will illustrate most of our concepts for a duapoly on a graph. For that purpose, define a graph $G = (V, E)$ with set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and set of edges $E = \{e_{ij} : v_i, v_j \in V\}$. The demand is assumed to occur only at vertices and is fixed (implying that we are dealing with an essential good), its magnitude is $w(v_i) \geq 0$, $\forall v_i \in V$. Let a positive distance be associated with each edge and the distance between two vertices v_i and v_j is then defined as the shortest path between v_i and v_j, denoted by d_{ij}. Competitors A and B are currently located at vertices v_A and v_B, respectively. Both charge mill prices, *i.e.* customers pick up the good at the facility and have to pay for the transport separately. The mill prices are p_A and p_B, respectively, so that, given unit transportation costs, the delivered prices of a customer at v_i are $p_A + d_{Ai}$ from facility A and $p_B + diB$ from facility B. Customers purchase from the facility that offers the lowest delivered price.

We can generally distinguish between two models concerning customer behavior.

(a) "Winner-take-all" means that a customer at a vertex v_i purchases $w(v_i)$ units from facility A, if $p_A + d_{iA} < p_B + d_{iB}$, and similarly from facility B. In case of equal delivered prices, customers satisfy their entire demand from the supplier with the lower mill price. This assumption is quite arbitrary and made solely for the reason of unequivocally describing the competitors' profit functions. Changing this assumption does in no way impede our arguments.

(b) In a "proportional model" customers purchase from both facilities. The proportion of their demand that they satisfy from a facility depends on the relation between the two delivered prices. In this paper we assume that a customer at v_i will purchase $\{(p_B + d_{iB})/[(p_A + d_{iA}) + (p_B + d_{iB})]\}\, w(v_i)$ from facility A and $\{(p_A + d_{iA})/[(p_A + d_{iA}) + (p_B + d_{iB})]\}\, w(v_i)$ from facility B. As an example, if the mill prices charged by A and B are 3 and 5, respectively, and a customer is located 7 miles from A and 3 miles from B, then the delivered prices are 10 and 8, respectively, and if the demand of the customer is 54, then in this model the customer will purchase [8/10 + 8] (54) = 24 units from A and [10/10 + 8] (54) = 30 units from facility$B$.

In both the winner-take-all and the proportional model, facilities A and B compete exclusively by adjusting their locations. More specifically, the planning facility will consider its opponent's location temporarily fixed and relocate to the vertex that offers the highest profit. We will refer to a locational game with *simultaneous* moves, if A and B (in a general n-person game: all players) optimize and relocate simultanously. Similarly, we call a locational game *sequential,* if facilities A and B optimize and relocate in an alternating sequence (in general n-person games: in a fixed order). In this paper we only consider simultaneous moves. Note that this also implies that we cannot prohibit location of the facilities at the same vertex as is possible in case of sequential moves. We assume inertia to prevail, *i.e.* if a competitor's highest profit is achieved at its current location as well as at some other vertex, it will not move.

With the above assumptions, we are now able to construct an index that measures the stability of locational arrangements on a given graph. Loosely speaking, the stability index developed in the following measures how much incentive, *i.e.* subsidy, it takes to destroy an equilibrium provided one exists; or how much disincentive, *i.e.* tax, it takes to create an equilibrium, given that none exists in the original model.

3. THE CASE OF EQUILIBRIA

We first consider the case in which equilibria exist. To facilitate the discussion, we consider any one of the equilibria. Denote the current profits of the facilities by $\wp_A^E$ and $\wp_B^E$, respectively. By definition, given facility B's current location, there is no place A could move to and achieve a profit level higher than $\wp_A^E$; the same is true for facility B. Let now $\wp_A^{E-2}$ $(\wp_B^{E-2})$ denote the highest profits $A\,(B)$ could possibly enjoy if it were forced out of the equilibrium under consideration; the superscript indicating a facility's move from equilibrium to the second-best solution, given its opponent's current location. Then $\Delta^+ \wp_A$ $(\Delta^+ \wp_B)$ is the smallest possible loss facility $A\,(B)$ would sustain if it were forced to move out of equilibrium, provided facility $B\,(A)$ does not move. Ignoring inertia for a moment, $\Delta^+ \wp_A = \wp_A^E - \wp_A^{E-2}$ is the dollar amount which leaves the decision maker at facility A indifferent between statying at its current location and moving to the next-best location. Defining $\Delta^+ \wp_B$ similarly, we can then compute $\Delta^+ = \min\{\Delta^+ \wp_A;\ \Delta^+ \wp_B\}$, so that $\Delta^+ + \varepsilon$ with $\varepsilon > 0$ but arbitrarily small, is the smallest subsidy which, if offered to each of the players for moving out of his current location, would be accepted by at least one of them, and thus destroys the equilibrium. Note that in case of multiple equilibria, we would have to compute Δ^+ values for each of them, the maximum value among all Δ^+ plus some small ε is then sufficient to destroy any of the equilibria in the model.

It stands to reason that a problem, in which it takes a lot effort or money to convince facilities to move out of an equilibrium, will be called stable. Similarly, if small amounts of subsidy are sufficient to make facilities move out of their equilibria, the problem will be considered fairly unstable. Hence, we use the expression Δ^+ as indicator for the stability of a problem's equilibria.

We should point out that our stability index as defined above is a purely local criterion in the sense that it only considers how to convince facilities to move out of an equilibrium situation; what happens after such a move is completely ignored. A *one-time* subsidy payment such as the one discussed here will temporarily move facilities out of an equilibrium, but in repeated subsequent moves the facilities may return to the same equilibrium as appears to be the case in many situations. The situation is different if the subsidy were offered in each step. However, if the facilities actually do return to an equilibrium or if the system is permanently thrown out of equilibrium is of no consequence here.

As an example for the computation of the stability index consider the case of two facilities locating at the vertices of a tree. Suppose that facilities locate at the same vertex, and assume that their mill prices p are fixed and equal. It is well known that an equilibrium is reached if both facilities locate at a median, *see*, e.g., Eiselt and Laporte (1991). Denote the median by v_q and let T_1^q, T_2^q, ... be subtrees that are generated by deleting from the given tree v_q and all edges incident to it. The weight of a tree or subtree is denoted by $w(T_i^q)$ and defined as the sum of weights of all vertices included in the tree. We can now order the subtress, so that $w(T_i^q) \geq w(T_j^q)$, $\forall\, i < j$. Finally, define v_q^1 as the unique vertex in T_1^q that is adjacent to the median v_q.

With A and B locating at v_q at equilibrium, the two facilities share the total demand $w(T)$ equally, *i.e.* sales of the two facilities are $S_A = S_B = \frac{1}{2} w(T)$ and profits are $\wp_A^E = \wp_B^E = \frac{1}{2} pw(T)$. Assume now that facility B were forced out of equilibrium to any other vertex on the tree. Eiselt and Laporte (1991) have shown that B's best option would be to locate at v_q^1, thus capturing sales of $S_A = w(T_q^1) \leq \frac{1}{2} w(T)$, leaving the remaining $w(T) - w(T_1^q)$ to A. Consequently, $\wp_A^{E-2} = p[w(T) - w(T_1^q)]$ and $\wp_B^{E-2} = pw(T_1^q)$. Facility B's loss is then $p\left[\frac{1}{2} w(T) - w(T_1^q)\right]$. The computations would be identical if facility A were to move out of equilibrium rather than B, thus our stability index is $\Delta^+ = p\left[\frac{1}{2} w(T) - w(T_1^q)\right]$. This implies that whenever the weight of the largest subtree generated by the median is small, our stability index Δ^+ assumes a large value. This coincides with the intuitive notion of stability as a small weight of the largest subtree implies that the weight of the entire tree is widely distributed among the subtrees. Such a distribution would be considered balanced, a situation that is usually associated with stability, confirming the above result.

4. THE CASE OF DISEQUILIBRIA

In this section we consider the case in which equilibria do not exist. Define $\wp_A^r(t)$ as the profit that is *realized* by facility A in period t, and similar for $\wp_B^r(t)$. Furthermore, let $\wp_A^a(t)$ denote the profit facility A *anticipates* to achieve in period $(t+1)$ while it plans in period t to maximize its profit in the succeeding period, assuming that facility B remains at its current location. Then $\Delta^- \wp_A(t) = \wp_A^a(t) - \wp_B^a(t)$ indicates the benefit derived by facility A when it relocates, and similar for $\Delta^- \wp_B(t)$. Clearly, if $\Delta^- \wp_A(t) > 0$, facility A will move, assuming zero relocation costs. In other words, it would take a fixed disincentive, or tax, of $\Delta^- \wp_A(t)$ to stop facility A from moving in period t. Neither facility will move provided that $\Delta^- \wp_A(t) \leq 0$ and $\Delta^- \wp_B(t) \leq 0$. Defining $\Delta^- \wp(t) = \max\{\Delta^- \wp_A(t); \Delta^- \wp_B(t)\}$, we can state neither facility will move in period t as long as $\Delta^- \wp(t) \leq 0$. Then a disincentive of $\Delta^- = \min_t\{\Delta^- \wp(t)\}$ assures that the relocation process stops eventually. We know, but for the moment ignore, the fact that the value Δ^- clearly depends on the initial location of the two facilities. Moreover, we will see that it also depends on the choice of tiebreaker. In other words, if a facility that considers relocation has more than one optimal choice, we have to specify the rule for choosing the next location. In essence, this leaves us with four cases to consider:

(a) The initial location is fixed and so is the tiebreaker rule.

(b) The initial location is fixed and ties are broken randomly.

(c) All initial locations are considered the tiebreaker rule is fixed.

(d) All initial locations are considered and ties are broken randomly.

For computational convenience we will denote the disincentives required to interrupt the process of relocation in the four above by Δ_a^-, Δ_b^-, Δ_c^-, and Δ_d^-. A few simple relations about the disincentives in the four cases are readily apparent. For instance, $\Delta_a^- \leq \Delta_c^-$ and $\Delta_b^- \leq \Delta_d^-$. Also, in case ($a$) there is only one cycle which has to be interpreted whereas in case (b), there may be many and since the idea is to stop movements eventually, it is sufficient to interrupt *any* of the cycles, hence $\Delta_a^- \geq \Delta_b^-$ and, similarly, $\Delta_c^- \geq \Delta_d^-$.

A simple example of a location problem on a linear market may explain cases (*c*) and (*d*). Let a linear market, *i.e.* a line segment, extend from 0 to L with $L/2$ denoting the center of the market. Two facilities A and B can locate anywhere on that market. Following Eiselt (1992), we can define *sufficient spatial separation* (SSS) as a distance equal to the price differential charged at the two facilities. If two facilities are at least SSS apart, then neither of them is cut out. The concept of SSS is similar to Eaton and Lipsey's (1975) "zero conjectural variation", the latter being, however, a behavioral assumption whereas SSS is simply a measure of distance. Formally, let p_A and p_B denote the fixed prices charged at facilities A and B, respectively, and assume that $p_A < p_B$. Now, whenever facilities A and B are less than $SSS = |p_A - p_B|$ distance units apart, then the cheaper facility A undercuts the more expensive facility B, leaving the latter with no market at all. On the other hand, if A and B are farther apart than SSS, then each captures its own hinterland (*i.e.* the area facing away from its opponent) and some of the competitive region between them. Consequently, A will always try to locate closer than SSS to B and gain the entire market, whereas B will locate $SSS + \varepsilon$, $\varepsilon > 0$ but arbitrarily small, distance units away from A towards the longer side of A. In doing so, B does not get anything in the competitive region but captures its own hinterland. If $SSS \geq L$, then A could locate at $L/2$, cut out B anywhere on the market, and capture everything. Consider now the case of $SSS < L$ and let x and y be two points given by $x = \frac{1}{2}[L - SSS]$ and $y = \frac{1}{2}[L + SSS]$ as shown in Figure 1. Suppose now that facilities A and B engage in sequential relocation, where one facility relocates so as to maximize its profit, then the other facility maximizes its own profit on the basis of its opponent's location. This process is then repeated until either an equilibrium is reached, or, in case no equilibrium exists, it continues indefinitely. We first show that, regardless of the facilities' initial locations, facility A will eventually locate between points x and y. Suppose this were not so. Without loss of generality, let A locate between 0 and x. Based on the above discussion, B then locates to the right of A at a

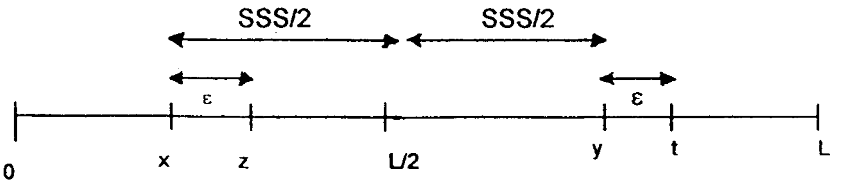

Figure 1.

distance of $SSS + \varepsilon$. B may now be located to the right or to the left of x, but it surely locates to the left of y. Assuming that A moves just as much as absolutely necessary, it moves to the right just enought to be closer than SSS to B. This move again pushes facility B to the right. The relocation continues until A locates at or to the right of x.

Suppose now that A is located between x and y at a point z as shown in Figure 1 and assume that it is now B's turn to move. As A has just moved, it controls the entire market and B's market share is currently zero. B now moves $SSS+\varepsilon$ distance units to the right to point t. At t, B captures its own hinterland which is $\frac{1}{2}[L - SSS] - \varepsilon$, which is also B's gain in the move as it captured nothing before. As $\varepsilon \in [0;\ SSS/2]$, B's gain is in the interval $[L/2 - SSS;\ L/2 - SSS/2]$. If there were a relocation cost or tax of this amount, then B would no longer move. This is our instability measure Δ^-. Note that for small values of SSS, *i.e.* similar prices, B's potential gain and the instability index is large. This is not really surprising as similar prices indicate a climate of active competition in which a slightly more expensive facility has a lot to lose when it is cut out.

Consider now the case of competitive locations on a network G. In order to compute values for Δ^-, it is useful to construct a *competition graph* $G^c = (N^c,\ A^c)$, as suggested by Eiselt and Bhadury (1993). The competition graphs for sequential and simultaneous moves are substantially different; here we are only concerned with the competition graph for simultaneous moves. The set of nodes $N^c = \{ (i,\ j) \}$ is defined for each possible pair of locations of the duopolists, *i.e.* the competition graph has $O(|V|^2)$ nodes. The set of arcs $A = \{ a_{ij,\,kl} \}$ is defined as follows. Assume that facility A currently locates at vertex v_i and facility B locates at v_j in the original graph G. Given B's location at v_j, A's optimal location is at vertex v_k, given A's location at v_i, B's optimal location is at vertex v_l. If ties exist for the maximum, arcs from v_j (v_i) to all such vertices v_k (v_l) exist. As an example, consider the graph in Figure 2 in which the double-digit numbers next to the vertices denote their weights and the single-digit numbers near the edges indicate the distances.

Given prices $p_A = 3$ and $p_B = 5$, Figure 3 *a* shows the competition graph associated with the graph in Figure 2 on the basis of the winner-take-all assumption. The competition graph in Figure 3 *b* is based on the proportional model.

As an example for its construction, consider A and B both locating at v_3. Presently, facility A undercuts B and hence captures all three vertices of

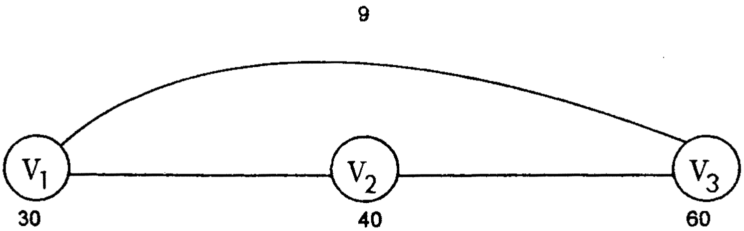

Figure 2.

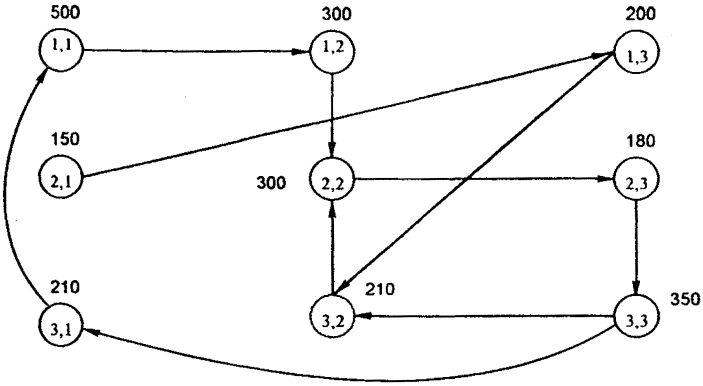

Figure 3 *a*.

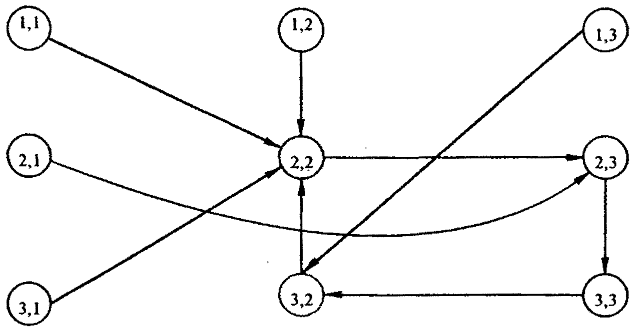

Figure 3 *b*.

the graph, whereas facility B gets nothing. The currently realized profits are $\wp_A^r(t)=390$ and $\wp_B^r(t)=0$. Given B's location at v_3, A's best option is to stay at v_3, still undercutting B and capturing the entire demand on the tree; its anticipated profit is $\wp_A^a=390$. Similarly, given A's current location at v_3, B's optimal strategy is to move to v_1 or v_2; in both cases B will capture the demand at vertices v_1 and v_2 for an anticipated profit of $\wp_B^a = 350$. Thus the competition graph has arcs leading from node (3, 3) to nodes (3, 1) and (3, 2). Note that if a node in the competition graph has outdegree zero, it constitutes an equilibrium. In the case under discussion in this section such a node does not exist. Finally, as discussed above, the smallest tax required to stop any further moves at some point where A is located at v_i and B at v_j is $\Delta^- \wp(t)$. In the above example, $\Delta^- \wp(t) = \max\{390\text{-}390; 350\text{-}0\} = 350$. In other words, if the two facilities were both located at v_3, then a tax of at least $350 would stifle any further moves and force an "equilibrium". In the competition graph, we assign such values $\Delta^- \wp(t)$ to all nodes. In the following these values will be referred to as *numerical labels.*

We are now able to discuss the four aforementioned cases in detail. As by assumption no equilibrium exists and the number of possible locations is finite, facilities A and B must eventually reach a pair of locations that they located at earlier. We will refer to any such path as a *relocation path* P_{ij}. In terms of the competition graph, each relocation path has the shape of a looking glass – a path leads from an original pair of locations (i, j) to some other pair of locations (k, l), and from there starts a cycle that eventually returns to node (k, l). In other words, relocation paths in G^c consists of a path and a cycle with the set of arcs on the path possibly empty. In case of a fixed tie-breaking rule, only one relocation path P_{ij} needs to be considered, in case of breaking ties randomly (with each choice having a strictly positive probability), all possible paths P_{ij} need to be known. The smallest numerical label of a node on any path P_{ij} is able to stop moves on that particular path and thus constitutes the desired tax Δ^-. We are now able to discuss the four cases introduced earlier.

Case a: The initial locations are fixed and there is a fixed tie-breaking rule. Only one relocation path P_{ij} exists and Δ^- is the minimum over all nodes on P_{ij}. As an example, consider the location pair (v_1, v_1) in the above example and suppose that ties are broken according to a maximum index rule. The relocation path is constructed directly from the competition graph and is shown in Figure 4 with the numerical labels $\Delta^- \wp(t)$ next to the nodes. In this example it takes a tax of $\Delta_a^- = \min\{500, 300, 300, 180, 350, 210\} = 180$ to stifle all moves.

Case b: The initial location is fixed and ties are broken randomly.

Given that ties are broken randomly with each choice having a positive probability, the goal is to interrupt the relocation process *eventually.* A simple approach to accomplish this is to determine all paths and cycles in

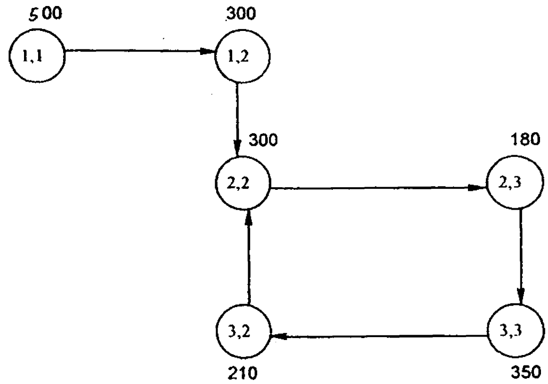

Figure 4.

the competition graph that can be reached from a given starting node (i, j); Δ_b^- is then the minimum taken over the Δ_a^- values of all relocation paths so determined. However, it is sufficient to determine all nodes in G^c that can be reached from the initial node (i, j). Then the smallest numerical label of any of these nodes will indicate the disincentive required to stop relocations at some time in the process. Finding the set of nodes reachable from the fixed starting point (i, j) can be achieved by any of the well-known shortest path methods. Then define S_{ij} as the set of nodes that can be reached from (i, j); the desired value Δ^- is the minimum taken over the numerical labels of all nodes in S_{ij}. As an example, consider the problem in Figure 2 with the competition graph in Figure 3 *a*. Let the starting node again be (1, 1). Detailed computations can be found in Table.

TABLE

α	Nodes reachable from (1, 4) in α steps	$\Delta^- \wp(t)$ values
0	(1, 1)	500
1	(1, 2)	300
2	(2, 2)	300
3	(2, 3)	180
4	(3, 3)	350
5	(3, 1), (3, 2)	210, 210

From the last column we infer that $\Delta_b^- = 180$, so that a tax of \$180 is sufficient to force a standstill in the relocation process.

Case c: All initial locations are considered and the tiebreaker rule is fixed. In principle, we could repeat the procedure described under case a for all possible pairs of locations, *i.e.* nodes in G^c, and select Δ^- as the maximum of all Δ^- values computed in case a. This task can, however, be simplified considerably. Consider again a relocation path that starts at node $(i,\ j)$, leads on a path to $(k,\ l)$, and the cycle returns to $(k,\ l)$. The minimum tax that stifles the relocation process is either determined by an arc on the path or on the cycle of the relocation path. However, not only relocations on the path anywhere between $(i,\ j)$ and $(k,\ l)$ must be stopped, but relocations on all other paths as well. In particular, consider the relocation path that starts at node $(k,\ l)$. In order to stop its moves, it is mandatory to stifle relocation on the cycle which is the same as the cycle part of the relocation path that starts at $(i,\ j)$. In other words, it is sufficient to consider a subgraph of G^c that contains only nodes that are located on a cycle. We define such a graph as the *reduced subgraph* $G^{rc} = (N^{rc},\ A^{rc})$ which is obtained from G^c by the following procedure:

– Delete all nodes with zero indegree and the arcs leading out of them.

– Repeat until no more nodes with zero indegree exist.

The reduced competition graph of Figure 3 a is shown in Figure 5.

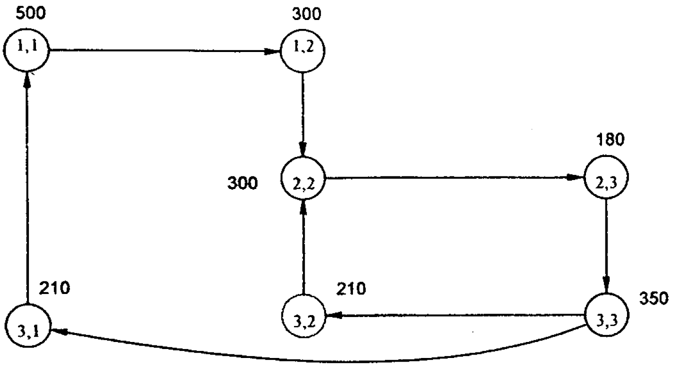

Figure 5.

We are now able to describe a procedure that finds Δ_c^- in polynomial time.

Step 1: For all nodes $(i,\ j)$ in G^{rc}, find the relocation path and label all its nodes that are also in G^{rc} with node labels "$(i,\ j)$".

Step 2: Create a list $L = \varnothing$ and let $(k,\ l)$ be the vertex with the smallest numerical label.

Step 3: Add all node labels of $(k,\ l)$ to L.

Step 4: If $L = N_{rc}$, STOP; else let (k, l) be the vertex with the next smallest label and go to step 3.

The general idea of the above procedure is as follows. If a node (i, j) has been assigned node labels (k_1, l_1), (k_2, l_2), ..., then node (i, j) is on relocations paths that start at nodes (k_1, l_1), (k_2, l_2), etc. In other words, stopping the relocation process at node (i, j) interrupts all relocation paths that start at any of its node labels. We now have to find the lowest cost to interrupt all paths, *i.e.* find a collection of nodes the union of whose labels equals N^{rc} and whose largest numerical label is as small as possible. This is achieved by the above algorithm.

Consider again the above example with the maximum index rule as tie breaker. Here, nodes (2, 2), (2, 3), (3, 2), and (3, 3) receive node labels [(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)] $= N^{rc}$ and the remaining labels are [(1, 1), (3, 1)] for node (1, 1), [(1, 1), (1, 2), (3, 1)] for node (1, 2), and [(3, 1)] for node (3, 1). The smallest numerical label of any node in N^{rc} is 180, it is associated with node (2, 3). As $L = N^{rc}$ already in the first iteration, the algorithm terminates with the conclusion that a tax of $\Delta_c^- = \$\,180$ is sufficient to stifle all further movements.

Case d: This case can be solved by applying the procedure described under (*b*) for $O(n^2)$ times. The tax Δ_d^- is then the maximum value of Δ_b^-, taken over all initial nodes in the competition graph. There does not appear to be an obvious shortcut to this method. In our example, $\Delta_d^- = 180$.

We are now finally able to define our *stability index* Δ. In case an equilibrium exists, we set $\Delta = +\Delta^+$, the positive sign indicates that at one of the facilities a subsidy payment is received. On the other hand, if an equilibrium does not exist, we set $\Delta = -\Delta^-$, the negative sign indicating that the facilities are charged a payment, or tax, for relocating.

5. COMPUTATIONAL RESULTS

A number of graphs were generated randomly and relocation paths starting at fixed initial locations were computed on the basis of winner-take-all and proportional models. A number of different combinations of prices were examined and stability indices were computed. Note that in the case of equal prices we have assumed the facilities to be distinguishable in the sense that A locating at vertex v_i and B at v_j is considered different from A locating at v_j and B at v_i. The graphs have between 15 and 25 vertices and between 24 and 44 edges.

In case of equal prices, both winner-take-all and the proportional model produced the same results: either an equilibrium exists with a 50 percent market share and profit for each facility, or no equilibrium exists. In those cases, we computed the average market share and profit on the cycle which turned out to be a 50-50 split as well. In cases of unequal prices, none of the examples showed an equilibrium. This behavior of the model is straightforward in winner-take-all models, as the less expensive facility will always move to the current location of its opponent where it can cut out its opponent and capture the entire market. At the same time the more expensive facility will move away from such an arrangement, so that the lack of equilibria in these cases is no surprise. The proportional model does not offer a similar, easy, solution.

An interesting observation concerns the proportional model with equal prices in which no equilibrium existed. We varied the prices from $p_A = p_B = 0.5$ to 10 or 20, and computed the Δ^- values, *i.e.* the "immobilization cost", as a function of the prices. These immobilization cost first rose with the prices as expected, but then eclipsed and decreased, and reached zero at which point an equilibrium existed. The largest Δ^- values occurred somewhere in the vicinity of $p_A = p_B = 1$. It is not known whether or not this is a provable property.

6. CONCLUSIONS

In this paper we have introducted a measure of stability for competitive location models. This measure indicates how much incentive is required to force facilities out of an equilibrium, given that one exists, or how much disincentive is needed to stop the relocation process and thus immobilize the facilities. It is then shown how to compute the index and the calculations were demonstrated by an example. Some computational results are also provided.

One strand of further research could investigate whether or not anything can be said for the stability index for trees or general graphs. Another question is if a specific stability in the winner-take-all model has any implications for the proportional model in the same graph or vice versa. Also, nothing much is presently known about the proportional model, not even on trees, as far as equilibria are concerned. In general, it appears that the proportional model is much more well-behaved and less volatile than the winner-take-all model, but we have to see if further studies bear this out.

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