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FRITZ JOHN'S TYPE CONDITIONS AND ASSOCIATED DUALITY FORMS IN CONVEX NON DIFFERENTIABLE VECTOR-OPTIMIZATION (*)

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Abstract. – *Necessary and sufficient conditions of the Fritz John's type as well as wolfe's type duality for convex non differentiable constrained multicriteria optimization problems have been studied by orthers under a rather strong constraint qualification. We first note that under the latter constraint qualification, every Pareto-optimal point is in fact a proper Pareto-optimal point in the sens presently commonly accepted. Furthermore, we show that with a weaker constraint qualification, simillar necessary and sufficient conditions and duality results as above still hold for proper Pareto-optimality. We also consider a dual derived from the Fritz John saddle point type problem.*

Keywords: Convexity, continuity, subgradient, subdifferential, vector-valued Lagrangian, saddle point, duality, Pareto-optimality.

Résumé. – *Des conditions nécessaires et suffisantes du type Fritz John aussi bien que la dualité du type Wolfe pour des problèmes d'optimisation multicritères convexes et non différentiables ont été étudiées par d'autres en présence d'une conditiion de contraintes plutôt forte. Nous montrons tout d'abord que sous cette même condition tout point optimal-Paréto est en fait proprement optimal-Paréto au sens présentement communément accepté. Ensuite, nous montrons qu'avec une condition de contrainte plus faible, des conditions nécessaires et suffisantes et une dualité similaires à celles ci-dessus tiennent encore pour l'optimalité-Paréto propre. Nous considérons aussi un dual dérivé d'un problème de point de selle du type Fritz John.*

Mots clés : Convexité, continuité, sous-gradient, sous-différentiel, Lagrangienne vectorielle, point de selle, dualité, optimalité-Paréto.

1. INTRODUCTION

Let Y be a locally convex real topological vector space in duality with Y^*, $f = (f_1, \ldots, f_k)$ be a vector function from Y to $\mathbb{R}^k$ and X a given non-empty subset of Y. For any x and $y \in \mathbb{R}^k$, $x \geq y$ (resp. $y \leq x$) means that $x_i \geq y_i$ (resp. $y_i \leq x_i$) for each i while $x > y$ (resp. $y < x$) means that

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$x_i > y_i$ (resp. $y_i < x_i$) for each i and $\langle x, y \rangle = xy = x_1 y_1 + \ldots + x_k y_k$. The product of a real matrix A with an euclidean vector z will be noted zA or Az depending on the one allowed. For two sets S and T in Y (resp. Y^*, $\mathbb{R}^k$), $S + T = \{x + y : x \in S, y \in T\}$, $S - T = \{x - y : x \in S, y \in T\}$, $S \backslash T = \{x \in S : x \notin T\}$ and for $\alpha \in \mathbb{R}$, $\alpha S = \{\alpha x : x \in S\}$, $(-1) S = -S = \{-x : x \in S\}$.

Definition 1: A point $x \in X$ is said to be Pareto-minimal (resp. Pareto-maximal) for f over X if for any other $y \in X$ such that (s.t.) $f(y) \leq f(x)$ (resp. $f(y) \geq f(x)$), we have $f(x) = f(y)$ in which case we also say that the objective value $f(x)$ is Pareto-minimal (resp. Pareto-maximal). □

A Pareto-optimal point is also said to be efficient. The problem of characterizing the set of Pareto-minimal (resp. Pareto-maximal) points is the vector-optimization primal problem which we note.

$$(P) : \min [(f(x) : x \in X] \, (\text{resp.} \max [f(x) : x \in X]).$$

Already in 1950, in their well known paper on nonlinear programming, Kuhn and Tucker observed in a differentiable bicriterion maximization problem the existence of an unstable qualified as non proper maximal point at which a decision maker may be willing to accept "a second order loss in one criterion to achieve a first order gain" in the other. later on, in 1968, in the nondifferentiable vector maximization problem, Geoffrion also observed the existence of an undesirable Pareto-maximal point at which the marginal gain in one criterion can be made arbitrarily large relative to each of the marginal losses in the others. He consequently introduced a new concept of proper Pareto-maximality to rule out such points. Others such as Borwein, Benson and Morin in 1977, Hartley in 1978, Benson again in 1979 also considered other concepts of proper optimality. It turns out that when X is convex and f convex (resp. concave) that is each component f_i convex (resp. concave), then all those concepts of proper-Pareto minimality (resp. Maximality) coincide with the following:

Definition 2: Let X be convex and f convex (resp. concave). $x^0 \in X$ is said to be a proper Pareto-minimal (resp. Maximal) solution for (P) if there exists $p \in \mathbb{R}^k$, $p > 0$ s.t. x° is optimal to the scalar convex (resp. concave) minimization (resp. maximization) problem (P_p) : $\min [pf(x) : x \in X]$ (resp. $\max [pf(x) : x \in X]$). In that case $f(x^\circ)$ is also said to be a proper Pareto-minimal (resp. maximal) objective value for (P). □

It is easy to verify that proper Pareto-optimal points are Pareto-optimal. A proper Pareto-optimal point is also said to be a proper efficient point. For the remaining of the paper, we suppose f convex with all components bounbed from above on a non empty open subset of Y. As the convexity of f is equivalent to the convexity of all the components which are then continuous, f is continuous. We also let $g = (g_1, \ldots, g_m)$ be a convex and continuous vector function from Y to $\mathbb{R}^m$ and C a non empty, closed and convex subset of Y so that the indicator function l_C of C defined on Y by

$$l_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}$$

is lower semi-continuous. We take $X = \{x \in C : g(x) \leq 0\}$ which we suppose non empty and we consider only the convex vector-minimization problem (P). We recall from [2] that if h is a numerical function on Y and $x^0 \in Y$, then any $y \in Y^*$ such that $h(x) \geq h(x^0) + \langle x - x^0, y\rangle$ for all $x \in Y$ is called a subgradient of h at x^0 the set of which noted $\partial h(x^0)$ is the subdifferential of h at x^0. If $\partial h(x^0) \neq \varnothing$, h is said to be subdifferentiable at x^0. It is well known [2] that $h(x^0) = \min[h(x) : x \in Y]$ if and only if (iff) $0 \in \partial h(x^0)$. Let us observe that $n_C(x^0) \equiv \partial l_C(x^0) = \{y \in Y^* : \langle x - x^0, y\rangle \leq 0$ for all $x \in C\}$ is the normal cone to C at x^0 if $x^0 \in C$ (empty if $x^0 \notin C$). In [4], Kanniappan considered necessary conditions for $x^0 \in X$ to be a Pareto-minimal solution for (P) under the rather strong constraint qualification that follows.

DEFINITION 3: X is said to satisfy the strong constraint qualification (SCQ) if each Pareto-minimal $x^0 \in X$ is s.t. for each component f_i of f, $f_i(x) < f_i(x^0)$ for all $j \in \{1, \ldots, k\}\backslash\{i\}$ and $g(x) < 0$ for some $x \in C$. □

In [1], Lai and Ho derived necessary and sufficient conditions for Pareto-minimality of the Fritz John stationary point type for (P) under the above SCQ. Specifically, their result says that $x^0 \in X$ is Pareto-minimal for (P) *iff* there exist a real $k \times k$-matrix $A = (a_{ij})$ and a real $k \times m$-matrix $U = (u_{ij})$ s.t. $a_{ii} = 1$. $a_{ij} \geq 0$ for $i \neq j$, $u_{ij} \geq 0$, $U\, g(x^0) = 0$ and $0 \in A\, \partial f(x^0) + U\, \partial g(x^0) + N_C(x^0)$ where $\partial f(x^0) = (\partial f_1(x^0), \ldots, \partial f_k(x^0)) = \{(y_1, \ldots, y_k) \in Y^* \times \ldots \times Y^* : y_i \in \partial f_i(x^0),\ i = 1, \ldots, k\}$, $\partial g(x^0) = (\partial g_1(x^0), \ldots, \partial g_m(x^0)) = \{(y_1, \ldots, y_m \in Y^* \times \ldots \times Y^* : y_i \in \partial g_i(x^0),\ i = 1, \ldots, m\}$, $N_C(x^0) = (n_C(x^0), \ldots, n_C(x^0))$ with

k components, $A\partial f(x^0) = \left(\sum_{j=1}^{k} a_{1j}\,\partial f_j(x^0), \ldots, \sum_{j=1}^{k} a_{kj}\,\partial f_j(x^0)\right)$ and

$$U\,\partial g(x^0) = \left(\sum_{j=1}^{k} u_{1j}\,\partial g_j(x^0), \ldots, \sum_{j=1}^{k} u_{mj}\,\partial g_j(x^0)\right).$$

In the remaining, $\mathcal{A}_0$ (resp. $\mathcal{U}$) is the set of matrices as A (resp. U) above. Based on their result that we just gave, Lai and Ho constructed a dual of Wolfe type to (P) on which we will be back in the third paragraph. Let us recall Slater's constraint qualification (*see* [5] for example).

DEFINITION 4: X (or g on C) is said to satisfy Slater's constraint qualification (noted CQ) if there is $x \in C$ s.t. $g(x) < 0$. □

CQ is obviously much weaker than SCQ and is used in convex scalar optimization. Is this paper, we will show that when X satisfies SCQ then every Pareto-minimal point is in fact a proper one. Further, with only CQ, we will show that similar results as Kanniappan's necessary condition result, Lai and Ho's necessary and sufficient condition result as well as their Wolfe's type duality results still hold with the concept of proper Pareto-minimality. We will finally also consider the Fritz John saddle point type problem and derive from it a dual to the convex vector minimization problem (P).

Let us take $k = 3$, $Y = \mathbb{R}^3$ and $f = (f_1, f_2, f_3)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ s.t. $f_1(x) = -2x_1 + x_2 - x_3$, $f_2(x) = x_1 - 2x_2 + x_3$ and $f_3(x) = 3x_1 - x_3$. Let us also take $m = 3$, $g = (g_1, g_2, g_3)$ s.t. $g_i(x) = x_i - 1$ for $i = 1, 2, 3$, $C = \{x \in \mathbb{R}^3 \mid x \geq 0\}$ the positive orthant of $\mathbb{R}^3$ which is closed, convex and non empty so that $X = \{x \in \mathbb{R}^3 \mid x \geq 0, x_i \leq 1, i = 1, 2, 3\}$ is non empty and satisfies CQ. With these hypotheses, we consider the linear vector minimization problem (P). It is well known that for such problems, the set of Pareto-optimal points coincides with the set of proper Pareto-optimal points.

Now let us observe that the linear problem $(\mathrm{LP}_1) : \min\{f_1(x) \mid x \in X\}$ has a unique optimum at $x^0 = (1, 0, 1)$ and this point is obviously a Pareto-optimal point thus a proper Pareto-optimal point for (P). However X does not satisfy SCQ for taking $i = 2$, there is no $x \in C$ s.t. $f_1(x) < f_1(x^0)$, $f_3(x) < f_3(x^0)$ and $g(x) < 0$ otherwise x^0 would not be optimal for (LP_1).

On the basis of this particular example, we see that under CQ, a proper Pareto-optimal point may exist at which SCQ fails to hold.

We now get back to the general problem (P).

2. OPTIMALITY CONDITIONS

THEOREM 5: *Let X satisfy* SCQ. *If $x^0 \in X$ is Pareto-minimal for (P), then it is a proper Parato-minimal solution for (P).* $\square$

Proof: For each $i \in \{1, \ldots, k\}$, x^0 is optimal to the scalar convex problem $\min [f_i(x) : x \in X,\ f_j(x) - f_j(x^0) \leqq 0,\ j \neq i]$ whose constraint functions $f_j - f_j(x^0)$ satisfy the usual Slater's constraint qualification over X. It follows from the usual Lagrange multiplier technique in scalar convex optimization that there exist $k-1$ reals $a_{ij} \geq 0$, $j \neq i$ s.t. for all $x \in X$,

$$f_i(x^0) + \sum_{j \neq i}^{k} a_{ij}\, f_j(x^0) \leq f_i(x) + \sum_{j \neq i}^{k} a_{ij}\, f_j(x).$$

Summing over i, we get :

$$\sum_{j=1}^{k} \left[1 + \sum_{i \neq j}^{k} a_{ij}\right] f_j(x^0) \leq \sum_{j=1}^{k} \left[1 + \sum_{i \neq j}^{k} a_{ij}\right] f_j(x).$$

Setting $p_j = 1 + \sum_{i \neq j}^{k} a_{ij}$ for $j = 1, \ldots, k$ and $p = (p_1, \ldots, p_k)$, we have $p > 0$ and $pf(x^0) = \min [pf(x) : x \in X]$. We conclude that x^0 is a proper Pareto-minimal solution for (P). $\square$

The following result improves Lemma 3.2 in [1].

LEMMA 6: *Suppose that X satisfies* CQ *and let $x^0 \in X$. Then x^0 is a proper minimal solution for (P) iff for each $i \in \{1, \ldots, k\}$ there exist $k-1$ reals $a_{ij} \geq 0$ $j \neq i$ and m reals $u_{il} \geq 0$, $l = 1, \ldots, m$* s.t.

$$u_{il}\, g_l(x^0) = 0 \qquad for \quad l = 1, \ldots, m \tag{1}$$

$$0 \in \partial f_i(x^0) + \sum_{j \neq i}^{k} a_{ij}\, \partial f_j(x^0) + \sum_{l=1}^{m} u_{il}\, \partial g_l(x^0) + n_C(x^0) \tag{2}$$

in which case we can choose each $a_{ij} > 0$. $\square$

Proof: Let x^0 be a proper Pareto-minimal solution. Then there exists $p \in \mathbb{R}^k$, $p > 0$ *s.t.* $pf(x^0) = \min [pf(x) : x \in X] = \min [pf(x) + l_C(x) :$

$g(x) \leq 0,\ x \in Y]$. It follows once again from the Lagrange multiplier technique in convex scalar optimization that there exists $v \in \mathbb{R}^m$, $v \geq 0$ and $vg(x^0) = 0$, hence $v_l\, g_l(x^0) = 0$ and $pf(x^0) + l_C(x^0) + vg(x^0) \leq pf(x) + l_C(x) + vg(x)$ for all $x \in Y$. It follows that x^0 is optimal in $\min pf(x) + l_C(x) + vg(x) : x \in Y]$, thus $0 \in \partial(pf + vg + l_C)(x^0)$. $\partial l_C(x^0) = n_C(x^0)$. From Theorem 6.4.6. in [2], for each i and l, $\partial f_i(x^0) \neq \emptyset$, $\partial g_l(x^0) \neq \emptyset$ and it comes from corollary 6.6.8 also in [2] that

$$\begin{aligned}\partial(pf + vg + l_C)(x^0) &= \partial p\, f(x^0) + \partial\, vg(x^0) + n_C(x^0)\\ &= p\,\partial f(x^0) + v\,\partial g(x^0) + n_C(x^0).\end{aligned}$$

So $0 \in p\,\partial f(x^0) + v\,\partial g(x^0) + n_C(x^0)$.

Now for each $i \in \{1, \ldots, k\}$, let us take $a_{ij} = p_j/p_i$ for $j \neq i$ and $u_{il} = v_l/p_i$ for all l. As $n_C(x^0)$ is a cone, $n_C(x^0)/p_i = n_C(x^0)$. We conclude that (1) and (2) of the theorem hold for each i.

Conversely let us suppose that (1) and (2) are satisfied for each i. Summing each over i and setting $p_j = 1 + \sum_{i \neq j}^{k} a_{ij}$ for each j and $v_l = \sum_{i=1}^{k} u_{il}$ for each l, $p = (p_1, \ldots, p_k)$ and $v = (v_1, \ldots, v_m)$, as the sum of k terms $n_C(x^0) + \ldots + n_C(x^0)$ is $n_C(x^0)$ since $n_C(x^0)$ is a convex cone, we get $vg(x^0) = 0$ and $0 \in \partial pf(x^0) + \partial vg(x^0) + n_C(x^0)$. Consequently x^0 is optimal in $\min [pf(x) + vg(x) : x \in C]$.

Since $x^0 \in X = \{x : x \in C,\ g(x) \leq 0\}$ and $vg(x^0) = 0$, x^0 is optimal in $\min [pf(x) : x \in X]$. The conclusion comes from the fact that $p > 0$. □

We now have the following necessary and sufficient conditions result for Pareto-minimality which is of the Fritz John stationary point type.

THEOREM 7: *if X satisfies* CQ, *then $x^0 \in X$ is a proper Pareto-minimal solution to (P) iff there exist $A \in \mathcal{A}_0$ and $U \in \mathcal{U}$* s.t.

$$U\,g(x^0) = 0 \tag{3}$$

$$0 \in A\,\partial f(x^0) + U\,\partial g(x^0) + N_C(x^0) \tag{4}$$

in which case in addition to having $a_{ii} = 1$ for each i, we can choose A s.t. *$a_{ij} > 0$ for $i \neq j$.* □

Proof: If x^0 is a proper Pareto-minimal point then (3) and (4) are the matrix forms for (1) with $i = 1, \ldots, k$ and $l = 1, \ldots, m$ and (2) with

$i = 1, \ldots, k$ respectively in the lemma and the last part of the theorem also follows from the lemma.

Conversely, given $A \in \mathcal{A}_0$ and $U \in \mathcal{U}$ satisfying (3) and (4) then for each i we have (2) and $\sum_{l=1}^{m} u_{il}\, g_l(x^0) = 0$. As $g_l(x^0) \leq 0$ and $u_{il} \geq 0$, we have $u_{il}\, g_l(x^0) = 0$ for $l = 1, \ldots, m$ that is (1) of the lemma for each i. Once again the conclusion comes from the lemma. □

Remarks 8: 1. Suppose that X satisfies CQ. If $x^0 \in X$ is a proper Pareto-minimal solution to (P), then as we showed in the first part of the proof of Lemma 6 above, there exists $p \in \mathbb{R}^k$, $p > 0$ and $v \in \mathbb{R}^m$, $v \geq 0$ s.t. $vg(x^0) = 0$ and $0 \in p\,\partial f(x^0) + v\,\partial g(x^0) + n_C(x^0)$. This is an improvement of Kanniappan's necessary theorem 3.4 in [4].

2. Let us define the Fritz John's type lagrangian L on $C \times \mathcal{A}_0 \times \mathcal{U}$ as follows. For any $(x, A, U) \in C \times \mathcal{A}_0 \times \mathcal{U}$, $A = (a_{ij})$ and $U = (u_{il})$ being as in Theorem 7 above, with $L(x, A, U) = A\,f(x) + U\,g(x)$, we have $L(x, A, U = f(x) + A \star f(x) + U\,g(x)$ where $A \star f(x) = ((A \star f(x))_1, \ldots, (A \star f(x))_k)$, $(A \star f(x))_i = \sum_{j \neq i}^{k} a_{ij}\, f_j(x)$ so that for each $i = 1, \ldots, k$, $L_i(x, A, U) = f_i(x) + \sum_{j \neq i}^{k} a_{ij}\, f_j(x) + \sum_{l=1}^{m} u_{il}\, g_l(x)$.

Thus for (A, U) fixed, L_i is a continuous convex function of x on C and consequently so is the vector-valued Lagrangian L. Now if $g(x) \nleq 0$, that is $x \in C \backslash X$, say $g_q(x) > 0$, then taking $u_{iq} = t > 0$ for all i and $u_{il} = 0$ for all i and all $l \neq q$, we get $L_i(x, A, U) = f_i(x) + \sum_{j \neq i}^{k} a_{ij}\, f_j(x) + t g_q(x) \to +\infty$ as $t \to +\infty$ so that $\max\,[L(x, A, U) : U \in \mathcal{U}] = \emptyset$.

On the other hand if $g(x) \leq 0$, that is $x \in X$, since $U\,g(x) \leq 0$, we have $\max\,[L(x, A, U) : U \in \mathcal{U}] = A\,f(x)$. We therefore deduce that

$$\min_{C} \max_{\mathcal{U}} L(x, A, U) = \min_{X} A\,f(x).$$

3. For the remaining of the paper, we set:

$$\mathcal{A} = \{A \in \mathcal{A}_0 : A > 0\}$$

where for $A = (a_{ij})$, $A > 0$ means each $a_{ij} > 0$. Recall that for $A \in \mathcal{A}_0$. $a_{ii} = 1$ for each i and each $a_{ij} \geq 0$. □

LEMMA 9: *If* $(x^0, A^0, U^0) \in X \times \mathcal{A}_0 \times \mathcal{U}$, *then* $U^0 g(x^0) = 0$ *iff* $L(x^0, A^0, U^0) = \max_{\mathcal{U}} L(x^0, A^0, U)$. □

Proof: Let $U^0 g(x^0) = 0$. Since $x^0 \in X$, thus $g(x^0) \leq 0$, then for all $U \in \mathcal{U}$, we have $A^0 f(x^0) + U g(x^0) \leq A^0 f(x^0) + U^0 g(x^0)$, so $L(x^0, A^0, U^0) = \max_{\mathcal{U}} L(x^0, A^0, U)$.

Conversely, if $L(x^0, A^0, U^0) = \max_{\mathcal{U}} L(x^0, A^0, U)$, then as $x^0 \in X$, it comes from the second part of Remarks 8 that $\max_{\mathcal{U}} L(x^0, A^0, U) = A^0 f(x^0)$, so that $A^0 f(x^0) + U^0 g(x^0) = A^0 f(x^0)$, hence $U^0 g(x^0) = 0$. □

We have the following definition of a Fritz John type saddle point.

DEFINITION 10: $(x^0, A^0, U^0) \in C \times \mathcal{A} \times \mathcal{U}$ is said to be a saddle point for L if $L(x^0, A^0, U^0) \in [\max_{\mathcal{U}} L(x^0, A^0, U)] \cap [\min_{C} L(x, A^0, U^0)]$. □

We can now prove the following Fritz John saddle point type necessary and sufficient conditions for minimality.

THEOREM 11: *Let* X *satisfy* CQ, $x^0 \in C$ *and suppose every Pareto-minimal solution for* (P) *is a proper Pareto-minimal one. Then* x^0 *is a Pareto-minimal solution for* (P) *iff there exists* $A^0 \in \mathcal{A}$ *and* $U^0 \in \mathcal{U}$ s.t. (x^0, A^0, U^0) *is a saddle point for* L *in which case* $U^0 g(x^0) = 0$. □

Proof: Let x^0 be Pareto-minimal for (P). Then $x^0 \in X$ and from Theorem 7, there exist $A^0 \in \mathcal{A}$ and $U^0 \in \mathcal{U}$ s.t. $U^0 g(x^0) = 0$ and $0 \in A^0 \partial f(x^0) + U^0 \partial g(x^0) + N_C(x^0)$. From Lemma 9, $L(x^0, A^0, U^0) = \max_{\mathcal{U}} L(x^0, A^0, U)$.

Now for each $i = 1, \ldots, k$, $0 \in \partial f_i(x^0) + \sum_{j \neq i}^{k} a_{ij}^0 \partial f_j(x^0) + \sum_{l=1}^{m} u_{il}^0 \partial g_l(x^0) + n_C(x^0)$ and we consequently have $L_i(x^0, A^0, U^0) = \min_{C} L_i(x, A^0, U^0)$ for each i so that $L(x^0, A^0, U^0) = \min_{C} L(x, A^0, U^0)$. It follows that (x^0, A^0, U^0) is a saddle point for L. Conversely, let (x^0, A^0, U^0) be a saddle point for L. Since $L(x^0, A^0, U^0) \in \max_{\mathcal{U}} L(x^0, A^0, U)$, the latter set is non empty and it comes from the second part of Remarks 8 that $x^0 \in X$ and $\max_{\mathcal{U}} L(x^0, A^0, U) = A^0 f(x^0)$. From Lemma 9 we also have $U^0 g(x^0) = 0$. Now $L(x^0, A^0, U^0) \in \min_C L(x, A^0, U^0)$ and since $x^0 \in X \subset C$, we have $L(x^0, A^0, U^0) \in \min_X L(x, A^0, U^0)$. We deduce that $A^0 f(x^0) \in \min_X A^0 f(x)$ for otherwise if $x^1 \in X$ is s.t. $A^0 f(x^1) \leq A^0 f(x^0)$ with $A^0 f(x^1) \neq A^0 f(x^0)$, since $U^0 g(x^1) \leq 0$ we would have $L(x^1, A^0, U^0) \leq L(x^0, A^0, U^0)$ with $L(x^1, A^0, U^0) \neq L(x^0, A^0, U^0)$ leading to a contraction since $L(x^0, A^0, U^0) \in \min_X L(x, A^0, U^0)$.

So we do have $A^0 f(x^0) \in \min_X A^0 f(x)$. This in turn implies that x^0 is Pareto-minimal for (P). Otherwise there exists $x^2 \in X$ s.t. $f(x^2) \leq f(x^0)$ with $f(x^2) \neq f(x^0)$ which would imply, since $A^0 > 0$, that $A^0 f(x^2) \leq A^0 f(x^0)$ with $A^0 f(x^2) \neq A^0 f(x^0)$, a contradiction to $A^0 f(x^0) \in \min_X A^0 f(x)$. □

Let us mention that in convex vector-optimization, hypotheses that X satisfy CQ and that Pareto-minimal solutions for (P) be proper Pareto-minimal ones that appear in the preceding theorem are very common in many interesting duality and saddle point results (*see* [3, 6] for example).

LEMMA 12: *Let X satisfy* CQ, $x^0 \in C$ *and every Pareto-minimal solution for (P) be a proper one. Then x^0 is a Pareto-minimal solution iff there exists* $A^0 \in \mathcal{A}$ s.t. $A^0 f(x^0) = \min [A^0 f(x) : x \in X]$. □

Proof: If $A^0 \in \mathcal{A}$ exists and is s.t. $A^0 f(x^0) \in \min [A^0 f(x) : x \in X]$, then proceeding as towards the end of the proof of the preceding theorem, we conclude that x^0 is Pareto-minimal for (P).

Conversely let x^0 be a Pareto-minimal solution for (P), hence a proper one. From Theorem 7, there exist $U^0 \in \mathcal{U}$ and $A^0 \in \mathcal{A}$ s.t. $U^0 g(x^0) = 0$ and $0 \in A^0 \partial f(x^0) + U^0 \partial g(x^0) + N_C(x^0)$. If A_i^0 (resp. U_i^0) is the i-th row of A^0 (resp. U^0) then $0 \in A_i^0 \partial f(x^0) + U_i^0 \partial g(x_0) + n_C(x^0)$. Using arguments already used in the first part of the proof of Lemma 6, we get $0 \in \partial (A_i^0 f + U_j^0 g + l_C)(x^0)$, that is x^0 is optimal in the scalar problem $\min [A_i^0 f(x) + U_i^0 g(x) : x \in C]$. Consequently x^0 is Pareto-minimal in $\min [A^0 f(x) + U^0 g(x) : x \in C]$. Since $U^0 g(x^0) = 0$, x^0 is Pareto-minimal in $\min [A^0 f(x) : x \in X]$. □

THEOREM 13: *Let X satisfy* CQ, $x^0 \in C$ *and every Pareto-minimal solution for (P) be a proper one. Then x^0 is Pareto-minimal iff there exists* $A^0 \in \mathcal{A}$ s.t. $\max_{\mathcal{U}} L(x^0, A^0, U) \cap \min_C \max_{\mathcal{U}} L(x, A^0, U) \neq \varnothing$. □

Proof: Let x^0 be Pareto-minimal. From Lemma 12, there exists $A^0 \in \mathcal{A}$ s.t. $A^0 f(x^0) \in \min [A^0 f(x) : x \in X]$. Further, from the second part of Remarks 8, $\max_{\mathcal{U}} L(x^0, A^0, U) = A^0 f(x^0)$ and $\min_C \max_{\mathcal{U}} L(x, A^0, U) = \min [A^0 f(x) : x \in X]$. It follows that $\max_{\mathcal{U}} L(x^0, A^0, U) \cap \min_C \max_{\mathcal{U}} L(x, A^0, U)$ is non empty as $A^0 f(x^0)$ belongs to it.

Conversely let $A^0 \in \mathcal{A}$ be s.t. the intersection of the lemma is non empty. Then once again from the second part of Remarks 8, $\max_{\mathcal{U}} L(x^0, A^0, U) = A^0 f(x^0) \in \min_C \max_{\mathcal{U}} L(x, A^0, U) = \min_X \max_{\mathcal{U}} L(x, A^0, U) = \min_X A^0 f(x)$ as for $x \in C \backslash X$, $\max_{\mathcal{U}} L(x, A^0, U) = \varnothing$. It follows from Lemma 12 that x^0 is Pareto-minimal. □

3. DUALITY

With $F = \{(x, A, U) \in C \times \mathcal{A}_0 \times \mathcal{U} : 0 \in A\,\partial f(x) + U\,\partial g(x) + N_C(x)\}$, Lai and Ho defined in [1] the following Wolfe type dual to the primal convex vector-minimization problem (P).

$$(D_{LH}) : \max [L(x, A, U) : (x, A, U) \in F].$$

Exception of their weak duality result (first part of the next remarks) which they prove in the absence of any constraint qualification, all their other duality results related to (D_{LH}) require the strong constraint qualification SCQ of Definition 3. As we saw in Theorem 5, under such a condition, every Pareto-minimal solution for (P) is a proper one. In the light of the preceding paragraph, necessary and sufficient conditions for proper Pareto-minimality require only the weaker constraint qualification CQ of Definition 4 which is the usual Slater's constraint qualification. Consequently Lai and Ho's duality results requiring SCQ may be improved as summerised in the following remarks.

Remarks 14: 1. Let $x^1 \in X$ and $(x^2, A, U) \in F$. Then $A f(x^1) \geq L(x^2, A, U)$. The reason is simple. Since $x^1 \in X$, $U g(x^1) \leq 0$ so $A f(x^1) \geq L(x^1, A, U)$. Now $(x^2, A, U) \in F$ implies that for $j = 1, \ldots, k$ and $l = 1, \ldots, m$, there exist $y_j \in \partial f_j(x^2)$, $z_l \in \partial g_l(x^2)$ and $w_j \in n_C(x^2)$ s.t. for each $i = 1, \ldots, k$, $y_i + \sum_{j \neq i}^{k} a_{ij}\, y_j + \sum_{l=1}^{m} u_{il}\, z_l + w_i = 0$. Using subdifferentiability at x^2, we get

$$\begin{aligned} L_i(x^1, A, U) - L_i(x^2, A, U) &= [f_i(x^1) - f_i(x^2)] \\ &+ \sum_{j \neq i}^{k} a_{ij} [f_j(x^1) - f_j(x^2)] + \sum_{l=1}^{m} u_{il} [g_l(x^1) - g_l(x^2)] \\ &\geq \langle y_i, x^1 - x^2 \rangle + \sum_{j \neq i}^{k} a_{ij} \langle y_j, x^1 - x^2 \rangle \\ &+ \sum_{l=1}^{m} u_{il} \langle z, x^1 - x^2 \rangle = -\langle w_i, x^1 - x^2 \rangle. \end{aligned}$$

Since $w_i \in n_C(x^2)$, we have $-\langle w_i, x^1 - x^2 \rangle \geq 0$. We conclude that $A f(x^1) - L(x^2, A, U) \geq 0$.

2. Let X satisfy CQ and $x^0 \in C$. If x^0 is a proper Pareto-minimal point for (P) then there exist $A^0 \in \mathcal{A}_0$ and $U^0 \in \mathcal{U}$ s.t. $(x^0, A^0, U^0) \in F$ and if (x^0, A^0, U^0) is optimal to (D_{LH}) then $L(x^0, A^0, U^0) = A^0 f(x^0)$. This is an evident consequence of Theorem 7.

3. Let X satisfy CQ, $x^0 \in X$, $(x^0, A^0, U^0) \in F$ for some A^0 and U^0. If $A^0 f(x^0) = L(x^0, A^0, U^0)$ then $(x^0, A^0 U^0)$ is optimal for (D_{LH}). Because we would have $U^0 g(x^0) = 0$ and the result comes again from Theorem 7. $\square$

Theorem 13 above suggests that we consider the following problem that we also call dual to the primal convex vector-minimization problem (P).

$$(D): \text{ Finding the set of } (A^0, U^0) \in \mathcal{A} \times \mathcal{U} \text{ s.t.}$$

$$\min_C L(x, A^0, U^0) \cap \max_{\mathcal{U}} \min_C L(x, A^0, U) \neq \varnothing.$$

Any (A^0, U^0) belonging to the desired set is said to be optimal for (D) in which case any $L(x^0, A^0, U^0)$ that belongs to the non empty intersection is an optimal objective value. We also consider the following auxiliary problem which, when we look at Theorem 13, is related to the primal minimization problem (P) under certain circumstances as we will see.

$$(P'): \text{ Finding the set of } (x^0, A^0) \in \mathcal{C} \times \mathcal{A} \text{ s.t.}$$

$$\max_{\mathcal{U}} L(x^0, A^0, U) \cap \min_C \max_{\mathcal{U}} L(x, A^0, U) \neq \varnothing.$$

Any $(x^0,\ A^0)$ belonging to the desired set is said to be optimal for (P') in which case any $L(x,\ A^0,\ U^0)$ that belongs to the non empty intersection is an optimal objective value. When X satisfies CQ and every Pareto-minimal solution for (P) is a proper one, then it comes from Theorem 13, the second part or Remarks 8 and Lemma 12 that (P') is equivalent to the minimization problem (P) minimal solutionwise.

LEMMA 15: *$(x^0,\ A^0,\ U^0)$ is a saddle point for L iff $(x^0,\ A^0)$ is optimal for (P') and $(A^0,\ U^0)$ is optimal for (D) with $A^0\ f\ (x^0) \in \min\limits_{C} L(x,\ A^0,\ U^0) \cap \max\limits_{\mathcal{U}} \min\limits_{C} L(x,\ A^0,\ U)$.* □

Proof: Let $(x^0,\ A^0,\ U^0)$ be a saddle point for L. Then $L(x^0,\ A^0,\ U^0) \in \max\limits_{\mathcal{U}} L(x^0,\ A^0,\ U)$ so $x^0 \in X$ according to the second part of Remarks 8 and $U^0\, g(x^0) = 0$ according to Lemma 9. On the other hand $A^0\ f\ (x^0) = L(x^0,\ A^0,\ U^0) \in \min\limits_{C} L(x,\ A^0,\ U^0)$ and since $x^0 \in X \subset C$, $A^0\ f\ (x^0) = L(x^0,\ A^0,\ U^0) \in \min\limits_{X} L(x,\ A^0,\ U^0)$. Further since $U^0\, g(x^0) = 0$, $A^0\ f\ (x^0) \in \min\limits_{X} A^0\ f(x)$ and from the second part of Remarks 8, $\min\limits_{X} A^0\ f\ (x) = \min\limits_{C} \max\limits_{\mathcal{U}} L(x,\ A^0,\ U)$. We deduce that $(x^0,\ A^0)$ is optimal to (P').

If $L(x^0,\ A^0,\ U^0) \notin \max\limits_{\mathcal{U}} \min\limits_{C} L(x,\ A^0,\ U)$, then there exists $U^1 \in \mathcal{U}$ s.t. for some $x^1 \in C$, we have $L(x^1,\ A^0,\ U^1) \in \min\limits_{C} L(x,\ A^0,\ U^1)$ and $L(x^1,\ A^0,\ U^1) \geq L(x^0,\ A^0,\ U^0)$, $L(x^1,\ A^0,\ U^1) \neq L(x^0,\ A^0,\ U^0)$. Since $U^1\, g(x^0) \leq 0$ and $U^0\, g(x^0) = 0$, we get $A^0\, f(x^1) + U^1\, g(x^1) \geq A^0\, f(x^0) + U^1\, g\ (x^0) - U^1\, g(x^0) \geq A^0 f(x^0) + U^1\, g\,(x^0)$ where the first inequality is not an equality. We get a contradiction to $L(x^1\ A^0,\ U^1) \in \min\limits_{C} L(x,\ A^0,\ U^1)$. Consequently $L(x^0,\ A^0,\ U^0) = A^0 f(x^0) \in \min\limits_{C} L\ (x,\ A^0,\ U^0) \cap \max\limits_{\mathcal{U}} \min\limits_{C} L(x,\ A^0,\ U)$ so that $(A^0,\ U^0)$ is an optimal solution to (D) with the required conditon. Conversely, let $(x^0,\ A^0)$ be optimal to (P') and $(A^0,\ U^0)$ s.t.

$A^0\ f\ (x^0) \in \min\limits_{C} L(x,\ A^0,\ U^0) \cap \max\limits_{\mathcal{U}} \min\limits_{C} L(x,\ A^0,\ U)$. From the second part of Remarks 8, we must have $x^0 \in X$ so that $U^0\, g(x^0) = 0$ and $L(x^0,\ A^0,\ U^0) \in \max\limits_{\mathcal{U}} L(x^0,\ A^0,\ U)$ according to Lemma 9. Since we also have $L(x^0,\ A^0,\ U^0) = A^0\, f\,(x^0) \in \min\limits_{C} L(x,\ A^0,\ U^0)$, we conclude that $(x^0,\ A^0,\ U^0)$ is a saddle point for L. □

THEOREM 16: *Let X satisfy* CQ, $x^0 \in C$ *and every Pareto-minimal solution for* (P) *be a proper one. Then* x^0 *is Pareto-minimal iff there exists an optimal solution* (A^0, U^0) *for* (D) *with* $A^0 f(x^0) \in \min_C L(x, A^0, U^0) \cap \max_{\mathcal{U}} \min_C L(x, A^0, U)$. □

Proof: From Theorem 11, x^0 is Pareto-minimal iff there exist $A^0 \in \mathcal{A}$ and $U^0 \in \mathcal{U}$ s.t. (x^0, A^0, U^0) is a saddle point for L. According to Lemma 15, this in turn means that (x^0, A^0) is optimal for (P') and (A^0, U^0) is optimal for (D) with

$$A^0 f(x^0) \in \min_C L(x, A^0, U^0) \cap \max_{\mathcal{U}} \min_C L(x, A^0, U) \; (\star).$$

On the other hand if (A^0, U^0) is optimal for (D) and $(\star)$ is satisfied, since $x^0 \in X$, we have $A^0 f(x^0) \in \min_X A^0 f(x)$ and x^0 is Pareto-minimal to (P) according to Lemma 12. □

Remark 17: Let X satisfy CQ, $x^0 \in C$ and every Pareto-minimal solution for (P) be a proper one. Then it comes from Theorem 16 and its proof that the following are equivalent:

a) x^0 is Pareto-minimal for (P);

b) There exist $A^0 \in \mathcal{A}$, $U^0 \in \mathcal{U}$ s.t. (x^0, A^0, U^0) is a saddle point for L;

c) There exist $A^0 \in \mathcal{A}$, $U^0 \in \mathcal{U}$ s.t. (x^0, A^0) is optimal for (P') and (A^0, U^0) is optimal for (D) with

$$A^0 f(x^0) \in \min_C L(x, A^0, U^0) \cap \max_{\mathcal{U}} \min_C L(x, A^0, U). \quad \square$$

4. CONCLUSION

We showed that under some sort of a stronger Staler's constraint qualification used by researchers to study Fritz John's type optimality conditions and Wolfe's type duality for convex multicriteria optimization problems, every Pareto-optimal solution is in fact a proper Pareto-optimal solution. Further, with a weaker constraint qualification which is the usual Slater's constraint qualification. similar results they obtained still hold with proper Pareto-optimality. Finally we considered a Fritz John saddle point type problem and another form of duality for the convex vector-optimization problem.

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