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SOLUTIONS OF TRANSFERABLE UTILITY COOPERATIVE GAMES (*)

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Abstract. – *A uniform competitive solution predicts a configuration of payoff vectors associated with coalitions which are likely to form in a cooperative game. This configuration satisfies the internal and external stability principles and the predicted payoffs are both individually and coalitionally rational. The uniform competitive solution is a modified version of the competitive solution concept introduced by R. D. McKelvey, P. C. Ordeshook and M. D. Winer (1978). A parallel study of these two solution concepts emphasizes their common properties and the existing differences. Existence theorems of uniform competitive solutions are provided under very general assumptions and it is proved that competitive solutions exists only for some special classes of transferable utility games.*

Keywords: Cooperative games, Characteristic function, Competitive solutions.

Résumé. – *Une solution compétitive uniforme prédit une configuration de vecteurs de paiements associés à certaines coalitions susceptibles de se former dans un jeu coopératif. Cette configuration est caractérisée par une stabilité interne et externe et les paiements associés répondent à des impératifs de rationalité individuelle et collective. La solution compétitive uniforme est une version modifiée de la solution compétitive proposée par R. D. McKelvey, P. C. Ordeshook and M. D. Winer (1978).*

On fait ici une étude parallèle de ces deux notions, en soulignant leurs propriétés communes ainsi que leurs différences. Les théorèmes d'existence pour la solution compétitive uniforme sont démontrés dans des conditions très générales. D'autre part, on prouve que l'existence d'une solution compétitive est restreinte à quelques classes particulières des jeux coopératifs à utilités transférables.

Mots clés : Jeux coopératifs, fonction caractéristique, solution compétitive.

1. INTRODUCTION

Since the first consistent solution concept for cooperative games was defined in the framework of the classical von Neumann-Morgenstern theory,

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several other solutions have been proposed. Much of them focus on predicting payoffs which will be received by the players after the bargaining process. But many situations in economics and political science require also to predict the coalitions that are likely to form.

Of course, each player can actually participate in at most one coalition, but generally, there are many potential coalitions that seem to be profitable for him. In such situations a solution should take into account the possibilities available to the players in different coalitions and predict a "stable" configuration of coalitions and payoffs.

The competitive solution was proposed by McKelvey, Ordeshook and Winer (1978) as an alternative to classical solution concepts. The main motivation behind this concept arises from its applications in political science [*see* also Ordeshook (1986)]. Although the original definition refers to the non-transferable utility case, it may be also interpreted in terms of transferable utility games.

The uniform competitive solution is a slightly modified version of the competitive solution and was introduced in Stefanescu (1993).

In sections 4 and 5 of the present study we discuss the common properties of these two concepts and clarify the relationships between them. Existence theorems are provided for different standard models of transferable utility games. In fact, the existence of the uniform competitive solutions is proved under very general assumptions, while the existence of competitive solutions is restricted to some special classes of games. Moreover, it is shown that in most situations any competitive solution must be an uniform competitive solution at the same time. Note also that the method of proof of the main theorem of Section 4 allows us to rediscover Shapley's theorem for the core of convex games as a particular case of the existence of uniform competitive solutions. The relationships between the uniform competitive solution and other solution concepts are discussed in sections 3 and 6.

2. DEFINITIONS AND NOTATIONS

Everywhere in this paper we deal with *n-person cooperative games.* The set of players $\{1, 2, \ldots, n\}$ is denoted by N and every subset of N is called *coalition.* The set of all coalitions (including the empty coalition $\varnothing$) is denoted by 2^N. If C is a nonempty coalition, then $|C|$ stands for the cardinality of C, but we will use for convenience the notation $\mathbf{R}^C$ instead of $\mathbf{R}^{|C|}$.

If $C \subset D$ and $u \in \mathbf{R}^D$ then $u(C) = \sum_{j \in C} u_j$ and the symbol u_C will be used for the vector $(u_j)_{j \in C}$ formed by the components of u indexed in C. More generally, if $A \subset \mathbf{R}^D$ then $pr_C\, A = \{u_C \in \mathbf{R}^C \mid u \in A\}$.

If $u,\ v \in \mathbf{R}^C$ for some $C \subseteq N$ then we write $u \geq v$ if $u_j \geq v_j$ for all $j \in C$ and $u > v$ if $u \geq v$ but $u \neq v$. For the case when $u_j > v_j$ for all $j \in C$ we will use the notation $u \gg v$.

A *transferable utility* (TU) cooperative game is defined by its characteristic function $\nu : 2^N \to \mathbf{R}$. As usual, $\nu(C)$ is the total payoff which the coalition C can guarantee for its members. Since the ouput space is $\mathbf{R}^N$ the condition $u(C) \leq \nu(C)$ if $u \in \mathbf{R}^N$ is called *the effectiveness condition* of the coalition C and therefore, the set $eff\, C = \{u \in \mathbf{R}^N \mid u(C) \leq \nu(C)\}$ is the set of all payoff vectors which are effective for C. Particularly, if $C = N$ the inequality $u(N) \leq \nu(N)$ shows that the payoff vector u may be achieved by the players at the end of a possible play. We will call the above relation *the feasibility* condition.

If the vector u is feasible and effective for a coalition C then the components of u_C may be interpreted as payoffs which can be achieved by the players in C even if the players not in C act against this coalition.

For the basical model of TU cooperative games, the set of feasible payoff vectors can be unbounded, but in many situations some boundedness conditions are required.

Let us denote by U the set of feasible payoff vectors. We will consider two cases:

(i) (basical model): $U = \{u \in \mathbf{R}^N \mid u(N) \leq \nu(N)\}$

(ii) (bounded model): $U = \{u \in \mathbf{R}^N \mid u(N) \leq \nu(N),\ u \geq a\}$, where $a \in \mathbf{R}^N$ is fixed.

In the classical von Neumann-Morgenstern theory, $a = u^0$ where $u_j^0 = \nu(\{j\})$ for all $j \in N$ and the nondominated feasible payoff vectors are called *imputations.*

Note also some properties of the characteristic function which may be cited:

monotonicity: $C \subset D \Rightarrow \nu(C) \leq \nu(D)$

superadditivity: $C \cap D = \varnothing \Rightarrow \nu(C) + \nu(D) \leq \nu(C \cup D)$ (if the equality always holds then ν is additive and if $C,\ D \neq \varnothing\ C \cap D = \varnothing \Rightarrow \nu(C) + \nu(D) < \nu(C \cup D)$ then ν is strictly superadditive).

convexity: $\nu(C) + \nu(D) \leq \nu(C \cup D) + \nu(C \cap D)$

In the following a TU cooperative game will be represented as the triple (N, ν, U) or, simply, (N, ν) when U is specified. Also, for every coalition C, we denote by $V(C)$ the set of all feasible effective payoffs $U \cap eff\ C$.

DEFINITION 1: *A proposal of the game (N, ν, U) is a pair (u, C) where C is a nonvoid coalition and $u \in V(C)$ (i.e. u is feasible and effective for C).*

Let $\mathcal{K} = \{(u^i, C_i); i = 1, \ldots, m\}$ be a finite set of proposals such that $C_i \neq C_j$ if $i \neq j$.

Following McKelvey, Ordeshook and Winer (1978) we will define the competitive and the strong competitive solutions of the game (N, ν, U).

DEFINITION 2: *$\mathcal{K}$ is a competitive solution (c.s.) if it satisfies the following properties:*

$$\text{There are no } i, j \text{ such that } u^i_{C_i \cap C_j} \gg u^j_{C_i \cap C_j} \tag{1}$$

and

$$\text{If } (u, C) \text{ is a proposal and } u_{C \cap C_i} \gg u^i_{C \cap C_i} \text{ for some } i,$$
$$\text{then there exists } j \text{ such that } u^j_{C \cap C_j} \gg u_{C \cap C_j} \tag{2}$$

If (1) is replaced by

$$\text{There are no } i, j \text{ such that } u^i_{C_i \cap C_j} \gg u^j_{C_i \cap C_j} \tag{3}$$

then $\mathcal{K}$ is called *strong competitive solution* (s.c.s.)

Now we slightly modify the previous definitions to obtain a new solution concept.

DEFINITION 3: *$\mathcal{K}$ is an uniform competitive solution (u.c.s.) if it satisfies:*

$$u^i_{C_i \cap C_j} = u^j_{C_i \cap C_j}, \quad \text{for any } i, j \tag{4}$$

and

$$\text{If } (u, C) \text{ is a proposal and } u_{C \cap C_i} > u^i_{C \cap C_i} \text{ for some } i,$$
$$\text{then there exist } j \text{ and } k \in C \cap C_j \text{ such that } u^j_k > u_k \tag{5}$$

There are some similarities between the c.s. and other classical solutions. In summary, a (strong) competitive solution is a set of proposals having two fundamental properties; internal stability and external stability. Hence, there is a close correspondence between the original definition of the c.s. and von Neumann-Morgenstern stable set. However, some significant differences must be pointed out. Firstly, in a c.s. (s.c.s.) each payoff vector is associated with a coalition for which it is effective. Therefore, a c.s. predicts not only the payoffs but the coalitions which can guarantee them too. On the other hand, the classical domination relation is replaced here by the strong preference manifested by the pivotal players, *i.e.* the players belonging to the coalitions associated to the payoff vectors to compare. An objection against a given proposal is a proposal which is strictly prefered by all pivotal players. According to this interpretation the internal stability follows from the absence of any objection within the c.s.

The external stability of a c.s. is expressed, like in the Aumann-Maschler bargaining theory, in terms of the counterobjections against any possible objection. But an important difference arises from the fact that the coalitions in a c.s. are not necessarily disjoint. Consequently, a c.s. may only predict the payoffs for a set of coalitions which seems to be profitable for the potential partners, but it cannot predict the coalitions which will be actually formed after the bargaining process.

The definition of the u.c.s. does not alter the main characteristics of the c.s. The internal stability, expressed by (4) is strengthened in order to avoid some contradictory situations in the original definition. To explain this assertion let us comment the external stability condition of the previous definition. Suppose that the proposal (u, C) is an objection against $(u^i, C_i) \in \mathcal{K}$. This means that the pivotal players, namely the players in $C \cap C_i$ support this objection since it improves their output. A counterobjection must be claimed by some players which would loose if the objection is accepted. But according to the definition of the c.s. some of these players might be at the same time supporting players, *i.e.* it is possible that $C \cap C_i$ and $C \cap C_j$ be not disjoint. In that case one or more players have two contradictory positions against the objection (u, C). Condition (4) of the definition of the u.c.s. makes impossible this situation. Consequently, any player must have one and only one position toward a given objection; he may support it, he may reject it or he is indifferent.

Finally, let us note that for some classes of cooperative games, the competitive solutions (c.s. or u.c.s) are also related to the core. Within the present formalization the definition of the core [denoted $C(N, \nu)$] may be restated as follows:

$$C(N, \nu) = \{u \in U \mid \text{there is no proposal } (x, C) \text{ such that } x_C \gg u_C\}$$

The relationships between the core and c.s. will be discussed in the next section.

3. THE COMPETITIVE SOLUTIONS AND THE CORE

As it was shown in McKelvey, Ordeshook and Winer (1978), if the core is not empty then a s.c.s. always exists. A similar statement holds for u.c.s. Moreover, a converse implication can be proved.

PROPOSITION 3.1: *Assume* $C(N, \nu) \neq \varnothing$ *and* $u \in C(N, \nu)$. *Then* $\mathcal{K} = \{(u, N)\}$ *is s.c.s. and u.c.s. at the same time.*

Proof: It is sufficient to show that there are not objections against (u, N). Obviously, if (x, C) is a proposal and $x_C > u_C$ then it is easy to see that there exists a proposal (y, C) such that $y(C) = x(C)$ and $y_C \gg u_C$. This contradicts the definition of the core.

PROPOSITION 3.2: *If* $\mathcal{K} = \{(u, N)\}$ *is a c.s. (or, u.c.s.) for some* $u \in \mathbf{R}^N$ *then* $u \in C(N, \nu)$ *and therefore* $C(N, \nu) \neq \varnothing$

Proof: If $\mathcal{K}$ is a c.s. (u.c.s.) and $u \notin C(N, \nu)$ then there exists a proposal (x, C) such that $x_C \gg u_C$. Clearly, there are no counterobjections against this objection, contradicting definitions 1 and 3.

As a consequence of the previous propositions,

$$C(N, \nu) = \{u \in \mathbf{R}^N \mid \{(u, N)\} \text{ is } c.s.\,(u.c.s.)\}$$

Moreover, for the u.c.s. a more complete statement easily follows:

PROPOSITION 3.3: *If* $\mathcal{K}$ *is a u.c.s. and* $(u, N) \in \mathcal{K}$ *for some* $u \in \mathbf{R}^N$ *then* $u \in C(N, \nu)$.

For the remainder of this section it will be convenient to represent a c.s. or a u.c.s. as a set $\mathcal{K} = \{(u^C, C) \mid C \in \mathcal{C}\}$ where $\mathcal{C}$ is a set of nonvoid coalitions and (u^C, C) is a proposal for every $C \in \mathcal{C}$.

DEFINITION 4: *The (uniform) competitive solution* $\mathcal{K}$ *is complete if* $\bigcup_{C \in \mathcal{C}} C = N$.

A complete competitive solution (c.c.s.), respectively a complete uniform competitive solution (c.u.c.s.) is based on the idea of the active participation of all players in the bargaining process. As we will see (Example 3) not all c.s. (u.c.s.) are necessarily complete. It seems to be reasonable to prefer the complete solutions if there exist.

Let us assume $\mathcal{K}$ be either a c.s. or a u.c.s. and denote by w the vector of components:

$$w_j = \max \{u_j^C \mid j \in C,\ C \in \mathcal{C}\}$$

We will call w *the ideal payoff vector associated to* $\mathcal{K}$. Obviously, the dimension of w equals the cardinality of $\bigcup_{C \in \mathcal{C}} C$, so that $w \in \mathbf{R}^N$ if $\mathcal{K}$ is complete.

Note that if $\mathcal{K}$ is a u.c.s. then $w_C = u_C^C$ for every $C \in \mathcal{C}$. For the c.u.c.s. the following characterization of w easily results from the definition.

Proposition 3.4: *If $\mathcal{K}$ is a c.u.c.s. then the ideal payoff vector w satisfies:*

$$w_C \in pr_C\, V(C), \quad \text{for every } C \in \mathcal{C} \tag{6}$$

$$\text{For any proposal } (x,\ D) \text{ it is impossible that } x_D > w_D \tag{7}$$

Conversely, if $w \in \mathbf{R}^N$ satisfies (6) and (7), where $\bigcup_{C \in \mathcal{C}} C = N$, then a c.u.c.s. $\mathcal{K}$ can be defined by taking, for each $C \in \mathcal{C}$ the vector $u^C \in \mathbf{R}^N$ as a possible extension of w_C up to a feasible payoff vector (i.e. $u^C \in U$, $u_C^C = w_C$).

According with this result the c.u.c.s. $\mathcal{K}$ could be represented as the pair $(w, \mathcal{C})$.

Remarks: 1. Obviously, w is not necessarily a feasible vector but if it is, then by Proposition 3.3 it follows that $w \in C(N, \nu)$.

2. If $\mathcal{C}(w) = \{C \subseteq N \mid w_C \in pr_C\, V(C)\}$ then $\mathcal{K}' = \{(u^C, C) \mid C \in \mathcal{C}(w)\}$, where $u^C \in V(C)$, $u_C^C = w_C$, is a maximal c.u.c.s. (Here the term "maximal" refers to the set-inclusion relation). This solution will be denoted by $(w, \mathcal{C}(w))$.

4. COMPETITIVE AND UNIFORM COMPETITIVE SOLUTIONS FOR THE GENERAL TU GAMES (BASICAL MODEL). PROPERTIES AND EXISTENCE THEOREMS.

Everywhere in this section, the set U of feasible payoff vectors will be considered in the case (i) of Section 2. Some interesting properties of the u.c.s. will be pointed out. These properties are common for c.s. and s.c.s. too. In fact, as we will prove, there are no c.s. which are not u.c.s. Moreover, the concept of c.s. is consistent only for the weakly-subadditive games or for the games having nonempty core. On the other hand, the existence of u.c.s. is proved for all games considered here.

Let us firstly remark that since U is not lower-bounded then each payoff vector which is effective for a coalition may be also considered to be feasible. More precisely, we have:

PROPOSITION 4.1: *If* $u \in eff\, C$ *for some* $C \subset N$, $C \neq \emptyset$, *then there exists* $x \in U$ *such that* $x_C = u_C$.

Since in the definition of the proposal only the payoffs assigned to the members of the associated coalition are employed we can neglect in the following the feasibility condition (*i.e.* we will always consider that if $u(C) \leq \nu(C)$ then u is one of the possible extension of u_C up to a feasible vector in $\mathbf{R}^N$). Moreover, for the sake of simplicity we will write $eff\, C$ but we will mean this as $V(C) = U \cap eff\, C$.

PROPOSITION 4.2: *Every u.c.s. is complete.*

Proof: Suppose that $\mathcal{K} = \{(u^i, C_i), i = 1, \ldots, m\}$ is a u.c.s. We are going to prove that $\bigcup C_i = N$. Let w be the ideal payoff vector associated with $\mathcal{K}$ and suppose that $\bigcup C_i \neq N$. Pick a $k \notin \bigcup C_i$. Set $C = \left(\bigcup C_i\right) \cup \{k\}$ and take $u_j = w_j + \varepsilon$ for $j \in \bigcup C_i$ and $u_k = \nu(C) - \sum_{j \neq k} u_j$, where ε is arbitrary positive. Of course, for a suitable extension $u \in \mathbf{R}^N$ of $u_C = (u_j)_{j \in C}$, the pair (u, C) is a proposal and $u_{C \cap C_i} \gg u^i_{C \cap C_i}$ for every $i = 1, \ldots, m$. This contradicts the definition of $\mathcal{K}$.

As it was already mentioned, a u.c.s. is not only stable, but also rational. The next two results establish the internal and the coalitional rationality of any u.c.s. Since the proof is in both cases straightforward it will be omitted here.

PROPOSITION 4.3: *If* $\mathcal{K}$ *is a (complete) u.c.s. and* $(u, C) \in \mathcal{K}$ *then* $u(C) = \nu(C)$. *Particularly,* u_C *is Pareto-optimum of* $pr_C\, eff\, C$.

PROPOSITION 4.4: *Let* w *be the ideal payoff vector associated with the u.c.s.* $\mathcal{K}$. *Then* $w_k \geq \nu(\{k\})$ *for every* $k \in N$. *Consequently, if* $C \in \mathcal{C}$ *then* $\nu(C) \geq \sum_{k \in C} \nu(\{k\})$.

Proposition 3.1 establishes sufficient conditions for the existence of the competitive solutions. Since any TU game with nonempty core admits s.c.s. and u.c.s. it follows that any sufficient condition for the nonemptiness of the core is a sufficient condition for the existence of s.c.s. and u.c.s. Particularly, the balanced and the convex games always admits s.c.s. and u.c.s. Let us emphazise another simple situation when the existence of the competitive solutions is guaranteed.

DEFINITION 5: *The characteristic function* ν *is weakly-subadditive if* $\nu(C) \leq \sum_{k \in C} \nu(\{k\})$ *for every* $C \subseteq N$, $C \neq \varnothing$

Denote by $A(\nu) = \{C \subseteq N \mid |C| \geq 2, \nu(C) = \sum_{k \in C} \nu(\{k\})\}$.

PROPOSITION 4.5: *If* ν *is weakly-subadditive, then* $\mathcal{K}_0 = \{(u^0, \{k\}) \mid k \in N\}$ *is s.c.s. and u.c.s. at the same time (Here* $u_j^0 = \nu(\{j\})$*). Moreover, for each subset* $\mathcal{A} \subseteq A(\nu)$, *the set of proposals* $\mathcal{K} = \mathcal{K}_0 \cup \{(u^0, C) \mid C \in \mathcal{A}\}$ *is s.c.s. and u.c.s.*

Proof: Obviously, $\mathcal{K}_0$ is internal stable in the sense of both definitions. If (u, C) is an objection against the proposal $(u^0, \{k\})$ then it is necessary that $|C| \geq 2$ and $u_k > \nu(\{k\})$. Since $u(C) \leq \nu(C) \leq \sum_{j \in C} \nu(\{j\})$ then there exists $j \in C$ such that $u_j < \nu(\{j\}) = u_j^0$. Hence, $(u^0, \{j\})$ is a counterobjection against (u, C).

For the second statement, we firstly note that if $\mathcal{K}$ is a c.s. (u.c.s.) and $\mathcal{K}_0 \subset \mathcal{K}$ then, for each proposal $(u, C) \in \mathcal{K}$ the equalities $u_k = \nu(\{k\})$, $k \in C$ must hold. Otherwise, the internal stability would be violated. Assume that $\mathcal{K} = \mathcal{K}_0 \cup \{(u, C) \mid C \in \mathcal{A}\}$ and pick $(u, C) \in \mathcal{K}$, $|C| \geq 2$. Then $\nu(C) = u(C) = \sum_{j \in C} \nu(\{j\})$. Suppose that there is a proposal (x, D) such that $x_{C \cap D} > u_{C \cap D}$. Then $x_j > u_j = \nu(\{j\})$ for some $j \in C \cap D$ but $x(D) \leq \nu(D) \leq \sum_{j \in D} \nu(\{j\})$. Hence, there exists $k \in D$ such that $x_k < \nu(\{k\}) = u_k^0$. Therefore, $(u^0, \{k\})$ is a counterobjection against (x, D).

Now we are able to precise the conditions under which a c.s. exists. As it will be shown, each c.s. is at the same time a u.c.s.

THEOREM 4.1: *The game (N, ν) admits c.s. if and only if one of the following two conditions is satisfied:*

a) $C(N, \nu) \neq \emptyset$

b) ν *is weakly-subadditive.*

In these situations the sets of c.s. are those described in the propositions 3.1 and 4.5, and each c.s. is at the same time a u.c.s.

Proof: Suppose that a c.s. $\mathcal{K} = \{(u^C, C) \mid C \in \mathcal{C}\}$ exists. Denote by $\mathcal{K} = \{k \mid \{k\} \in \mathcal{C}\}$ and let w be the associated ideal payoff vector. We must analyze three situations:

1. $\mathcal{C} = \{N\}$. (*i.e.* $\mathcal{K} = \{(u, N)\}$ for some $u \in \mathbf{R}^N$).

Then by Proposition 3.1 $u \in C(N, \nu)$ and by Proposition 3.2 $\mathcal{K}$ is a u.c.s.

2. $\mathcal{C} \neq \{N\}$, $K \neq N$.

Choose $C \in \mathcal{C}$ and $k \notin K$ such that $k \notin C$. (If $K \neq \emptyset$ then C may be $\{j\}$ for any $j \in K$ and $k \notin K$ is arbitrary. If $K = \emptyset$ pick $C \in \mathcal{C}$, $C \neq N$ and $k \in N \backslash C$.) Take $u \in \mathbf{R}^N$ of components:

$$u_j = \begin{cases} w_j + \varepsilon & \text{if } \quad j \neq k \\ \nu(N) - \sum\limits_{j \neq k} u_j & \text{if } \quad j = k \end{cases}$$

where ε is arbitrary positive.

Since $u(N) = \nu(N)$ then (u, N) is a proposal and $u_C \gg u_C^C$. Moreover, there is no $D \in \mathcal{C}$ such that $u_D^D \gg u_D$ (since $\{k\} \notin \mathcal{C}$). Therefore, (u, N) is an objection against (u^C, C) but there does not exist any counterobjection. Consequently, $\mathcal{K}$ is not a c.s.

3. $\mathcal{C} \neq \{N\}$, $K = N$

Clearly, $w_j = \nu(\{j\})$ for every $j \in N$ and if $(u^C, C) \in \mathcal{K}$ then $u_j^C = w_j = \nu(\{j\})$, $j \in C$. Consequently, $\sum\limits_{j \in C} \nu(\{j\}) = u^C(C) \leq \nu(C)$.

Moreover, if $\nu(C) > \sum\limits_{j \in C} \nu(\{j\})$ for some $C \subseteq N$, then it is easy to find $u \in eff\, C$ such that $u_j > \nu(\{j\})$ for all $j \in C$. Then the definition of $\mathcal{K}$ would be contradicted. Hence, $\mathcal{K}$ is a c.s. only if ν is weakly-subadditive. In this case the conclusion of the theorem follows from Proposition 4.5. Indeed, since if $K \neq N$ there are no c.s., it follows that the c.s. listed in Proposition 4.5 are all c.s. of the game.

Remark: An important conclusion of the previous theorem is that every c.s. is at the same time u.c.s. But, as it will be shown in the following example, there are u.c.s. which are not c.s. Moreover, even in the special cases considered in the propositions 3.1 and 4.5, there are u.c.s. which are not c.s.

EXAMPLE 1: $n = 3 \cdot \nu(\{i\}) = 1$, $i = 1, 2, 3$: $\nu(\{i, j\}) = 2$, *for every* $\{i, j\} \subset \{1, 2, 3\}$; $\nu(N) = 3$.

This is a simple example of additive game. The core consists in a single point $C(N, \nu) = \{u^0\}$, where $u^0 = (1, 1, 1)$. The following set of proposals is a u.c.s. but not a c.s.:

$$\mathcal{K} = \{(u^0, \{1, 2\}), (u^0, \{1, 3\}), (u^0, \{2, 3\})\}$$

Indeed, $((1.5, 1.5, 0), N)$ is an objection against $(u^0, \{1, 2\}) \in \mathcal{K}$, but there does not exist counterobjections in the sense of the definition of c.s.

EXAMPLE 2: $n = 4 \cdot \nu(\{i\}) = 1$, $i = 1, 2, 3, 4$; $\nu(\{1, 4\}) = 2$; $\nu(\{1, 2, 3\}) = \nu(\{1, 2, 4\}) = \nu(\{2, 3, 4\}) = 4$; $\nu(C) = 3$, *otherwise.*

Let $\mathcal{K} = \{((1, 2, \, , \,), \{1, 2\}), ((1, \, ,2, \,), \{1, 3\}), ((1, \, , \, , 1), \{1, 4\})\}$. (The blanks can be arbitrary completed up to feasible payoffs).

It is easy to verify that $\mathcal{K}$ is a u.c.s. However, it is not c.s. Indeed, the proposal $((2, 1, 1, 0), \{1, 2, 3\})$ is an objection against the last proposal of $\mathcal{K}$, but there are no counterobjections satisfying the definition of c.s.

The main result of this section establishes the existence of the u.c.s.

THEOREM 4.2: *Every TU cooperative game* (N, ν) *[case (i)] has a (complete) uniform competitive solution.*

LEMMA 1: *Let* $\mathcal{K}$ *be a c.u.c.s. and* (x, D) *a proposal such that* $x(D) < \nu(D)$. *Then there exist* $(u, C) \in \mathcal{K}$ *and* $k \in C \cap D$ *such that* $u_k > x_k$.

Proof: Let $(y, E) \in \mathcal{K}$ such that $D \cap E \neq \varnothing$. Suppose $x_{D \cap E} \geq y_{D \cap E}$. Define $z \in \mathbf{R}^D$ by $z_t = x_t + \varepsilon / |D|$, $t \in D$, where $\varepsilon = \nu(D) - x(D) > 0$.

Obviously, $z(D) = x(D) + \varepsilon = \nu(D)$, and therefore $z \in eff\, D$. Since $z_{D \cap E} \gg x_{D \cap E}$ there exist a proposal $(u, C) \in \mathcal{K}$ and $k \in C \cap D$ such that $u_k > z_k > x_k$.

Proof of the Theorem:

By induction on $n = |N|$.

For $n = 1$ the truth of the theorem is obvious; $\{(\nu(\{1\}), \{1\})\}$ is a u.c.s.

Suppose that the theorem holds for every game having at most $n - 1$ players and let (N, ν) be a game with n players. Set $M = N \backslash \{n\}$ and define the reduced game (M, ν_M) by its characteristic function:

$$\nu_M (C) = \max \{\nu (C), \nu (C \cup \{n\}) - z_n\}, \qquad C \subseteq M$$

where $z_n = \nu(\{n\})$.

Since $|M| = n-1$, the game (M, ν_M) has a u.c.s., say $\mathcal{K}_M = \{(y^i, C_i); i = 1, \ldots, m\}$.

Set:

$$C'_i = \begin{cases} C_i & \text{if} \quad \nu (C_i \cup \{n\}) - z_n < \nu (C_i) \\ C_i \cup \{n\} & \text{if} \quad \nu (C_i \cup \{n\}) - z_n \geq \nu (C_i) \end{cases} \tag{8}$$

$$u^i = \begin{cases} (y^i, y^i_n) & \text{where} \quad y^i_n \leq \nu (N) - \sum_{j \in M} y^i_j \qquad \text{if} \quad C'_i = C_i \\ (y^i, z_n) & \text{if} \quad C'_i = C_i \cup \{n\} \end{cases} \tag{9}$$

$$\mathcal{K} = \{(u^i, C'_i); i = 1, \ldots, m\}$$

and

$$\mathcal{K}_N = \begin{cases} \mathcal{K} & \text{if there exists } i \text{ such that } n \in C'_i \\ \mathcal{K} \cup \{(u^{m+1}, \{n\})\} & \text{otherwise} \end{cases}$$

(Here u^{m+1} is arbitrary with $u^{m+1}_n = z_n$, and for convenience, the pair $(u^{m+1}, \{n\})$ will be denoted by (u^{m+1}, C'_{m+1})).

We are ready to prove that $\mathcal{K}_N$ is a u.c.s. of the game (N, ν).

Obviously, $\bigcup C'_i = N$ and $u^i \in eff\, C'_i$ for all $i \in N$. To verify (4) suppose $(u^i, C'_i), (u^j, C'_j) \in \mathcal{K}_N$ and $u^i_k > u^j_k$ for some $k \in C'_i \cap C'_j$. Clearly k must differs on n and then $y^i_k > y^j_k$ for some $k \in C_i \cap C_j$, contradicting the properties of $\mathcal{K}_M$.

Let us verify (5). Pick a coalition C and $x \in eff\, C$ and suppose that $x_{C \cap C'_i} > u^i_{C \cap C'_i}$ for some i. We will consider three possibilities:

(i) If $n \notin C$ then (x_M, C) is a proposal of (M, ν_M) and $x_{C \cap C_i} > y^i_{C \cap C_i}$. Since $\mathcal{K}_M$ is a u.c.s. of (M, ν_M) then there exist $j \leq m$ and $k \in C \cap C_j$ such that $y^j_k > x_k$ or, equivalently, $u^j_k > x_k$ for some $k \in C \cap C'_j$.

(ii) If $C = \{n\}$, then $x_n \leq z_n$ and the initial assumption fails.

(iii) Suppose that $C = Q \cup \{n\}$ where $\varnothing \neq Q \subset M$. There are three possible situations: $x_n > z_n$, $x_n = z_n$ or $x_n < z_n$. In the first case, $x(Q) = x(C) - x_n < \nu(C) - z_n \leq \nu_M(Q)$. Hence, (x_M, Q) is a proposal of (M, ν_M). From the previous lemma, there exist j and $k \in Q \cap C_j$ such that $y_k^j > x_k$. That is, $u_k^j > x_k$ for some $k \in C \cap C_j'$. If $x_n = z_n$, the initial assumption implies $x_{Q \cap C_i} > y^i_{Q \cap C_i}$ (if $n \in C_i'$ then $u_n = z_n$). Since $x(Q) \leq \nu(Q \cup \{n\}) - z_n \leq \nu_M(Q)$ it follows that (x_M, Q) is a proposal of (M, ν_M). (In fact, x_M stands for a $(n-1)$-dimensional extension of x_Q.) But $\mathcal{K}_M$ is a u.c.s. of (M, ν_M) so that, there exist j and $k \in Q \cap C_j$ such that $y_k^j > x_k$ *i.e.* $u_k^j > x_k$ for some $k \in C \cap C_j'$. In the third situation $u_n^j = z_n$ for every j such that $n \in C_j'$, so that the desired conclusion follows with $k = n$.

COROLLARY 4.1: *If ν is convex and $\nu(\varnothing) = 0$ then the game (N, ν) has a u.c.s. of the form $\mathcal{K} = \{(u, N)\}$.*

Proof: Note firstly that the proof of Theorem 4.2 is constructive. A u.c.s. may be obtained in n steps, each step extending an existing u.c.s. to a solution of a game with one more player. This extension may be made on two ways; by adding a new proposal to the existing solution, or keeping the same number of proposals. In the last case, according to (8) and (9) a new player is added to some coalition-component of the proposals in the previous solution. The process starts at the first step with the solution of an one-person game. Obviously, this solution would be extended to a one-proposal solution at the next step if (8) picks on the second alternative. It is easy to see that this happens if the corresponding characteristic function is superadditive. In summary, if at each step the characteristic function of the current game is superadditive, then the final solution will consist of a single proposal. The completeness implies that the corresponding coalition is just N. Now, it is sufficient to prove that if ν satisfies the assumptions of the corollary, then ν_M is also convex. Obviously, since $\nu(\varnothing) = 0$, the convexity implies the superadditivity (we can always take $\nu_M(\varnothing) = 0$ by definition).

Let $C, D \subseteq M$. We are going to prove that $\nu_M(C) + \nu_M(D) \leq \nu_M(C \cup D) + \nu_M(C \cap D)$. Since this relation is obviously verified if one of the two coalitions is $\varnothing$ or M, we can consider both C and D to be different of $\varnothing$ and M. Since ν is superadditive then $n \notin C$ implies $\nu(C) + \nu(\{N\}) \leq \nu(C \cup \{n\})$. Hence, $\nu_M(C) = \nu(C \cup \{n\}) - z_n$. Then, $\nu_M(C) + \nu_M(D) = \nu(C \cup \{n\}) + \nu(D \cup \{n\}) - 2z_n \leq \nu(C \cup D) \cup \{n\}) + \nu((C \cap D) \cup \{n\}) - 2z_n = \nu_M(C \cup D) + \nu_M(C \cap D)$.

Corollary 4.2: *[Shapley (1971)]. If* ν *is convex and* $\nu(\varnothing) = 0$ *then* $C(N, \nu) \neq \varnothing$.

Proof: It simply follows from the previous result and Proposition 3.2.

5. BOUNDED TU GAMES. EXISTENCE THEOREMS

We will consider now the case (ii) of Section 2. That is, the set of all feasible payoff vectors is $U = \{u \in \mathbf{R}^N \mid n(N) \leq \nu(N),\ u \geq a\}$, for a fixed $a \in \mathbf{R}^N$. Without loss of generality we can take $a = 0$. This is an immediate consequence of the following proposition.

Proposition 5.1: *Let us consider the games* (N, ν, U), (N, ν', U') *where*

$$\left.\begin{array}{c}\nu'(C) = \alpha\nu(C) + r(C)\ \ for\ every\ C \subseteq N,\ \ for\ fixed\ \alpha > 0,\\ r \in \mathbf{R}^N\end{array}\right\} \quad (10)$$

and

$$U = \{u \in \mathbf{R}^N \mid u(N) \leq \nu(N),\ u \geq a\};$$
$$U' = \{y \in \mathbf{R}^N \mid y(N) \leq \nu'(N),\ y \geq \alpha a + r\}$$

Then (u, C) *is a proposal of* (N, ν, U) *if and only if* (y, C) *is a proposal of* (N, ν', U'), *where* $y = \alpha u + r$. *Moreover,* $\mathcal{K} = \{(u^i, C_i);\ i = 1, \ldots, m\}$ *is a c.s. (u.c.s.) of* (N, ν, U) *if and only if* $\mathcal{K}' = \{(y^i, C_i);\ i = 1, \ldots, m\}$, $y^i = \alpha u^i + r$ *is a c.s. (respectively, a u.c.s.) of* (N, ν', U').

Remarks: This proposition extends to the competitive solutions the classical property of the *strategical equivalence*. As it is known, since ν and ν' are strategically equivalent, [as a consequence of (10)], then the games (N, ν, U) and (N, ν', U') are isomorphic and the one-to-one correspondence $h : \mathbf{R}^N \mapsto \mathbf{R}^N$, $h(u) = \alpha u + r$ maps U onto U' preserving the preference relations.

For the remainder of this section, the set of feasible payoff vectors will have the form $U = \{u \in \mathbf{R}_+^N \mid u(N) \leq \nu(N)\}$ and we will refer to the game (N, ν, U) as the pair (N, ν). Clearly, $U \neq \varnothing$ if and only if

$$\nu(N) \geq 0$$

and this condition will be imposed in the following to ensure the consistency of the solution concepts.

Moreover, $V(C) = U \cap eff\, C \neq \varnothing$ if and only if $\nu(C) \geq 0$ and it is easy to see that in this case $pr_C\, V(C) = \{u_C \mathbf{R}^C \mid u(C) \leq \min\{\nu(C), \nu(N)\}\}$

Let us define $\nu' : \mathbf{R}^N \mapsto \mathbf{R}$ by $\nu'(C) = \min\{\nu(C), \nu(N)\}$. Then,

PROPOSITION 5.2: *$\mathcal{K} = \{(u^i, C_i);\ i = 1, \ldots, m\}$ is a c.s. (u.c.s.) of the game (N, ν) if and only if it is a c.s. (u.c.s.) of the game (N, ν')*

As a consequence of this result we can consider that $\nu(C) \leq \nu(N)$ for every coalition C, *i.e.* every payoff which is effective for a coalition C must be also feasible. (Otherwise, the original characteristic function can be replaced by ν' as above).

Refering to the properties of the u.c.s. proved in the previous section, it is easy to verify that Propositions 4.3 and 4.4 remain still valid, but it is not the case of Proposition 4.2. The next example will show that within the present model a u.c.s. (as well as a c.s.) is not necessarily complete.

EXAMPLE 3: *$n = 3$. $\nu(\{i\}) = 1$, $i = 1, 2, 3$; $\nu(\{i, j\}) = k$, if $\{i, j, k\} = \{1, 2, 3\}$; $\nu(N) = 4$.*

Obviously, $\mathcal{K} = \{((2, 1, 1), \{1, 2\})\}$ is a u.c.s. (and c.s.) which is not complete.

The following example emphasizes other significant differences between the considered models; a c.s. is not necessarily u.c.s.

EXAMPLE 4: *$n = 3$. $\nu(\{1\}) = \nu(\{2\}) = 1.05$; $\nu(\{3\}) = 0$; $\nu(\{1, 2\}) = \nu(N) = 2$; $\nu(\{1, 3\}) = \nu(\{2, 3\}) = 0.9$.*

$\mathcal{K} = \{((0.9, 1.1, 0), \{1, 2\}), ((1.1, 0.9, 0), N)\}$ is a s.c.s. but not a u.c.s. [condition (4) is violated].

The main results of this section concern the existence of the u.c.s. In fact, we will prove the existence of a c.u.c.s. For the sake of the simplicity it will be assumed that the characteristic function satisfies $\nu(C) \leq \nu(N)$ for every C and that $U = \{u \in \mathbf{R}^N \mid u(N) \leq \nu(N)\}$. As it was shown on above this does not restrict the generality.

THEOREM 5.1: *If the characteristic function ν is non-negative, then the game (N, ν) admits a c.u.c.s.*

Proof: The proof follows the same way as for Theorem 4.2; the completeness of the u.c.s. is obvious at the first step and may be assumed inductively for ν_M.

We will pay more attention for a special case which includes the simple games. As it is known, in a simple game $eff\,C$ is nonempty only for the winning coalitions and the class of these coalitions is closed under the set-inclusion relation. Since in the present case $eff\,C \neq \varnothing \Leftrightarrow \nu(C) \geq 0$ we can characterize the set of all winning coalition as:

$$\mathcal{W} = \{C \mid \nu(C) \geq 0\}$$

THEOREM 5.2: *Assume ν be monotonic on $\mathcal{W}$ and $\mathcal{W} \neq \varnothing$ satisfy the condition:*

$$C \in \mathcal{W},\ C \subset D \Rightarrow D \in \mathcal{W} \tag{11}$$

Then the game (N, ν) has a c.u.c.s.

Proof: If $\mathcal{W} = 2^N \setminus \{\varnothing\}$, then we are in the case of Theorem 5.1.

Let us assume that $\mathcal{W} \neq 2^N \setminus \{\varnothing\}$. Pick $T \in \mathcal{W}$ such that $|T| = \min\{|C| \mid C \in \mathcal{W}\}$. Take $z \in eff\,C$ such that $z(T) = \nu(T)$ and set $M = N \setminus T$. Define $\nu_M : 2^M \mapsto \mathbf{R}$ by:

$$\nu_M(C) = \begin{cases} \nu(\varnothing) & \text{if } C = \varnothing \\ \max\{\nu(C \cup S) - z(S) \mid S \subseteq T\} & \text{if } \varnothing \neq C \subseteq M. \end{cases}$$

Clearly, $\nu_M(C) \geq 0$ for every $C \neq \varnothing$. Indeed, $\nu_M(C) \geq \nu(C \cup T) - z(T) = \nu(C \cup T) - \nu(T) \geq 0$. Note also that $\nu_M(C) \leq \nu_M(M)$ for every $C \subset M$. Then, Theorem 5.1 implies the existence of a c.u.c.s. $\mathcal{K}_M$ of (M, ν_M); $\mathcal{K}_M = \{(y^i, C_i);\ i = 1, \ldots, m\}$. Let $y \in \mathbf{R}^M$ be the associated ideal payoff vector.

Since $y^i(C_i) = \nu_M(C_i)$ there exists $S_i \subseteq T$ such that $y^i(C_i) + z(S_i) = \nu(C_i \cup S_i)$. Obviously, $(y^i_{C_i}, z_{S_i}) \in \mathbf{R}^{C_i \cup S_i}$ and it can be extended up to a feasible payoff vector $u^i \in \mathbf{R}^N$. This means that $u^i(N) \leq \nu(N)$ and $u^i_{C_i \cup S_i} = (y_{C_i}, z_{S_i})$

Set $C'_i = C_i \cup S_i$, $\mathcal{C} = \{C'_1; \ldots, C'_m\}$ and

$$\mathcal{C}' = \begin{cases} \mathcal{C} & \text{if } \bigcup S_i = T \\ \mathcal{C} \cup \{T\} & \text{otherwise} \end{cases}$$

The existence of a c.u.c.s. of the game (N, ν) will be proved by using the proposition 3.4. In fact we will prove that the pair $(u, \mathcal{C}')$ satisfies (6) and (7), where $u = (y, z_T)$. Clearly, $\bigcup_{C' \in \mathcal{C}'} C' = N$ and $u_{C'} \in pr_{C'}\, V(C')$ for every $C' \in \mathcal{C}'$. Let us verify (7).

Suppose $C \in \mathcal{W}$ and $x \in eff\, C$ such that $x_C > u_C$. Of course, $C = T$ or $C \cap M \neq \emptyset$. If $C = T$ then $x \in eff\, T$ and $x_T > z_T$ which is impossible because $x(T) \leq \nu(T) = z(T)$. Assume $C = Q \cup S$ where $\emptyset \neq Q \subseteq M$ and $S \subset T$. We have either $x_Q > y_Q$ or $x_S > z_S$. In the first case, since $x_S \geq z_S$ it follows that $x(Q) + z(S) \leq x(Q \cup S) \leq \nu(Q \cup S)$. Consequently, $x(Q) \leq \nu_M(Q)$ *i.e.* (x_M, Q) is a proposal of (M, ν_M) and the initial inequality contradicts the properties of y.

In the second case, $x_Q \geq y_Q$ and $x_S > z_S$. Then, $x_t > z_t$ for a least one $t \in S$. Denote by $\varepsilon = x_t - z_t$ and construct $v \in \mathbf{R}^N$ such that:

$$v_k = \begin{cases} x_k + \varepsilon / \,|\, Q \,| & \text{if} \quad k \in Q \\ z_t & \text{if} \quad k = t \\ x_k & \text{otherwise} \end{cases}$$

Obviously, $v(Q) + z(S) \leq x(Q \cup S)$ and $v_Q > y_Q$. Hence, (v_M, Q) is a proposal of the game (M, ν_M) and the previous inequality contradicts the properties of y. This complete the proof of the theorem.

Remark: If ν is monotonic then ν' is monotonic too. Therefore the assumptions of Theorem 5.2 refer to the original characteristic function of the game.

The next result establishes that for an important class of TU games, a c.s. exists only when it coincides with a u.c.s.

THEOREM 5.3: *Suppose ν be strictly superadditive and $\nu(C) \geq 0$ for every $C \subseteq N$. Then a c.s. exists if and only if $C(N, \nu) \neq \emptyset$. Moreover, in this case, every c.s. is a u.c.s. at the same time.*

Proof: It follows from the propositions 3.1 and 3.2 that $C(N, \nu) \neq \emptyset$ if and only if there exists $u \in \mathbf{R}^N$ such that $\{(u, N)\}$ is c.s. and u.c.s. All that we must do is to prove that there are no other c.s.

To the contrary, let assume $\mathcal{K} = \{(u^C, C) \,|\, C \in \mathcal{C}\}$ be a c.s. and $\mathcal{C} \neq \{N\}$. This will lead us to a contradiction.

We begin by proving that $C_i \cap C_j \neq \emptyset$ whenever $C_i, C_j \in \mathcal{C}$. Suppose that this is not true and denote by $C = C_i \cup C_j$. Then, $\nu(C) > \nu(C_i) + \nu(C_j)$. Now, it is easy to find a proposal (x, C) such that $x(C) = \nu(C)$ and $x_{C_i} \gg u^i_{C_i}$, $x_{C_j} \gg u^j_{C_j}$. (Here, $u^i = u^{C_i}$). Since $\mathcal{K}$ is c.s. there exists $(u^k, C_k) \in \mathcal{K}$ such that $u^k_{C \cap C_k} \gg x_{C \cap C_k}$. Consequently, at least one of the following two inequalities: $u^k_{C_k \cap C_i} \gg u^i_{C_k \cap C_i}$, $u^k_{C_k \cap C_j} \gg u^j_{C_k \cap C_j}$ holds. But this contradicts the properties of $\mathcal{K}$.

Now, let choose $(u, C) \in \mathcal{K}$, $C \neq N$. Clearly, $\nu(N) > \nu(C) = u(C)$ and a proposal (x, N) can be found such that $x_C \gg u_C$. Then there would exist a proposal $(y, D) \in \mathcal{K}$ such that $y_D \gg x_D$. But, as it follows from the above, $D \cap C \neq \emptyset$. This implies $y_{C \cap D} \gg u_{C \cap D}$ and condition (1) is violated.

6. UNIFORM COMPETITIVE SOLUTIONS AND ASPIRATIONS

Although the definition of u.c.s. is derived from the original concept of competitive solution, it seems to be closely related to other solution concepts. Particularly, there is a strong similarity between the ideal payoff vector associated to a c.u.c.s. and the "aspiration" [see Bennett (1983)]. In a slighty different form the definition of the aspirations can be adapted for all game models considered in the present paper.

DEFINITION 6: *The n-vector $u \in \mathbf{R}^N$ is an aspiration if*

$$\bigcup \{C \subseteq N \mid u \in eff\, C\} = N \tag{12}$$

and

$$\text{There does not exist } x \in \mathbf{R}^N \text{ such that } x \in eff\, C$$
$$\text{and } x_C > u_C \text{ for some } \emptyset \neq C \subseteq N \tag{13}$$

Note that an aspiration is not restricted to satisfy any feasibility condition and this is the main difference between this concept and the u.c.s. (compare the above definition with Proposition 3.4). However, as it was shown, for the basical model of TU games [case (i)] the feasibility is a consequence of the effectiveness. In this case, the relationships between the c.u.c.s. and the aspirations follow from Proposition 3.4.

PROPOSITION 6.1: *Assume the game (N, ν) satisfy (i). Let $\mathcal{K}$ be a c.u.c.s. and w the associated ideal payoff vector. Then w is an aspiration. Conversely, if u is an aspiration then $(u, C(u))$ is a c.u.c.s.*

For the following algebraic characterization of the c.u.c.s. [case (i)] we can also refer to Lemma 2.1 of Bennett (1983).

PROPOSITION 6.2: *The pair $(u, \mathcal{C})$, where $\bigcup \{C \mid C \in \mathcal{C}\} = N$ is a c.u.c.s. if and only if*

$$u(C) \geq \nu(C), \quad for\ every\ C \subseteq N, \qquad C \neq \emptyset \tag{14}$$

and

$$u(C) = \nu(C), \quad for\ every\ C \in \mathcal{C} \tag{15}$$

Proof: Indeed, if $(u, \mathcal{C})$ is a c.u.c.s. then (15) follows from Proposition 4.3. Moreover, if $u(C) < \nu(C)$ for some C, then a proposal (x, C) can be found such that $x(C) = \nu(C)$ and $x_C \gg u_C$ contradicting the definition. Consequently, (14) must be satisfied too.

Conversely, if $u \in \mathbf{R}^N$ satisfies (13) and (14) then it also satisfies (6) and (7) and the second statement of the present proposition follows from Proposition 3.4.

For the bounded model [case (ii)] only the first statement of the proposition 6.1 remains valid. Since the aspirations are not subjected to the feasibility condition it is not always possible to identify an aspiration with the ideal payoff of a c.u.c.s.

EXAMPLE 5: $n = 3$. $\nu(\{i\}) = 0.5$; $\nu(\{1, 2\}) = 2.5$; $\nu(\{1, 3\}) = 3$; $\nu(\{2, 3\}) = 1$; $\nu(N) = 2$.

Clearly, $u = (2, 0.5, 1)$ is an aspiration and $\mathcal{C}(u) = \{\{2\}, \{1, 2\}, \{1, 3\}\}$. But the pair $(u, \mathcal{C}(u))$ does not represent a u.c.s. (If $C = \{1, 2\}$ then $u_C = (2, 0.5)$ cannot be extended to a feasible payoff vector).

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