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T. DOHI
A. WATANABE
S. OSAKI

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A NOTE ON RISK AVERSE NEWSBOY PROBLEM (*)

by T. Dohi (1), A. Watanabe (1) and S. Osaki (1)

Abstract. – *This paper considers the maximization of expected utility functions in the classical newsboy problem. First, the general theory for the risk averse utility functions is developed. Then, the necessity of approximation occurs when the logarithmic utility, which is familiar to the risk theory, is used in the newsboy problem. Applying the Taylor approximation, we consistently define the logarithmic utility in the context of the newsboy problem. Secondly, we discuss an alternative problem maximizing the upper and lower bounds of the expected utility function, instead of the direct maximization of it. Some numerical examples show that the approximation procedure and the alternative objective proposed here are useful from the practical viewpoint.*

Keywords: Newsboy problem, Expected utility, Risk aversion, Approximation, Upper and lower bounds.

Résumé. – *Nous examinons la maximisation de l'espérance de la fonction d'utilité dans le problème classique dit du « distributeur de journaux » (« newsboy problem »). Nous développons tout d'abord la théorie générale pour les fonctions d'utilité dans le cas d'aversion du risque. La nécessité d'opérer une approximation se présente lorsque l'utilité logarithmique, qui est familière dans la théorie du risque, est appliquée à notre problème. Appliquant l'approximation de Taylor, nous définissons en conséquence l'utilité logarithmique dans le contexte de notre problème. En deuxième lieu, nous examinons le problème de la maximisation de la borne supérieure et de la borne inférieure de l'espérance de la fonction d'utilité, au lieu de sa maximisation directe. Des expériences numériques montrent que la procédure d'approximation et les autres maximandes proposés ici sont utiles du point de vue pratique.*

Mots clés : Problème du distributeur de journaux, utilité moyenne, aversion du risque, approximation, bornes supérieures et inférieures.

1. INTRODUCTION

The newsboy problem is a single-item single-period inventory problem characterized by a newsboy facing an uncertain daily demand for newspapers. If he carries only a small quantity of newspaper, he misses out on a profit. Conversely, if he buys too much newspapers, he must pay for a penalty. The problem is to determine the optimal order quantity of newspaper maximizing an objective function. This kind of inventory model is applicable to many

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(1) Dept. of Industrial and Systems Engineering, Hiroshima University, Kagamiyama 1-4-1, Higashi-Hiroshima 724, Japan.

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practical situations (*e. g.* the overbooking in hotel reservations). One of the most traditional approaches to the newsboy problem was to simply maximize the expected profit. It is, however, pointed out by many authors that this kind of approach takes account of only *risk neutrality* of the decision maker, but never common *risk aversion.*

As the risk averse newsboy problem, Lau [8] and Ward *et al.* [13] provided a utility function consisting of the mean and the standard deviation of stochastic profit. Anvari [1] and Chang [6] discussed the newsboy problem in the context of CAPM (Capital Asset Pricing Model), which is familiar to the financial theory. Recently, Bouakiz and Sobel [5] analyzed a multi-period newsboy problem with an exponential utility function by applying the theory of Markov decision process. It should be noted, however, that the inventory control problems discussed in earlier contributions specified the type of utility function. Although Atkinson [2] used the utility functions of HARA (Hyperbolic Absolute Risk Aversion) class to explain the relationship between incentives, uncertainty and risk in the newsboy problem, the more general theory of risk aversion in the newsboy problem is needed.

The main purpose of this paper is to develop the optimality condition for the risk averse manager facing the newsboy problem. We analyze the optimization problem with the risk averse utility function. Then, it is necessary to approximate the most common utility function as a *logarithmic utility function.* We propose an approximation procedure for the utility function. In numerical examples, we investigate the precision of the approximation and carry out the sensitivity analysis. Next, we suggest an alternative objective of the newsboy problem. Concretely speaking, the maximization problem of the upper and lower bounds of risk averse expected utility functions is discussed. These two optimality criteria seem to be useful to realize a flexible decision making, when the Schmeiser-Deutsch distribution [2] is assumed as a demand one. We numerically examine the optimal ordering policy and refer to the applicability of this alternative optimization problem. Finally, we conclude with a discussion and some directions for future research.

2. SINGLE-PERIOD NEWSBOY PROBLEM

Notation and assumption

Following Lau [8], let us make some basic assumptions on the classical single-period newsboy problem. A single item is considered. The amount of item Q $(0 \leq Q < \infty)$ is ordered and instantaneously delivered at the initial time. The goods are delivered with cost C per unit and its unit selling price is R. The demand $\tilde{D}$ for the period is a random variable, where the probability density function and the cumulative density function are $g(\cdot)$ and $G(\cdot)$, respectively. At the moment we assume $\tilde{D} \in [0, \infty)$. Let V be the salvage value per unit at the end of period. If the demand is less than Q, the manager disposes of the extra goods at the cost V, otherwise, the cost S is suffered for the shortage penalty per unit. Without loss of generality, suppose that $R > C > V > 0$.

Then the actual sales and the stochastic profit are

$$\tilde{A} = \min(Q, \tilde{D}), \tag{1}$$

$$\begin{aligned} Y(Q) &= R\tilde{A} + V\max(0, Q - \tilde{D}) - S\max(0, \tilde{D} - Q) - CQ \\ &= (R - V + S)\tilde{A} - S\tilde{D} - (C - V)Q. \end{aligned} \tag{2}$$

Let E be the expectation operator. The mean and the variance of profit are

$$E[Y(Q)] = (R - V + S)E[\tilde{A}] - SE[\tilde{D}] - (C - V)Q, \tag{3}$$

$$\begin{aligned} \operatorname{Var}[Y(Q)] =&(R - V + S)^2 \operatorname{Var}[\tilde{A}] \\ &+ S^2 \operatorname{Var}[\tilde{D}] - 2S(R - V + S)\operatorname{Cov}[\tilde{A} \cdot \tilde{D}], \end{aligned} \tag{4}$$

where the mean and the variance of demand and actual sales are

$$E[\tilde{D}] = \int_0^\infty Dg(D)\,dD, \tag{5}$$

$$E[\tilde{A}] = E_Q[\tilde{D}] + Q\{1 - G(Q)\}, \tag{6}$$

$$\operatorname{Var}[\tilde{D}] = E[\tilde{D}^2] - \{E[\tilde{D}]\}^2, \tag{7}$$

$$\operatorname{Var}[\tilde{A}] = E_Q[\tilde{D}^2] + Q^2\{1 - G(Q)\} - \{E[\tilde{A}]\}^2, \tag{8}$$

respectively, and the covariance of them is

$$\operatorname{Cov}[\tilde{A} \cdot \tilde{D}] = E_Q[\tilde{D}^2] - (Q + E[\tilde{D}])E_Q[\tilde{D}] + QG(Q)E[\tilde{D}]. \tag{9}$$

Also the first and second order partial moments are

$$E_Q[\tilde{D}] = \int_0^Q Dg(D)\,dD, \tag{10}$$

$$E_Q\,[\tilde{D}^2] = \int_0^Q D^2 g\,(D)\,dD, \tag{11}$$

respectively.

Motivation for risk averse decision making

First, let us consider the risk neutral newsboy problem. This problem discussed in the standard text book on the inventory theory is to derive the optimal order quantity simply maximizing the expected profit as follows;

$$\left.\begin{array}{rl} & \max_{Q\geq 0} E\,[Y\,(Q)], \\ \text{s.t.} & Y\,(Q) = (R - V + S)\,\min\,(Q,\,\tilde{D}) - S\tilde{D} - (C - V)\,Q, \\ & R > C > V > 0. \end{array}\right\} \tag{12}$$

We summarize the following standard result.

PROPOSITION 1: *Under the condition of $R > C > V$, $E\,[Y\,(Q)]$ is a unimodal function of Q and there is a finite and unique solution of the risk neutral problem in Eq. (12).*

Proof: Since $d^2\,E\,[Y\,(Q)]/dQ^2 < 0$, $\lim_{Q\to 0} dE\,[Y\,(Q)]/dQ > 0$ and $\lim_{Q\to\infty} dE\,[Y\,(Q)]/dQ < 0$ from Eq. (3) and $R > C > V$, $dE\,[Y\,(Q)]/dQ$ is a monotonically decreasing function in $0 \leq Q < +\infty$ and $E\,[Y\,(Q)]$ is a concave function of Q. Thus there is a unique optimal order quantity maximizing $E\,[Y\,(Q)]$. $\square$

It is clear that the optimization problem above never takes account of the manager's risk aversion, which is considered as the most general attitude toward the risk. Lua [8] and Ward *et al.* [13] considered the following expected utility function consisting of mean-standard deviation tradeoff;

$$EU\,(Y\,(Q)) = E\,[Y\,(Q)] - k\,\sqrt{\mathrm{Var}\,[Y\,(Q)]}, \tag{13}$$

where k is the parameter reflecting a manager's individual degree of risk aversion. This objective function is essentially the same as quadratic utility function. Pulley [11] proposed the mean-variance approximation to the expected utility by using a similar formula to Eq. (13). It is noted, however, that the quadratic utility displays increasing absolute risk aversion in the sence of Arrow-Pratt and that the marginal utility of the quadratic one is not always positive. Further, the manager's preference is not always represented only by the mean and variance of profit. These properties make us questionable to use the quadratic utility function.

In general, the utility functions called HARA (Hyperbolic Absolute Risk Aversion) class are used in the risk theory to illustrate the properties of risk aversion. The HARA class of utility functions satisfy

$$\frac{1}{r(y)} = a + by, \tag{14}$$

where a and b are the constant parameters presenting the risk sensitivity and where $r(y) = -U''(y)/U'(y)$ is called the absolute risk aversion measure. Atkinson [2] applied this kind of utility functions to the newsboy problem, which is different from our model.

As mentioned in Section 1, the purpose of this paper is to develop the optimal policy and to propose it for the risk averter facing the newsboy problem. Then, there are three problems which should be overcome as following:

1) Proof for the existence of optimal order quantity when the risk averse utility function is adopted.

2) Calculation procedure deriving the optimal policies and their associated expected utility functions.

3) Approximation procedure of the common risk averse utility function.

Especially, it is noted that the approximation procedure is needed in order to apply the plausible utility function to the objective one, since the profit variable in Eq. (2) can be negative.

Moreover, as an interesting and extended problem, we can consider the nearly optimal ordering policy so as to maximize the upper and lower bounds of expected utility. This alternative decision problem is discussed in Section 4. In the next section, we develop the general theory for the risk averse manager.

3. UTILITY THEORY IN THE NEWSBOY PROBLEM

Risk aversion

The manager who faces the newsboy problem has his or her own utility $U(\cdot)$, which is well-defined on $Y(Q)$, and wishes to find the optimal order quantity Q^* maximizing it. The manager is risk averse and the utility $U(\cdot)$ has the properties as follows;

(A-1) $U(\cdot)$ is twice continuously differentiable.
(A-2) $U(\cdot)$ is strictly increasing.
(A-3) $U(\cdot)$ is concave.

Then the manager's objective is to solve the following problem;

$$\left.\begin{array}{c} \max_{Q \geq 0} EU(Y(Q)), \\ \text{s.t.} \quad Y(Q) = (R - V + S) \min(Q, \tilde{D}) - S\tilde{D} - (C - V) Q, \\ R > C > V > 0. \end{array}\right\} \tag{15}$$

Since, from Eq. (2),

$$Y(Q) = \left\{\begin{array}{ll} (R - V + S) Q - S\tilde{D} - (C - V) Q; & (Q \leq \tilde{D}), \\ (R - V + S) \tilde{D} - S\tilde{D} - (C - V) Q; & (Q > \tilde{D}), \end{array}\right\} \tag{16}$$

we have

$$dY(Q)/dQ = \left\{\begin{array}{ll} R + S - C\,(> 0); & (Q \leq \tilde{D}), \\ V - C\,(< 0); & (Q > \tilde{D}), \end{array}\right. \tag{17}$$

and

$$d^2 Y(Q)/dQ^2 = 0. \tag{18}$$

Then we obtain the analytical results as follows;

LEMMA 2: *Suppose that* $R > C > V$, $U'(\cdot) > 0$, $U''(\cdot) < 0$ *and* $dU(Y(Q))/dQ$ *is finite for* $0 \leq Q < \infty$. *Then,*

(i)

$$dE[U(Y(Q))]/dQ = E\left[\frac{d}{dQ} U(Y(Q))\right]. \tag{19}$$

(ii) $U(Y(Q))$ *is a concave function of* Q, *provided* $U'''(\cdot) \geq 0$.

Proof: (i) From the definition of differentiation,

$$dE[U(Y(Q))]/dQ = \lim_{h \to 0} E\left[\frac{U(Y(Q+h)) - U(Y(Q))}{h}\right]. \tag{20}$$

Since $Y(Q)$ and $U(Y)$ are strictly monotonically increasing functions of Q and Y for a fixed $\tilde{D}$ ($\geq Q$), respectively, we have

$$\begin{aligned}\frac{U(Y(Q+h))-U(Y(Q))}{h} &= \frac{U(Y(Q+h))-U(Y(Q))}{Y(Q+h)+Y(Q)} \\ &\quad\times \frac{Y(Q+h)-Y(Q)}{h} \\ &= (R+S-C) \\ &\quad\times \frac{U(Y(Q+h))-U(Y(Q))}{Y(Q+h)+Y(Q)} \\ &> 0. \end{aligned} \tag{21}$$

From the concavity of $U(\cdot)$, $\{U(Y(Q+h))-U(Y(Q))\}/h$ is a monotonically decreasing function of h. Since $dU(Y(Q))/dQ$ is finite for $0 \leq Q < \infty$, Lebesgue's monotone convergence theorem gives

$$\begin{aligned} dEU(Y(Q))/dQ &= -\lim_{h\to 0} E\left[\frac{U(Y(Q))-U(Y(Q+h))}{h}\right] \\ &= -E\left[\lim_{h\to 0}\frac{U(Y(Q))-U(Y(Q+h))}{h}\right] \\ &= E\left[\frac{d}{dQ}U(Y(Q))\right]. \end{aligned} \tag{22}$$

The proof for the case of $Q > \tilde{D}$ is similar.

(ii) Using the result above gives

$$\frac{d^2}{dQ^2}EU(Y(Q)) = \frac{d}{dQ}E\left[\frac{d}{dQ}U(Y(Q))\right]. \tag{23}$$

In a similar fashion to (i), when $Q \leq \tilde{D}$, we have

$$\begin{aligned}\frac{d^2}{dQ^2}EU(Y(Q)) &= (R+S-C)\lim_{h\to 0} E \\ &\quad\times\left[\frac{U'(Y(Q+h))-U'(Y(Q))}{h}\right]. \end{aligned} \tag{24}$$

Since $U'(\cdot)$ is non-negative and a decreasing function, $\{U'(Y(Q+h)) - U'(Y(Q))\}/h$ is a monotonically decreasing function of h from $Y'(Q) > 0$ and $U'''(\cdot) \geq 0$. By the monotone convergence theorem,

$$\frac{d^2}{dQ^2} EU(Y(Q)) = (R+S-C)^2 E[U''(Y(Q))] < 0. \tag{25}$$

The case of $Q > \tilde{D}$ is also similar. Hence, $U(Y(Q))$ is a concave function of Q. □

PROPOSITION 3: *Under the condition of Lemma 2, there exists at least a finite optimal order quantity Q^*, which maximizes $E[U(Y(Q))]$.*

Proof: From Lemma 2, $dEU(Y(Q))/dQ$ is a decreasing function of Q. Then the first order condition of optimality is

$$\begin{aligned} dEU(Y(Q))/dQ &= E\left[\frac{d}{dQ} U(Y(Q)) \,\middle|\, Q \leq \tilde{D}\right] \\ &\quad + E\left[\frac{d}{dQ} U(Y(Q)) \,\middle|\, Q > \tilde{D}\right] \\ &= \int_Q^\infty (R+S-C)\, U'(Y(Q))\, g(D)\, dD \\ &\quad - \int_0^Q (C-V)\, U'(Y(Q))\, g(D)\, dD = 0. \end{aligned} \tag{26}$$

On the other hand, for $0 \leq Q < \infty$,

$$\lim_{Q \to 0} \frac{d}{dQ} EU(Y(Q)) = (R+S-C)\, EU'(Y(Q)) > 0, \tag{27}$$

$$\lim_{Q \to \infty} \frac{d}{dQ} EU(Y(Q)) = -(C-V)\, EU'(Y(Q)) < 0. \tag{28}$$

This completes the proof. □

The non-negativity of $U'''(\cdot)$ is a simple consequence of decreasing absolute risk aversion. In fact, we note that the exponential, the power and the logarithmic utilities are considered to be important in applications. We can also obtain the same result even if the demand follows the distribution in the class of truncated distribution with support $[0, D_{\max}]$, where $D_{\max}$ is an upper limit of demand.

Calculation of expected utility

Given a utility function $U(\cdot)$, the calculations of the expected utility and the corresponding optimal order quantity Q^* are numerically carried out. Following Lau [8], we have the distribution function for the random variable $Y(Q)$ as follows;

$$\Pr\{y \geq Y\} = \left\{\begin{array}{l} 1 - G(H_1(y)); \quad (y \leq -(C-V)Q), \\ 1 - G(H_1(y)) + G(H_2(y)); \\ \quad (-(C-V)Q < y \leq (R-C)Q), \\ 1; \quad ((R-C)Q < y), \end{array}\right\} \tag{29}$$

where,

$$H_1(y) = \frac{(R-C+S)Q - y}{S}, \tag{30}$$

$$H_2(y) = \frac{(C-V)Q + y}{R-V}. \tag{31}$$

Its probability density function is

$$\left.\begin{array}{l} f(y) = d\Pr\{y \geq Y\}/dy \\ = \left\{\begin{array}{l} g(H_1(y))/S; \quad (y \leq -(C-V)Q), \\ g(H_1(y))/S + g(H_2(y))/(R-V); \\ \quad (-(C-V)Q < y \leq (R-C)Q), \\ 0; \quad ((R-C)Q < y). \end{array}\right. \end{array}\right\} \tag{32}$$

Figure 1 shows the behavior of $f(y)$ for fixed order quantities, when the demand follows the truncated normal distribution;

$$g(D) = \frac{1}{\Lambda} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(D-\mu)^2}{2\sigma^2}\right), \tag{33}$$

where

$$\Lambda = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(D-\mu)^2}{2\sigma^2}\right) dD, \tag{34}$$

and where μ and σ are the constant parameters, respectively. It it obvious that the distribution of $Y(Q)$ is truncated at the point of $(R-C)Q$ and is unsymmetrical. Eq. (32) clearly satisfies $\int_{-\infty}^{(R-C)Q} f(y)\,dy = 1$.

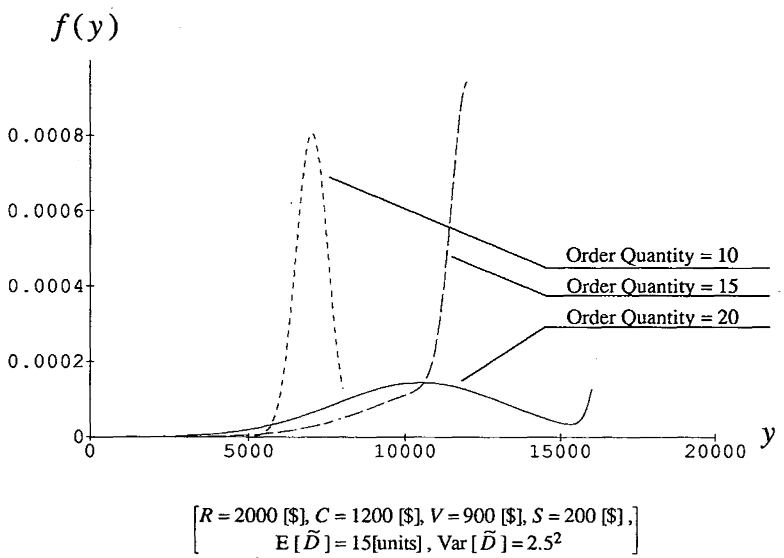


Figure 1. – Behavior of $f(y)$ for Fixed Order Quantities (Truncated Normal Distribution).

Thus the problem defined in Eq. (15) is presented by

$$\left.\begin{array}{c} \max\limits_{Q \geq 0} \displaystyle\int_{-\infty}^{(R-C)\,Q} U(y)\, f(y)\, dy, \\ \text{s.t.} \quad Y(Q) = (R - V + S) \min(Q, \tilde{D}) - S\tilde{D} - (C - V)\, Q, \\ R > C > V > 0. \end{array}\right\} \tag{35}$$

If $U(\cdot)$ is well-defined on $Y(Q)$, we can obtain the maximum expected utility and the optimal order quantity by carrying out the numerical integration.

Approximation procedure of logarithmic utility

In the previous discussion we assumed that the utility function $U(\cdot)$ is well-defined over the range of $Y(Q)$. However, since $Y(Q) \in (-\infty, (R-C)\,Q]$ from $\tilde{D} \in [0, \infty)$ and Eq. (16), the most common utility functions are undefined for $Y(Q) \leq 0$. Thus, it is necessary to append a concave segment to $U(\cdot)$ in the range of $(-\infty, \omega]$ for a *approximation point* ω (> 0). Especially, we focus on the logarithmic utility function whose absolute risk aversion measure is monotonically decreasing. Ziemba [14] defined the approximated logarithmic utility in the context of portfolio selection problem by using the following linear segment;

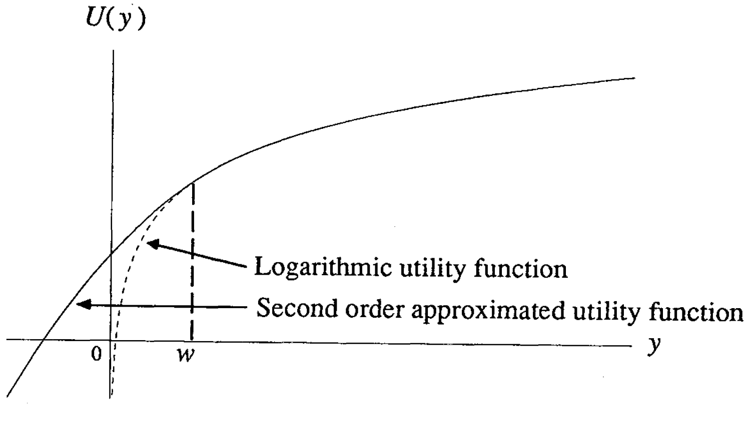


Figure 2. – **Schematic Illustration of Approximation.**

$$U(y) = \left\{ \begin{array}{ll} \log y; & (y \geq \omega), \\ \dfrac{y}{\omega} + \log \omega - 1; & (y < \omega). \end{array} \right\} \quad (36)$$

On the other hand, we consider the second order Taylor approximation to guarantee the concavity of the utility function as follows;

$$U(y) = \left\{ \begin{array}{ll} \log y; & (y \geq \omega), \\ \dfrac{-y^2}{2\,\omega^2} + \dfrac{2\,y}{\omega} + \log \omega - \dfrac{3}{2}; & (y < \omega). \end{array} \right\} \quad (37)$$

Figure 2 shows the schematic illustration to represent the concept of the second order approximation procedure given in Eq. (37). The approximated functions given by Eqs. (36) and (37) are continuous and monotonically increasing with respect to the profit. Further, note that the value of expected utility depends on the approximation point ω. This fact tells us that the choice of approximation point presents the manager's preferences to the risk in some degree.

Numerical example

Tables 1 and 2 present the dependence of the approximation point and the model parameters of demand for the optimal order quantity, respectively. Moreover, we compare the risk averse optimal order quantity Q_a^* with the risk neutral one Q_n^*. Data set is as follows: $R = 2\,000$ [\$], $C = 1\,200$ [\$], $V = 900$ [\$] and $S = 200$ [\$]. The demand is supposed to follow the truncated normal distribution with $\mu = 15$ and $\sigma^2 = 2.5^2$.

TABLE 1
Optimal Order Quantity for Various Approximation Points

ω	Risk Neutral Q_n^*	Risk Averse Q_a^*	
		First Order	Second Order
0.001		10.00	5.66
0.010		13.10	5.70
0.100	16.80	15.30	5.80
1.000		16.20	10.90
10.000		16.40	15.60

In Table 1, we find that the risk neutral optimal order quantity Q_n^* is 16.80 [units], which is always larger than Q_a^*, and that the optimal order quantity Q_a^* becomes small as the approximation point approaches to zero (as the property of the approximated utility approaches to one of the logarithmic utility). Also the second order approximation makes Q_a^* smaller than one in the first order quantity. This means that the first order approximation overestimates the order quantity comparing with one by the second order approximation. In other words, the manager using the second order approximation is more risk averse than using the first order one. Hence the manager should apply the second order approximation method with a relatively small approximation point in order to evaluate the logarithmic utility function accurately.

In Table 2, we find that Q_a^* is always smaller than Q_n^* for the set of (μ, σ). Especially, note that Q_a^* and Q_n^* become small and large, respectively, as σ becomes large. This result means that the risk averse manager should not

TABLE 2
Optimal Order Quantity for Model Parameters

Model Parameter		Risk Neutral Q_n^*	Risk Averse Q_a^*	
μ	σ		First Order	Second Order
10.0	2.0	11.5	10.0	4.1
10.0	3.0	12.2	4.6	3.9
15.0	2.0	16.5	16.2	15.9
15.0	3.0	17.2	14.5	5.8
20.0	2.0	21.5	21.3	21.3
20.0	4.0	22.9	18.9	7.6

order too much quantities under uncertainty. On the other hand, the risk neutral manager makes the order quantity increase as μ and/or σ increase. Thus, if he or she does not prefer to high-risk-high-return, we conclude that the ordinary risk neutral criterion in the newsboy problem is very questionable.

4. ALTERNATIVE OPTIMIZATION OBJECTIVE

Upper and lower bounds

In practical situations, it is often useful for the manager to know some candidates for the optimal order quantity rather than a unique order quantity. In order to carry out such a flexible decision making, it seems to be important to provide the range of the order quantities which the manager should choice. In this section, we consider the order quantities maximizing the upper and lower bounds of expected utility function, and derive the interval consisting of them. We apply Ben-Tal and Hochman's bounds [3, 4] as upper and lower bounds. The upper and lower bounds are well known to be more tight than Jensen's and Madansky's bounds, respectively (*see* [3]).

Let us define the mean, the mean absolute deviation and the probability as follows;

$$\mu(Q) = E[Y(Q)], \tag{38}$$

$$\alpha(Q) = E|Y(Q) - \mu(Q)|, \tag{39}$$

$$\beta(Q) = \Pr\{Y(Q) \geq \mu(Q)\}. \tag{40}$$

Then, for a concave utility function $U(\cdot)$ and the class of distributions of $Y(Q)$ with support $[a, b]$, we have

$$B(Q) \leq EU(Y(Q)) \leq A(Q), \tag{41}$$

where

$$\begin{aligned} A(Q) = \beta(Q)\, U\left(\mu(Q) + \frac{\alpha(Q)}{2\beta(Q)}\right) \\ + (1-\beta(Q))\, U\left(\mu(Q) - \frac{\alpha(Q)}{2(1-\beta(Q))}\right), \end{aligned} \tag{42}$$

$$B(Q) = \frac{\alpha(Q)}{2(\mu(Q) - a)} U(a) + \frac{\alpha(Q)}{2(b - \mu(Q))} U(b) + \left(1 - \frac{\alpha(Q)}{2} \frac{b - a}{(b - \mu(Q))(\mu(Q) - a)}\right) U(\mu(Q)). \quad (43)$$

Note that the upper bound above is independent of the support $[a, b]$. On the other hand, if a and/or b are infinite, the finite lower bound for an arbitrary distribution of $Y(Q)$ can be obtained under the additional restriction on $U(\cdot)$ as follows;

(B-1) Existence and finiteness of $\lim_{y \to \infty} U(y)/y$ for $y \in [a, \infty]$.

(B-2) Existence and finiteness of $\lim_{y \to -\infty} U(y)/y$ for $y \in [-\infty, b]$.

(B-3) Existence and finiteness of $\lim_{y \to \pm\infty} U(y)/y$ for $y \in [-\infty, \infty]$.

Since we assumed $\tilde{D} \in [0, \infty)$ in the previous section, the utility function must satisfy the condition (B-2) to guarantee the finite lower bound for $Y(Q) \in (-\infty, (R - C)Q]$. For the approximated logarithmic utility in Eq. (37), its lower bound is not finite (although it is finite for the first order approximated logarithmic utility). The exponential utility $U(y) = -a \exp(-y/a)$ ($a \geq 0$; constant) does not also satisfy (B-2). Thus we have to make a decision by using the only upper bound of expected utility, if the three conditions above are not satisfied.

As a special case, let us consider the case of $\tilde{D} \in [0, D_{\max}]$, where $D_{\max}$ is a constant. In a similar fashion to Eq. (32), we have the corresponding probability density function of $Y(Q)$ to $\tilde{D} \in [0, D_{\max}]$ as follows;

$$\left. \begin{aligned} f(y) &= d\Pr\{y \geq Y\}/dy \\ &= \begin{cases} g(H_2(y))/(R - V); & (\text{Case } 1), \\ g(H_1(y))/S + g(H_2(y))/(R - V); & (\text{Case } 2), \\ g(H_1(y))/S; & (\text{Case } 3), \\ 0; & (\text{otherwise}), \end{cases} \end{aligned} \right\} \quad (44)$$

where

(Case 1); $0 \leq D_{\max} \leq Q$ or $-(C - V)Q \leq y < (R - C + S)Q - SD_{\max}$ (when $Q < D_{\max} < (R - V + S)Q/S$).

(Case 2); $(R - C + S)Q - SD_{\max} \leq y \leq (R - C)Q$ (when $Q < D_{\max} < (R - V + S)Q/S$) and $-(C - V)Q < y \leq (R - C)Q$ (when $(R - C + S)Q/S \leq D_{\max}$).

(Case 3); $(R - C + S)\, Q - SD_{\max} \leq y \leq -(C - V)\, Q$ (when $(R - C + S)\, Q/S \leq D_{\max}$).

Thus, specifying the density function $g(\cdot)$, we can obtain the optimal order quantity for the direct expected utility maximization from Eqs. (35) and (44) by carrying out the numerical integration. It is, however, remarked that there are some problems on the overflow of computer and on the long time required in the calculation to solve Eq. (35). These facts motivates to introduce the upper and lower bound criteria in the risk averse newsboy problem.

Schmeiser and Deutsch distribution

A versatile distribution family so-called *S-D distribution* (Schmeiser-Deutsch distribution) is well-known to be useful in some practical inventory calculations. Schmeiser and Deutsch [12] developed a probability distribution with the p. d. f. and c. d. f. as follows;

$$\left.\begin{array}{c} g(D) = \left|\dfrac{\lambda_1 - D}{\lambda_2}\right|^{(1-\lambda_3)/\lambda_3} / \lambda_2\, \lambda_3, \\ (\lambda_1 - \lambda_2\, \lambda_4^{\lambda_3} \leq D \leq \lambda_1 + \lambda_2\, (1 - \lambda_4)^{\lambda_3}), \end{array}\right\} \tag{45}$$

$$G(D) = \left\{\begin{array}{c} \lambda_4 - \left(\dfrac{\lambda_1 - D}{\lambda_2}\right)^{1/\lambda_3}; \\ (\lambda_1 - \lambda_2\, \lambda_4^{\lambda_3} \leq D \leq \lambda_1), \\ \lambda_4 - \left(\dfrac{D - \lambda_1}{\lambda_2}\right)^{1/\lambda_3}; \\ (\lambda_1 < D \leq \lambda_1 + \lambda_2\, (1 - \lambda_4)^{\lambda_3}), \end{array}\right\} \tag{46}$$

where λ_1 and λ_4 are the mode and the percentile of the S-D distribution, and λ_2 and λ_3 are the shape parameters, respectively (*see* [7, 12]).

If the demand follows the S-D distribution and $\tilde{D} \in [0, D_{\max}]$ the relationship of $\lambda_1 - \lambda_2\, \lambda_4^{\lambda_3} = 0$ and $\lambda_1 + \lambda_2\, (1 - \lambda_4)^{\lambda_3} = D_{\max}$ are satisfied (*see* [12]). Then we have the corresponding mean and the partial moments as follows;

$$E[\tilde{D}] = \lambda_1 + \frac{\lambda_2 \{-\lambda_4^{\lambda_3+1} + (1 - \lambda_4)^{\lambda_3+1}\}}{\lambda_3 + 1}, \tag{47}$$

$$E_Q[\tilde{D}] = \left\{ \begin{array}{l} \dfrac{\lambda_2}{\lambda_3+1}\left\{\lambda_3\,\lambda_4^{\lambda_3+1} - \dfrac{Q+\lambda_1\,\lambda_3}{\lambda_2}\left(\dfrac{\lambda_1-Q}{\lambda_2}\right)^{1/\lambda_3}\right\}; \\ \qquad\qquad (Q \leq \lambda_1); \\ \dfrac{\lambda_2}{\lambda_3+1}\left\{\lambda_3\,\lambda_4^{\lambda_3+1} + \dfrac{Q+\lambda_1\,\lambda_3}{\lambda_2}\left(\dfrac{Q-\lambda_1}{\lambda_2}\right)^{1/\lambda_3}\right\}; \\ \qquad\qquad (\lambda_1 < Q). \end{array} \right\} \tag{48}$$

Thus we get $\mu(Q)$, $\alpha(Q)$ and $\beta(Q)$ as follows.

(1) Mean $\mu(Q)$:

$$\begin{aligned} \mu(Q) = (R-V+S)\,\{E_Q[\tilde{D}] + Q\,(1-G(Q))\} \\ - SE[\tilde{D}] - (C-V)\,Q, \end{aligned} \tag{49}$$

where $E_Q[\tilde{D}]$ and $E[\tilde{D}]$ are given by Eqs. (47) and (48).

(2) Mean absolute deviation $\alpha(Q)$:

(2-1) Case of $0 \leq D_{\max} \leq Q$;

$$\begin{aligned} \alpha(Q) = (R-V)\,\{\xi(Q)\,K(0,\,\xi(Q)) - J(0,\,\xi(Q)) \\ + J(\xi(Q),\,D_{\max}) - \xi(Q)\,K(\xi(Q),\,D_{\max})\}, \end{aligned} \tag{50}$$

where

$$\xi(Q) = \frac{(C-V)\,Q + \mu(Q)}{R-V}, \tag{51}$$

$$J(i,\,j) = \int_i^j Dg(D)\,dD, \tag{52}$$

$$K(i,\,j) = \int_i^j g(D)\,dD, \tag{53}$$

respectively.

(2-2) Case of $Q < D_{\max} \leq (R-V+S)\,Q/S$;

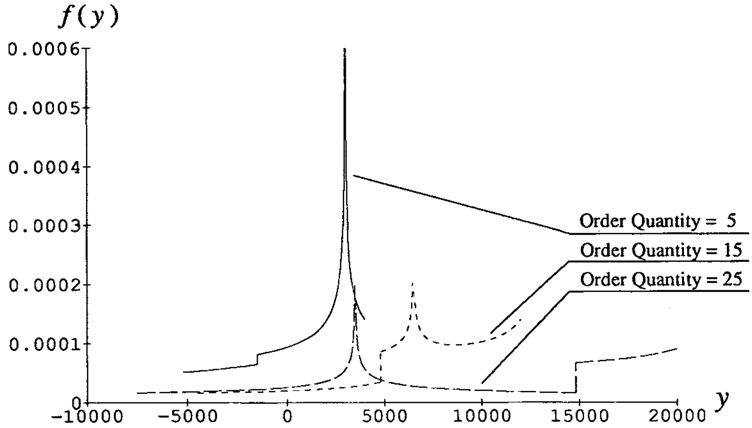
$$\alpha(Q) = \left\{ \begin{array}{l} (R-V)\{\xi(Q) K(0, \xi(Q)) - J(0, \xi(Q)) \\ \quad + J(\xi(Q), Q) - \xi(Q) K(\xi(Q), Q)\} \\ \quad + S\{\phi(Q) K(Q, D_{\max}) - J(Q, D_{\max})\}; \\ \qquad (-(C-V) Q \leq \mu(Q) \\ \qquad\quad < (R-C+S) Q - S D_{\max}), \\ (R-V)\{\xi(Q) K(0, \xi(Q)) - J(0, \xi(Q)) \\ \quad + J(\xi(Q), Q) - \xi(Q) K(\xi(Q), Q)\} \\ \quad + S\{\phi(Q) K(Q, \phi(Q)) - J(Q, \phi(Q)) \\ \quad + J(\phi(Q), D_{\max}) - \phi(Q) K(\phi(Q), D_{\max})\}; \\ ((R-C+S) Q - S D_{\max} \leq \mu(Q) \leq (R-C) Q), \end{array} \right\} \tag{54}$$

where

$$\phi(Q) = \frac{(R-C+S) Q - \mu(Q)}{S}. \tag{55}$$

(2-3) Case of $(R-V+S) Q/S < D_{\max}$;

$$\alpha(Q) = \left\{ \begin{array}{l} (R-V)\{J(0, Q) - \xi(Q) K(0, Q)\} \\ \quad + S\{\phi(Q) K(Q, \phi(Q)) - J(Q, \phi(Q)) \\ \quad + J(\phi(Q), D_{\max}) - \phi(Q) K(\phi(Q), D_{\max})\}; \\ \qquad ((R-C+S) Q - S \cdot D_{\max} \\ \qquad\quad \leq \mu(Q) \leq -(C-V) Q), \\ (R-V)\{\xi(Q) K(0, \xi(Q)) - J(0, \xi(Q)) \\ \quad + J(\xi(Q), Q) - \xi(Q) K(\xi(Q), Q)\} \\ \quad + S\{\phi(Q) K(Q, D_{\max}) - J(Q, D_{\max}) \\ \quad + J(\phi(Q), D_{\max}) - \phi(Q) K(\phi(Q), D_{\max})\}; \\ \qquad (-(C-V) Q < \mu(Q) \leq (R-C) Q). \end{array} \right\} \tag{56}$$



$\left[\begin{array}{l} R=2000\ [\$], C=1200\ [\$], V=900[\$], S=200\ [\$], \\ \lambda_1=10.00, \lambda_2=74.1788, \lambda_3=1.6644, \lambda_4=0.30 \end{array} \right]$

Figure 3. – Behavior of $f(y)$ for Fixed Order Quantity (S-D Distribution).

(3) Probability $\beta(Q)$:

(3-1) Case of $0 \leq D_{\max} \leq Q$;

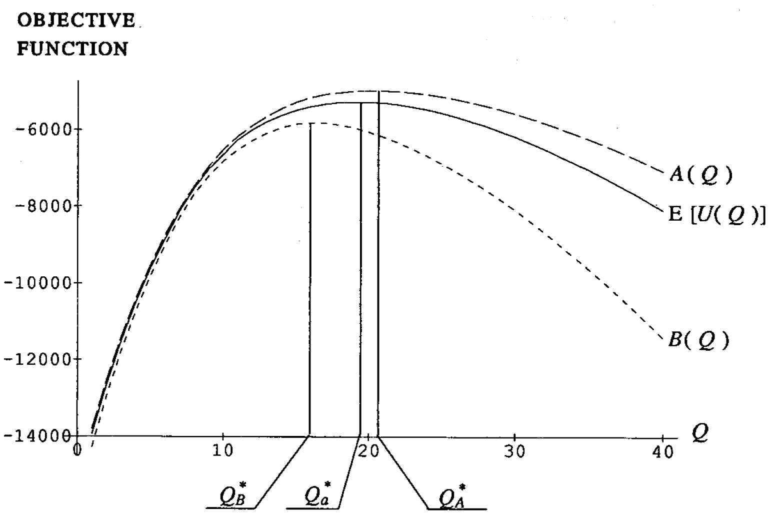
$$\beta(Q) = 1 - G(H_2(\mu(Q))). \tag{57}$$

(3-2) Case of $Q < D_{\max} \leq (R - V + S)\, Q/S$;

$$\beta(Q) = \left\{ \begin{array}{c} 1 - G(H_2(\mu(Q))); \\ (-(C - V)\, Q \leq \mu(Q) \\ < (R - C + S)\, Q - SD_{\max}), \\ G(H_1(\mu(Q))) - G(H_2(\mu(Q))); \\ ((R - C + S)\, Q - SD_{\max} \leq \mu(Q) \leq (R - C)\, Q). \end{array} \right\} \tag{58}$$

(3-3) Case of $(R - V + S)\, Q/S < D_{\max}$;

$$\beta(Q) = \left\{ \begin{array}{c} G(H_1(\mu(Q))); \\ ((R - C + S)\, Q - SD_{\max} \\ \leq \mu(Q) \leq -(C - V)\, Q), \\ G(H_1(\mu(Q))) - G(H_2(\mu(Q))); \\ (-(C - V)\, Q < \mu(Q) \leq (R - C)\, Q). \end{array} \right\} \tag{59}$$



$\left[\begin{array}{c} R = 2000\ [\$],\ C = 1200\ [\$],\ V = 900[\$],\ S = 200\ [\$],\ a = 10000, \\ \lambda_1 = 10.00,\ \lambda_2 = 74.1788,\ \lambda_3 = 1.6644,\ \lambda_4 = 0.30 \end{array}\right]$

Figure 4. – **Behavior of** $A\ (Q)$, $B\ (Q)$ **and** $EU\ (Y(Q))$.

Optimal interval

From Eqs. (49)-(59), we can analytically obtain the upper and lower bounds of expected utility function in Eqs. (42) and (43). Hence, the problem is to get the following order quantities;

$$Q_A^* = \{Q | \max_{Q \geq 0} A\,(Q)\}, \tag{60}$$

$$Q_B^* = \{Q | \max_{Q \geq 0} B\,(Q)\}. \tag{61}$$

Unfortunately, it is difficult to analytically examine the concavity of $A\,(Q)$ and $B\,(Q)$ for $Q \geq 0$. Figure 4 numerically illustrates the concavity of the bounds and presents that the interval $[Q_B^*,\ Q_A^*]$ includes the risk averse optimal order quantity Q_a^*, where the parameters are same as the case of Figure 3 and the exponential utility function with the risk sensitivity parameter a is assumed. We simply call $[Q_B^*,\ Q_A^*]$ *optimal interval.*

By applying the result of Ben-Tal and Hochman [4] directly, we have the following useful result on the optimal interval without the proof.

PROPOSITION 4: *If a utility function* $U\ (Q,\ \tilde{D})$ *satisfies*

(i) $U\ (Q,\ \cdot)$ *is concave for* $Q \geq 0$,

(ii) $U(\cdot, \tilde{D})$ *is strictly concave for* $\tilde{D} \in [0, D_{\max}]$,

(iii) $\partial U(Y(Q_a^*))/\partial Q_a^*$ *is concave,*

then

$$Q_B^* \leq Q_a^* \leq Q_A^*. \tag{62}$$

In particular, it is difficult to check whether the condition (iii) is always saisfied. Thus, we show that the inequalities of Eq. (62) are satisfied for various parameters in the next discussion.

Numerical example

It is interesting to examine the sensitivity of the upper and lower bounds for the risk sensitive parameter a, when the exponential utility function and the numerical data in Figure 4 are assumed. Table 3 shows the dependence of a in the optimal order quantities and their associated evaluation functions. It is clear that Q_a^* is always included in the optimal interval. Also, as a increases, Q_B^*, Q_a^* and Q_A^* are nondecreasing and the optimal interval becomes more tight. This result tells us that the number of feasible order quantities including $[Q_B^*, Q_A^*]$ increases as the manager becomes more risk averse. Table 4 presents the dependence of the approximation point in the optimal order quantities when the logarithmic utility function is adopted. The data set is same as the case above. Here the second order approximation is used. We have that the most optimal order quantities are insensitive to ω. In other words, the arbitrary parameter determined by the manager is almost independent of the optimal order quantity and the optimal interval.

TABLE 3

Optimal Order Quantity's Interval (Exponential Utility)

a	Risk Neutral	Lower Bound $B(Q)$		Risk Averse $EU(Y(Q))$		Upper Bound $A(Q)$	
	Q_n^*	Q_B^*	$B_{\max}$	Q_a^*	$EU_{\max}$	Q_A^*	$A_{\max}$
50		7.8	-2.85×10^{21}	7.9	-2.49×10^{19}	15.2	-1.23×10^{-30}
100		7.9	-3.97×10^{11}	8.0	-6.10×10^{9}	15.2	-1.05×10^{-14}
1000		7.9	−2184.35	9.7	−402.86	15.2	−12.33
10000	31.1	16.3	−5795.79	19.2	−5255.56	20.3	−4966.57
50000		25.6	−42545.50	26.1	−42157.30	27.7	−42067.70
100000		28.1	−91677.40	28.4	−91373.30	29.3	−91358.90

TABLE 4

Optimal Order Quantity's Interval (Logarithmic Utility)

ω	Risk Neutral	Lower Bound $B(Q)$		Risk Averse $EU(Y(Q))$		Upper Bound $A(Q)$	
	Q_n^*	Q_B^*	$B_{\max}$	Q_a^*	$EU_{\max}$	Q_A^*	$A_{\max}$
0.01		7.9	-5.68×10^9	9.3	-6.71×10^8	17.2	8.77
0.10		7.9	-3.97×10^{11}	9.3	-6.71×10^6	17.2	8.77
1.00		7.9	−2184.35	9.3	-6.73×10^4	17.2	8.77
10.00	31.1	7.9	−2413.12	9.3	−680.93	17.2	8.77
100.00		7.9	−5795.79	9.4	−0.56	17.2	8.77
1000.00		11.3	−42545.50	11.3	8.08	17.2	8.77

Thus, the numerical examples above illustrate that the inequalities of Eq. (62) are satisfied in some situations and that the optimal interval consists of the range of rather practical order quantities. This result shows that the alternative criterion proposed here is of practical use. In fact, the calculations to get Q_B^* and Q_A^* are rather easier than one to do Q_a^* and the problem on the overflow of computer did not happen in our calculations.

5. CONCLUSION

In this paper we examined the analytical properties of the expected utility function for the risk averse manager facing the single-period newsboy problem. In addition, we considered the approximation procedure of the expected utility function in this problem, and alternatively proposed the upper and lower bound criteria. The mathematical results which we developed in this paper are rather straightforward but are systematically arranged. These are valid from the financial theoretical standpoint as well as useful for the practioner in evaluating the inventory risk. In numerical examples, the sensitivity analyses were carried out and some interesting and significant insight for the newsboy problem were given.

In future, the multi-item newsboy problem should be considered in the context of this paper. Then, the result will never be simple as one for the single-item. Also, the problems having the more complex model structure such that Lau and his co-authors [9, 10] discussed is desired to analyze.

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