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TWO-DIMENSIONAL COMBINATION WARRANTY POLICIES (*)

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Abstract. – *A combination warranty policy is an extension of two simple warranty policies – the free replacement warranty (FRW) and the pro-rata warranty (PRW) – in the sense that it either contains additional feactures or combines the terms of two simple policies. The paper deals with the cost analysis of five different two dimensional combination policies.*

Key-words: Warranty policy, cost analysis, replacement.

Résumé. – *Une politique de garantie combinée est une extension de deux politiques simples de garantie – la politique de renouvellement libre (FRW) et la politique de garantie au prorata (PRW) – en ce sens que, ou bien elle contient des caractéristiques supplémentaires, ou bien elle combine les termes de deux politiques simples. L'article traite de l'analyse des coûts de cinq politiques combinées à deux dimensions.*

Mots-clés : Politique de garantie, analyse des coûts, renouvellement.

1. INTRODUCTION

A warranty is a contractual obligation incurred by a manufacturer in connection with the sale of a product. A variety of warranty policies have been proposed and studied by researchers over the last two decades. (*See* Blischke and Murthy [1992] for a taxonomy of these.) Most one dimensional policies are based on either time or age of items but could be based only on usage. Many consumer durables (*e. g.*, televisions, electrical appliances) are sold with one dimensional warranty policies. In contrast, a typical two dimensional policy is based on both time and usage. An example is an automobile warranty policy where the components are covered till a specified age and usage.

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Various aspects of warranty have been studied by researchers from different disciplines. Murthy and Blischke [1992 *a*] gives a unified framework which integrates this literature. Of particular interest to this paper is the manufacturer's viewpoint. Offering warranty results in additional cost due to the servicing of the warranty. This in turn affects the profits. The two simple policies are the free replacement warranty (FRW) and the pro-rata warranty (PRW) policies. The former tends to favour the consumer whilst the latter tends to favour the manufacturer. A combination warranty combines the terms of two or more simple policies and tends to achieve a sensible balance between the consumer's interests and the manufacturer's interests. For the one dimensional case, a variety of mathematical models have been developed to obtain the expected cost of servicing warranty for the different types of simple and combination policies. A review of these models can be found in Murthy and Blischke [1992 *b*].

For the two dimensional case, mathematical models have been developed for cost analysis of free replacement (*see*, Moskowitz and Chun [1988], Singpurwalla [1987], Murthy and Wilson [1991], Murthy, Iskandar and Wilson [1990]) and pro-rata (or rebate) policies (*see*, Iskandar *et al.* [1991]). In contrast, there has been no study of two dimensional combination policies. In this paper we formulate a variety of renewing and non-renewing two dimensional combination policies and carry out the cost analysis. In a renewing policy, when ever an item fails under warranty, it is replaced by a new item with a new warranty replacing the old one. In constrast, in the case of a non-renewing policy, replacement of a failed item does not alter the original warranty. The outline of the paper is a follows. In Section 2 we give the details of the different two dimensional combination policies that are studied in this paper. Section 3 deals with the model formulation for the cost analysis. The analysis of non-renewing combination policies is carried out in Section 4. In Section 5, we consider the cost analysis of the renewing combination policies. Section 6 deals with numerical results for an example and we conclude with a brief discussion of some extensions in Section 7.

2. TWO-DIMENSIONAL COMBINATION POLICIES

A two dimensional warranty is characterized by a region in a two dimensional plane with one axis representing time or age and the other representing item usage. As a result, we have different types of warranty policies based on the shape of the warranty coverage region. We shall confine our attention to policies where the warranty region is a rectangle Ω given by

$$\Omega = [0,\ K) \times [0,\ L).$$

We assume that the product is non-repairable so that all failed items need to be replaced by new ones. We first define the free replacement and pro-rata policies before proceeding to stating the different combination policies that are analysed in this paper.

2.1. Free Replacement Warranty Policies

Non-renewing Policy :

Under this policy the manufacturer agrees to provide a replacement for failed items free of charge up to a time K or up to a usage L, which ever occurs first, from the time of the intitial purchase. K is called the warranty period and L the usage limit. The original warranty terms are not affected with replacement. Thus, should a failure occur at time t ($<K$) with total usage x ($<L$), then the replaced item has a warranty for a time period $(K-t)$ and for usage $(L-x)$.

Renewing Policy:

Under this policy the manufacturer agrees to provide a replacement for failed items free of charge should the item fail before age K and its usage at failure is less than L. Each replacement comes with a new warranty identical to the initial warranty.

2.2. Pro-Rata Policies

Non-renewing Policy:

Under this policy the manufacturer agrees to refund a fraction of the original purchase price should the item fail before time K from the time of the initial purchase and if the total usage at failure is below L and the buyer is not constrained to buy a replacement item. If the age at failure is given by t and the usage at failure by x, then the amount refunded is a function of $(K-t)$ and $(L-x)$ as well as the sale price S. We denote this function by $\phi(t, x)$. $\phi(t, x)$ is non-negative over the warranty region and zero elsewhere. Different forms for $\phi(t, x)$ define a family of warranty policies.

Renewing Policy:

Under this policy the manufacturer agrees to provide, at a pro-rated cost, a replacement item should the item fail before age K and its usage at failure is less than L. If the age at failure is given by t and the usage at failure by x, then the cost to the buyer of a replacement is $[S-\phi(t, x)]$, if $(t, x) \in \Omega$. The replacement item comes with a new warranty.

2.3. Combined Free and Pro-rata Policies

In this section we define five different combination policies which combine the terms of the simple free replacement warranty and the pro-rata warranty policies. The analysis is carried out in the subsequent sections.

Define Ω_1 as a rectangle contained in Ω and given by (see *fig.* 1)

$$\Omega_1 = [0, K_1) \times [0, L_1)$$

with $K_1 \leq K$ and $L_1 \leq L$, and Ω_2 be the region with

$$\Omega_2 \cup \Omega_1 = \Omega \qquad \text{and} \qquad \Omega_1 \cap \Omega_2 = \emptyset.$$

At failure, let t_0 and x_0 denote the total time and total usage respectively since the initial purchase and let t and x denote the age and usage respectively of the item to fail. [For first failure, $t_0 = t$ and $x_0 = x$.]

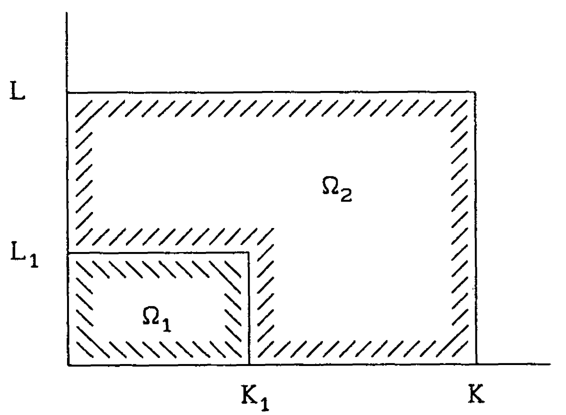

Figure 1. – Warranty Regions: Rectangular.

Policy 1 [Non-renewing FRW and PRW Combination]:

Under this policy the manufacturer agrees to provide a free replacement for any failure with $(t_0, x_0) \in \Omega_1$ and to refund an amount $\phi(t_0, x_0)$ for any failures with $(t_0, x_0) \in \Omega_2$. Replacement items come with a reduced warranty region as the original warranty is unaffected.

Policy 2 [Non-renewing FRW Combination]:

Under this policy, the first failure under warranty results in a refund and the warranty ceases. As a result, $t_0 = t$ and $x_0 = x$. The manufacturer agrees to refund the sale price for any failure if $(t_0, x_0) \in \Omega_1$, and to refund an amount $\phi(t_0, x_0)$ if $(t_0, x_0) \in \Omega_2$.

Comments

(i) Policy 2 can be viewed as money back guarantee policy for item failures in Ω_1.

(ii) For both Policies 1 and 2, when ever an item fails in Ω_2, the refund is unconditional for failures in Ω_2, as there is no requirement for the consumer to buy a replacement item.

Policy 3 [Renewing FRW and Renewing PRW Combination]:

Under this policy, the manufacturer agrees to provide a free replacement with a new warranty for any failures with $(t, x) \in \Omega_1$ and to provide a replacement with a new warranty at pro-rated cost $[S - \phi(t, x)]$ for any failure with $(t, x) \in \Omega_2$.

Policy 4 [Renewing FRW and Non-renewing PRW Combination]:

Under this policy the manufacturer agrees to provide a free replacement with a new warranty for any failure with $(t, x) \in \Omega_1$ and to refund an amount $\phi(t, x)$ for any failure with $(t, x) \in \Omega_2$.

Policy 5 [Non-renewing PRW and Renewing PRW Combination]:

Under this policy the manufacturer agrees to refund the sale price for any failure with $(t, x) \in \Omega_1$ and to provide a replacement with a new warranty at a pro-rated cost $[S - \phi(t, x)]$ for any failure with $(t, x) \in \Omega_2$.

Comments:

(iii) Policy 5 can be viewed as money back guarantee policy for item failure in Ω_1.

(iv) For both Policies 3 and 5, whenever an item fails in Ω_2, the refund is conditional – *i.e.*, it is tied to a repeat purchase.

(v) If $K_1 = K$ and $L_1 = L$ or if $K_1 = L_1 = 0$ the policies reduce to appropriate simple policies defined earlier.

3. MODEL FORMULATION

In this section we discuss the modelling of item failures and the expected warranty servicing cost per item to the manufacturer.

3.1. Modelling Item Failures

Items fail randomly and failures can be modelled using two different approaches. In the first approach, the failure of items is modelled using a one dimensional point process. This approach has been used by Singpurwalla (1987), Moskowitz and Chun (1988) and Murthy and Wilson (1991). In the second approach, the failures are modelled using a two dimensional point process. Murthy, Iskandar and Wilson (1991) use this approach for the analysis of free replacement policies and Iskandar, Wilson and Murthy (1991) use it for the analysis of pro-rata (or rebate) policies.

In this paper, we follow the second approach and item failures are characterized through a bivariate distribution function $F(t, x)$. Let T_i and X_i denote the age and usage of item i at failure. Since all replacement items are identical to the original item sold, T_i and X_i, $i \geq 1$ is a sequence of independent and identically distributed bivariate random variables with joint distribution function $F(t, x)$. The form of $F(t, x)$ must be such that the conditional expectation $E[X_i | T_i = t]$ is increasing in t. Two possible forms of $F(t, x)$ are as follows:

(i) *Beta-Stacy Distribution*

The density function $f(t, x)$ for (T_i, X_i), $i \geq 1$, is given by

$$f(t, x) = \frac{1}{\Gamma(\alpha) B(\theta_1, \theta_2)} \times \frac{c}{a^{\alpha c} \phi} t^{(\alpha c - \theta_1 - \theta_2)} (x/\phi)^{(\theta_1 - 1)} (t - x/\phi)^{\theta_2 - 1} e^{[-(t/a)^c]}$$

where $t > 0$; $0 < x < \phi t$; and α, c, a, ϕ, θ_1, $\theta_2 > 0$. The above is a slightly modified version of the Beta-Stacy distribution proposed by Mihran and Hultquist. Expressions for the first and second moments are given in Johnson and Kotz [1972] and of particular interest is the conditional expectation, $E[X_i|T_i = t]$, given by

$$E[X_i|T_i = t] = \{\phi\,\theta_1/(\theta_1 + \theta_2)\}\, t.$$

This is a linear function in *t* implying that the usage increases with age in a linear manner in an expected sense.

(ii) *Multivariate Pareto Distribution of the Second Kind* (Type 2)

The density function $f(t, x)$ for (T_i, X_i), $i \geq 1$, is given by

$$f(t, x) = \frac{a_1 a_2}{tx(1-\rho^2)}\left[(\theta_1/t)^{a_1}(\theta_2/t)^{a_2}\right]^{1/(1-\rho^2)} \times I_0\left[\frac{2\rho\sqrt{a_1 a_2 (\log (t/\theta_1)\log (x/\theta_2))}}{1-\rho^2}\right]$$

with $t > \theta_1$; $x > \theta_2$; a_1 and $a_2 > 2$; $\rho^2 < 1$ and $I_0(.)$ is a modified Bessel function of order zero. Expressions for the first and second moments are given in Johnson and Kotz [1972] and of particular interest is the conditional expectation, $E[X_i|T_i = t]$

$$E[X_i|T_i = t] = \{a_2\theta_2/(a_2 - 1 + \rho^2)\}\,(t/a_1)^{a_1\rho^2/(a_1-1+\rho^2)}.$$

In contrast to the earlier case, this is a nonlinear function in *t*.

3.2. Warranty Servicing Cost per Item

We assume the following:

1. All failures under warranty are claimed.
2. All claims are valid.
3. The claims are made immediately after failures.
4. The time to replace is small so that it can treated as being zero.
5. The manufacturing cost per item is $C (< S)$.
6. The warranty handling cost is small relative to C and S and hence ignored.

For Policy 1, the warranty cost to the manufacturer for each failure under warranty depends on the total age and usage since initial purchase (*i. e.*, t_0 and x_0) and is given by

$$R(t_0, x_0) = \begin{cases} c_1 & \text{if } (t_0, x_0) \in \Omega_1 \\ c_2 + \phi(t_0, x_0) & \text{if } (t_0, x_0) \in \Omega_2 \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

with

$$c_1 = C \qquad \text{and} \qquad c_2 = 0. \tag{2}$$

For Policy 2, the first failure under warranty implies a refund is given and the warranty ceases. Hence, the warranty cost to the manufacturer is given by (1) with

$$c_1 = S \qquad \text{and} \qquad c_2 = 0. \tag{3}$$

For Policies 3-5, the warranty cost to the manufacturer depends on the age and usage at failure (*i. e.*, on t and x) and not on t_0 and x_0 and is given by

$$R(t, x) = \begin{cases} c_1 & \text{if } (t, x) \in \Omega_1 \\ c_2 + \phi(t, x) & \text{if } (t, x) \in \Omega_2 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where

$$c_1 = C \qquad \text{and} \qquad c_2 = C - S \quad \text{(Policy 3)}, \tag{5}$$

$$c_1 = C \qquad \text{and} \qquad c_2 = C \quad \text{(Policy 4)} \tag{6}$$

and

$$c_1 = S \qquad \text{and} \qquad c_2 = C - S \quad \text{(Policy 5)}. \tag{7}$$

For policies 3-5, since replacements under warranty come with a new warranty, the total warranty servicing cost per item sold at full price is the sum of the costs associated with supply of new items (at no or pro-rated cost to the buyer) and the refund. Note that the warranty ceases only when an item survives for a time period K or the usage reaches L before failure or the item fails under warranty with the warranty ending with a refund. As a result, the total warranty servicing cost per item sold at full price is a random variable and its expected value depends on the parameters of the policy. We denote this expected cost by $EC(\theta)$ where θ represents the set

$$\theta = \{K_1, L_1, K, L\}. \tag{8}$$

4. ANALYSIS OF NON-RENEWING POLICIES [Policies 1 and 2]

In this section we derive expressions for the expected warranty cost per item for the two non-renewing policies.

4.1. Policy 1:

Let $EC_1\,(K_1,\, L_1,\, K,\, L)$ denote the expected warranty cost. We derive an expression for this by conditioning on $(T_1,\, X_1)$, the age and usage at first failure. If $T_1 = t$ and $X_1 = x$ with $(t, x) \in \Omega_1$, then the new item comes with a warranty region $[t,\, K) \times [x,\, L)$. The cost to the manufacturer is C. If the replacement item fails in $[0,\, K_1 - t) \times [0,\, L_1 - x)$, it is replaced by a new item. Since the first replacement is a renewal point, the subsequent expected warranty cost is given by $EC_1\,(K_1 - t,\, L_1 - x,\, K - t,\, L - x)$. If $(t,\, x) \in \Omega_2$, then the manufacturer refunds an amount $\phi\,(t,\, x)$. Finally, if $(t,\, x)$ is outside the warranty region, the manufacturer incurs no cost. As a result,

$$
\left.\begin{array}{l}
EC_1\,(K_1,\, L_1,\, K,\, L) \\
= \left\{\begin{array}{c}
EC_1\,(K_1 - t,\, L_1 - x,\, K - t,\, L - x) + C \\
\text{if} \quad (t,\, x) \in \Omega_1 \\
\phi\,(t,\, x) \qquad \text{if} \;\; (t,\, x) \in \Omega_2 \\
0 \qquad \text{otherwise}
\end{array}\right.
\end{array}\right\} \tag{9}
$$

On removing the conditioning, we have

$$
\begin{aligned}
EC_1\,&(K_1,\, L_1,\, K,\, L) \\
&= \iint_{\Omega_1} [EC_1\,(K_1 - t,\, L_1 - x,\, K - t,\, L - x) + C]\, dF\,(t,\, x) \\
&\quad + \iint_{\Omega_2} \phi\,(t,\, x)\, dF\,(t,\, x)
\end{aligned} \tag{10}
$$

and this is a two dimensional renewal integral equation. In general, one would need to use a computational approach to obtain the solution.

4.2. Policy 2:

Only one failure is possible under this policy. If the age and usage at this failure is $(t_0,\, x_0) \in \Omega$, the cost to the manufacturer, $R\,(t_0,\, x_0)$, is given by

(1) with c_1 and c_2 given by (3) and the warranty ceases. As a result, the expected warranty cost per item is given by

$$EC_2(\theta) = SF(K_1, L_1) + \iint_{\Omega_2} \phi(t, x)\, dF(t, x). \qquad (11)$$

5. ANALYSIS OF RENEWING POLICIES [Policies 3-5]

We propose a unified approach which yields the results for the three policies as special cases.

5.1. Unified Approach

Consider the following:

If item i, $i \geq 1$ ($i=1$ corresponds to the first sale at full price and $i \geq 2$ correspond to replacement items supplied at possibly less than full price) fails under warranty with ($T_i = t$, $X_i = x$) the buyer has the choice of either (i) buying a replacement item (at less than full cost) or (ii) obtaining a refund. If the buyer chooses alternative (i), the replacement item is covered by a new warranty identical to that of the original item. If the buyer chooses alternative (ii), a refund is given and the warranty is terminated. The warranty also terminates when an item fails outside the warranty for the first time. The cost to the manufacturer for a single failure, $R(t, x)$, is given by (4) with c_1 and c_2 given by (5)-(7) depending on the policy.

We assume that the buyer's decision to choose (i) or (ii) occurs randomly. Let U_i denote the buyer's decision at the failure of the i-th item. U_i is a random variable defined as

$$U_i = \begin{cases} 1 & (T_i, X_i) \in \Omega, \text{ the buyer buys a new one} \\ 0 & (T_i, X_i) \in \Omega, \text{ the buyer does not buy or } (T_i, X_i) \notin \Omega \end{cases}$$

Given $\{T_i = t, X_i = x\}$, we assume that U_i is independent of previous failure history and that the probability of $U_i = 1$ conditional on $T_i = t$, $X_i = x$, is given by

$$\Pr\{U_i = 1 | T_i = t, X_i = x\} = p(t, x). \qquad (12)$$

Also, note that $0 \leq p(t, x) \leq 1$ if $(t, x) \in \Omega$ and $p(t, x) = 0$ if $(t, x) \notin \Omega$. We assume that $p(t, x)$ is constrained to satisfy

$$\int_0^\infty \int_0^\infty p(t, x)\, dF(t, x) \leq 1$$

[which will be true if $F(K, L) < 1$].

If $p(t, x)$ is of the form

$$p(t, x) = \begin{cases} p_1 & \text{if} \quad (t, x) \in \Omega_1 \\ p_2 & \text{if} \quad (t, x) \in \Omega_2 \\ 0 & \text{if} \quad (t, x) \notin \Omega \end{cases} \tag{13}$$

then the special cases are:

(1) $p_1 = 1$ and $p_2 = 1$ for Policy 3 with $R(t, x)$ given by (4) and (5).

(2) $p_1 = 1$ and $p_2 = 0$ for Policy 4 with $R(t, x)$ given by (4) and (6).

(3) $p_1 = 0$ and $p_2 = 1$ for Policy 5 with $R(t, x)$ given by (4) and (7).

Finally, Policy 2 can be viewed as corresponding to the special case $p_1 = p_2 = 0$; $t = t_0$ and $x = x_0$ and $R(t_0, x_0)$ given by (1) and (3).

5.2. Cost Analysis

We now carry out the analysis to obtain the expected warranty cost per item sold at full price for the general case. Note that the warranty ceases if an item fails under warranty and the buyer does not buy a replacement or if the item fails outside warranty for the first time. In other words, the warranty terminates whenever $U_i = 0$ for $i \geq 1$. Let

$$N(\theta) = \{n : U_1, U_2, \ldots, U_{n-1} = 1;\ U_n = 0\} \tag{14}$$

be the number of items needed to service warranty, that is, the original item plus the number of replacement items under warranty..

Since (T_i, X_i, U_i), $i \geq 1$ is a sequence of independent trivariate random variables and since the event $\{N = n\}$ only depends on the information $\{(T_i, X_i, U_i), i = 1, 2, \ldots, n\}$ then [from Ross (1970), p. 37], N is a stopping time for (T_i, X_i, U_i), $i = 1, 2, \ldots$

Let $Q\ [= Q(\theta)]$ denote the total warranty servicing cost to the manufacturer per item sold at full price. Q is equal to the total cost associated with the servicing of the original item and all subsequent replacement items under warranty. It is given by

$$Q = \sum_{i=1}^{N} R(T_i, X_i). \tag{15}$$

The expected value of Q is given by

$$E[Q] = E\left[\sum_{i=1}^{N} R(T_i, X_i)\right] \tag{16}$$

and represents the expected warranty service cost per item sold at full price.

Since N is a stopping time, using Wald's equation [Ross (1970), p. 38], we have,

$$E[Q] = E[N]\, E[R(T_1, X_1)] \tag{17}$$

We now derive the expressions for $E[N]$ and $E[R(T_1, X_1)]$. Since item failures occur independently with an identical warranty for each replacement item, we have

$$\Pr\{N = n\} = q^{n-1}(1-q), \qquad n = 1, 2, 3, \ldots \tag{18}$$

where,

$$q = \int_0^\infty \int_0^\infty p(t, x)\, dF(t, x). \tag{19}$$

This is a geometric distribution and as a result, we have

$$E[N] = \sum_{n=1}^{\infty} n.\,\text{Prob.}\,\{N = n\} = 1/(1-q). \tag{20}$$

$E[R(T_1, X_1)]$ is given by

$$E[R(T_1, X_1)] = \int_0^\infty \int_0^\infty R(t, x)\, dF(t, x). \tag{21}$$

[*Note*: $R(t, x)$ is zero outside Ω for the cases considered later.]

Using (13) in (19), we have from (20)

$$E[N] = 1/[1 - p_1 F(K_1, L_1) - p_2 \{F(K, L) - F(K_1, L_1)\}]. \tag{22}$$

As a result, from (17), the expected warranty cost per item sold at full price, is given by

$$E[Q] = \left[\iint_{\Omega} R(t, x)\, dF(t, x)\right] / [1 - p_1 F(K_1, L_1) - p_2\{F(K, L) - F(K_1, L_1)\}]. \tag{23}$$

For Policy 3, $p_1 = p_2 = 1$ and the expected warranty cost per unit

$$EC_3(\theta) = \left[CF(K_1, L_1) + \iint_{\Omega_2} \{C - S + \phi(t, x)\}\, dF(t, x)\right] / [1 - F(K, L)]. \tag{24}$$

For Policy 4, $p_1 = 1$ and $p_2 = 0$ and the expected warranty cost per unit

$$EC_4(\theta) = \left[CF(K_1, L_1) + \iint_{\Omega_2} \phi(t, x)\, dF(t, x)\right] / [1 - F(K_1, L_1)]. \tag{25}$$

For Policy 5, $p_1 = 0$ and $p_2 = 1$ and the expected warranty cost per unit

$$EC_5(\theta) = \left[S\, F(K_1, L_1) + \iint_{\Omega_2} \{C - S + \phi(t, x)\}\, dF(t, x)\right] / [1 - \{F(K, L) - F(K_1, L_1)\}]. \tag{26}$$

6. NUMERICAL EXAMPLE

In general, it is not possible to obtain analytical expressions for $EC_i(\theta)$, $1 \leq i \leq 5$. One needs to use a computational scheme to obtain them. The expected warranty cost for Policy 1 requires solving a two-dimensional renewal type equation. The authors are currently looking at efficient and accurate numerical methods for solving such equations. The expected warranty costs for the remaining policies can be obtained using a standard two dimensional numerical integration package. As such, we confine our attention to Policies 2-5.

We consider the following form for

$$\phi(t, x) - \phi(t, x) = \left\{ \begin{array}{c} S \quad \text{if } 0 \leq t \leq K_1 \text{ and } 0 \leq x \leq L_1 \\ S(K - t)/(K - K_1) \quad \text{if } K_1 \leq t \leq K \text{ and } 0 \leq x \leq L_1 \\ S(L - x)/(L - L_1) \quad \text{if } 0 \leq t \leq K_1 \text{ and } L_1 \leq x \leq L \\ S(K - t)(L - x)/(K - K_1)(L - L_1) \\ \text{if } K_1 \leq t \leq K \text{ and } L_1 \leq x \leq L \\ 0 \quad \text{otherwise} \end{array} \right\} \tag{27}$$

Note that $\phi(t, x)$ is a continuous function in t and x over $\mathbb{R}^2_+$. We assume that the item is a part of an automobile and the units for age and usage are 1 year and 10^4 km respectively. Let $F(t, x)$ be given by a Beta-Stacy distribution with the following parameter values:

$$a = 0.6, \quad \alpha = 5, \quad c = 1, \quad \phi = 7/3, \quad \theta_1 = 18 \quad \text{and} \quad \theta_2 = 24.$$

These were selected so that the mean age at failure ($E[T_i]$) equals 2 (years) and the mean usage at failure ($E[T_i]$) equals 2 (10^4 km). We consider a range of values for K and L varying from 0.5 to 2 and consider the case where $K_1 = 0.5\ K$ and $L_1 = 0.5\ L$.

The expected profit per item sold at full price, $EP_i(\theta)$, $1 \leq i \leq 5$, is given by

$$EP_i(\theta) = S - C - EC_i(\theta) \tag{28}$$

and the expected profit as a fraction of the manufacturing cost is given by

$$EP_i(\theta)/C = [S/C - 1 - EC_i(\theta)/C]. \tag{29}$$

As a result, we present the results in terms of $EC_i(\theta)/C$, S/C and $EP_i(\theta)/C$.

Table I shows $EC_2(\theta)/C$ for different (K, L) and S/C combinations for Policy 2. As can be seen, for a given S/C, the $EC_2(\theta)/C$ increases with K and L as to be expected since better warranty terms imply greater expected warranty costs. For a given (K, L) combination, $EC_2(\theta)/C$ also increases

TABLE I

$EC_2(\theta)/C$ for different S/C and (K, L) values ($K_1 = 0.5\ K$ and $L_1 = 0.5\ L$).

S\c	K\L	0.50	1.00	1.50	2.00
1. 1	0.50	0.00039	0.00095	0.00101	0.00101
	1.00	0.00066	0.00743	0.01359	0.01566
	1.50	0.00067	0.01122	0.03418	0.05182
	2.00	0.00067	0.01199	0.04647	0.08875
1. 4	0.50	0.00050	0.00121	0.00128	0.00128
	1.00	0.00085	0.00946	0.01730	0.01993
	1.50	0.00085	0.01428	0.04350	0.06595
	2.00	0.00085	0.01526	0.05914	0.11295
1. 8	0.50	0.00064	0.00156	0.00165	0.00165
	1.00	0.00109	0.01216	0.02224	0.02562
	1.50	0.00109	0.01836	0.05592	0.08480
	2.00	0.00109	0.01962	0.07603	0.14522

with S/C. This is again to be expected, as bigger S/C implies greater payout under warranty.

$EP_2(\theta)/C$ depends on (K, L) and S/C. It is interesting to compare the following three cases:

(a)	$K = L = 1.0$	and	$S/C = 1.1$
(b)	$K = L = 1.5$	and	$S/C = 1.4$
(c)	$K = L = 2.0$	and	$S/C = 1.8$

They represent increasingly better warranty terms with increasing sale price. $EC_2(\theta)$ is the smallest for (a) and largest for (c). This follows as the increase in the expected warranty servicing cost (with better warranty terms) is more than offset by the increase in the revenue generated due to increase in the sale price. As a result, from the manufacturer's viewpoint the best strategy is (c) – that is, sell the product with $S/C=1.8$ and with warranty terms $K=L=2$. Alternatively, K and L could be increased with S/C being increased to have the same profit to cost ratio.

In general, as the warranty terms get better (*i. e.*, as K and/or L increases) the total sales increase. This is due to two reasons – (i) manufacturers using better warranty terms as promotional tool for maketing and (ii) the consumers perceptions that better warranty terms imply better product. In contrast, as the sale price increases, the total sales decreases due to reasons such as value for money, budgetary constraints of consumers etc. Hence, to compare options (a)-(b) in a more meaningful manner, one needs to develop a model for total sales as a function of the sale price and the warranty terms (and other relevant variables such as, for example, advertising). This aspect is not pursued in this paper and is an open topic for research. A model dealing with this issue in the context of one dimensional warranty policy can be found in Murthy [1990].

Table II shows $EC_3(\theta)/C$ for different (K, L) combinations for Policy 3. For $S/C=1.1$, the expected warranty cost is always positive and increases with K and/or L increasing as to be expected. For $S/C=1.4$, the expected warranty cost is negative for $K=L=0.5$ and positive for the remaining combinations of (K, L). The reason for this happening is that for some values of $(t, x) \in \Omega_2$, $\phi(t, x)$ is small and $R(t,x)$ is negative because $S/C=1.4$. This implies that the manufacturer makes a profit rather than incur a loss due to replacement under warranty as the compensation under warranty claim is less than the difference between the sale price and the manufacturing cost. This becomes more striking when $S/C=1.8$ where for all

TABLE II

$EC_2(\theta)/C$ for different S/C and (K, L) values ($K_1 = 0.5\,K$ *and* $L_1 = 0.5\,L$).

S/C	K\L	0.50	1.00	1.50	2.00
1. 1	0.50	0.00025	0.00071	0.00076	0.00076
	1.00	0.00050	0.00528	0.01049	0.01254
	1.50	0.00050	0.00871	0.02751	0.04508
	2.00	0. 00050	0.00950	0.03996	0.08574
1. 4	0.50	–0.00005	0.00022	0.00029	0.00029
	1.00	0.00017	0.00036	0.00354	0.00587
	1.50	0.00018	0.00336	0.00738	0.01918
	2. 00	0.00018	0.00436	0.01771	0.03636
1. 8	0.50	–0.00045	–0.00042	–0.00034	–0.00034
	1.00	–0.00026	–0.00618	–0.00573	–0.00302
	1.50	–0.00025	–0.00377	–0.01946	–0.01534
	2.00	–0.00025	–0.00249	–0.01195	–0.02949

warranty combinations considered, the expected warranty costs are negative – in other words, the manufacturer is making a profit rather than loss.

For $S/C = 1.8$, the expected profit the maximum for $K = L = 2$. As a result, the manufacturers strategy should be to sell the item at a price $S/C = 1.8$ and with $K = L = 2$. Note that as K and L increase beyond 2, the profits will initially increase and then start decreasing and finally become negative to imply costs rather than profits.

Table III shows $EC_4(\theta)/C$ for different (K, L) combinations for Policy 4. The results are very similar to that for Policy 2. For $S/C = 1.1$, the expected costs for Policy 4 are smaller than that for Policy 2 for all combinations of (K, L) and S/C. This is to be expected for any item failure with $(t, x) \in \Omega_1$, as under Policy 2, the cost to the manufacturer is S whilst, under Policy 4, it is $C\,(< S)$. Although the policy is renewing for failures with $(t, x) \in \Omega_1$, the probability of the original and the replaced item failing in Ω_1 is very small. Again by offering different warranty combinations to different types of user, the expected cost becomes smaller.

Table IV shows $EC_5(\theta)/C$ for different (K, L) combinations for Policy 5. The results are similar to Policy 3. However, it is interesting to note that the manufacturers profits are maximum when items are sold with $S/C = 1.8$ and $K = L = 1.5$. This is in contrast to Policy 3, where warranty terms for maximum profit are $K = L = 2$.

TABLE III

$EC_4(\theta)/C$ for different S/C and (K, L) values ($K_1 = 0.5\,K$ and $L_1 = 0.5\,L$).

S/C	K\L	0.50	1.00	1.50	2.00
1. 1	0.50	0.00038	0.00094	0.00099	0.00100
	1.00	0.00066	0.00731	0.01339	0.01545
	1.50	0.00066	0.01107	0.03368	0.05127
	2.00	0.00066	0.01184	0.04598	0.08848
1. 4	0.50	0.00047	0.00117	0.00123	0.00124
	1.00	0.00082	0.00893	0.01640	0.01899
	1.50	0.0082	0.01363	0.04082	0.06210
	2.00	0.00082	0.01461	0.05602	0.10626
1.8	0.50	0.00059	0.00147	0.001551	0.00156
	1.00	0.00103	0.01109	0.02041	0.02371
	1.50	0.00104	0.01704	0.05034	0.07654
	2.00	0.00104	0.01830	0.06940	0.12997

TABLE IV

$EC_5(\theta)/C$ for different S/C and (K, L) values ($K_1 = 0.5\,K$ and $L_1 = 0.5\,L$).

S/C	K\L	0.50	1.00	1.50	2.00
1.1	0.50	0.00026	0.00072	0.00077	0.00077
	1.00	0.00050	0.00541	0.01070	0.01276
	1.50	0.00051	0.00886	0.02810	0.04578
	2.00	0.00051	0.00965	0.04056	0.08614
1. 4	0.50	−0.00002	0.00027	0.00033	0.00034
	1.00	0.00020	0.00092	0.00450	0.00688
	1.50	0.00021	0.00404	0.01057	0.02407
	2.00	0.00021	0.00505	0.02157	0.04665
1. 8	0.50	−0.00040	−0.00033	−0.00025	−0.00024
	1.00	−0.00020	−0.00506	−0.00037	−0.00096
	1.50	−0.00019	−0.00238	−0.01279	−0.00487
	2.00	−0.00019	−0.00110	−0.00377	−0.00625

7. CONCLUSIONS

In this paper, we have carried out the cost analysis of five different two-dimensional combination warranty policies with rectangular warranty regions. For the non-renewing case, the warranty gives a maximum cover for a time period K and for a usage L. We presented numerical results for the case where the refund function is given by (27) and item failures are modelled by the Beta-Stacy distribution.

As mentioned earlier, different shapes for the warranty regions Ω and different rebate functions $\phi(t, x)$ define a family of two-dimensional combination warranty policies. Figure 2 shows a different shape for the warranty region. For the non-renewing case, the warranty gives a minimum cover for a time period K and for a usage L. The analysis for each of these can be carried out in a manner indicated in this paper. The form of the final expressions would be different.

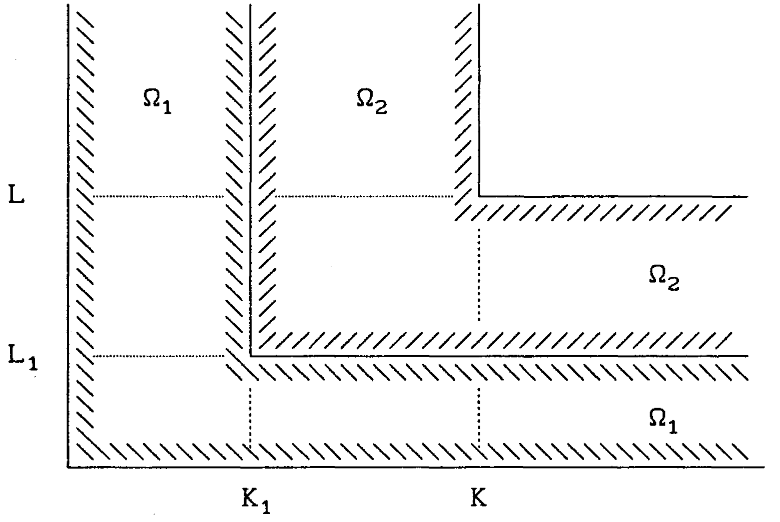

Figure 2. – Warranty Regions: Infinite Dimensional Strips.

In the policies studied in this paper, the warranty region Ω is comprised of two-sub-regions with warranty terms different for each region. One can define warranty policies with more than two sub-regions and the warranty terms varying from sub-region to sub-region. The cost analysis of such combination policies would be more complex but the approach would essentially be similar to that indicated in this paper.

In our model formulation, we have assumed that all warranty claims are valid and exercised. Often this is not the case. One can model this through an additional variable $\psi(t, x)$ which represent the probability that a claim is exercised if the age and usage at failure is t and x respectively. As a result, the probability that a claim is not exercised is given by $[1 - \psi(t, x)]$. The case where not all claims are valid poses additional problems. One needs to incorporate a resolution mechanism which models the outcome should a dispute arise. This is a topic for future research.

The results obtained in Section 5 can be used to study two-dimensional combination policies where the consumer has the choice of either choosing a refund or buying a replacement (at less than full price) for items which fail under warranty. In particular, the probability that the consumer buys a replacement [given by $p(t, x)$] can be more complex than the cases studied here [given by (13)].

Finally, as mentioned in Section 6, the optimal decisions with regard to warranty terms must take into account the total sales as a function of warranty terms and sale price. This is yet another open topic for further research.

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