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LAGRANGEAN DECOMPOSITION FOR INTEGER PROGRAMMING: THEORY AND APPLICATIONS (*)

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Abstract. — *Given a mixed-integer programming problem whose constraint set is the intersection of several specially structured constraint sets, it is possible to artificially induce decomposition in the Lagrangean relaxation problems by introducing copies of the original variables for a subset of constraints and dualizing the equivalence conditions between the original variables and the copies. We study duality for Lagrangean decomposition and compare it with conventional Lagrangean duality. The implications of Lagrangean decomposition can be quite profound for integer programming problems containing special classes of constraints such as subtour elimination constraints or matching inequalities. Several application problems exemplify the use of Lagrangean decomposition.*

Keywords : decomposition; Lagrangean relaxation; Lagrangean decomposition; Lagrangean duality; integer programming; variable splitting; layering; subtour elimination constraints; b-matching; implicit constraints; facets; integer polytopes.

Résumé. — *Considérons un programme en nombres entiers dont l'ensemble des solutions réalisables est l'intersection de plusieurs ensembles possédant des structures particulières. Il est possible d'introduire dans ce problème une structure décomposable en remplaçant les variables initiales par des « copies » dans chacun des sous-ensembles de contraintes, sauf un, et en dualisant les conditions d'identité entre originaux et copies. Nous étudions la dualité en décomposition lagrangienne classique. Nous montrons l'importance de la décomposition lagrangienne pour des problèmes contenant des sous-structures spéciales du type contraintes d'élimination de circuits ou inégalités de couplage. Nous démontrons sur plusieurs exemples l'application de ce principe.*

Mots clés : décomposition; relaxation lagrangienne; décomposition lagrangienne; dualité lagrangienne; programmation en nombres entiers; découplage de variables; contraintes d'élimination de circuits; b-couplage; contraintes implicites; facettes; polytopes entiers.

1. INTRODUCTION

Given a mixed-integer programming problem whose constraint set is the intersection of several specially structured constraint sets, it is possible to define a Lagrangean relaxation whose problems decompose into several

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subproblems, each one over one of the special structures. The technique used consists in creating one (or more) identical copies of the vectors of decision variables, in using one of these copies in each set of constraints and in dualizing the condition(s) that they should be identical.

The new Lagrangean scheme, called Lagrangean decomposition, is important as well as interesting in that the Lagrangean subproblems keep all the original constraints; conventional Lagrangean relaxation inevitably loses all but at most one specially structured constraint set. Under a condition discussed in section 2, Lagrangean decomposition may yield a stronger bound than conventional Lagrangean relaxation, emphasizing the importance of keeping all the problem constraints.

The idea of creating "copies" of some subset of the original variables has been used earlier, in particular in the context of layering strategies for networks by [Glover and Mulvey (1975, 1980)]. It was then used for Lagrangean relaxation by [Shepardson and Marsten (1980)] for the two-duty period scheduling problem, by [Ribeiro (1983)] for constrained shortest path problems and by [Jörnsten and Näsberg (1986)] for the generalized assignment problem. Yet it does not seem that the implications of this new way of applying Lagrangean relaxation have been studied or fully exploited for lack of understanding of the primal-dual relationship.

In the first part of this paper, the "decomposed" Lagrangean dual will be shown to be at least as strong as the standard dualization of a portion of the constraints. Duality theory specialized to Lagrangean decomposition implies that the Lagrangean decomposition dual is equivalent to an *LP* problem in the original variable space whose feasible set is defined on the intersection of the convex hulls of the feasible solutions of the corresponding blocks. It furnishes a necessary condition under which stronger bounds than the conventional Lagrangean bounds can be obtained. In addition to providing bound improvement, Lagrangean decomposition is also important because it permits the implicit use of facets of integer polytopes without requiring their explicit construction. When one or more integer constraint sets can be described by an exponential number of equations, yet yield easily solvable subproblems, Lagrangean decomposition provides the same bound as the Lagrangean relaxation in which all these constraints are dualized. Independent sets in a graphic matroid (which can be described by subtour elimination constraints) and b-matching polytopes are examples of such a situation. Lagrangean decomposition permits the implicit use of all facets describing such integer polytopes.

In the second part of the paper, examples are introduced to demonstrate how Lagrangean decomposition can be used. They are the generalized assignment problem, the resource constrained arborescence problem, the capacitated plant location problem, and the symmetric traveling salesman problem.

2. LAGRANGEAN DECOMPOSITION

We shall use the following notation. Given a constrained optimization problem $(*)$, $(\bar{*})$ will denote its continuous relaxation, $FS(*)$ its feasible set, $OS(*)$ its optimal set, i.e. the set of all its optimal solutions, and $V(*)$ its optimal value. $\mathrm{Co}(S)$ will denote the convex hull of a set S of R^n, and $\mathrm{Co}\{(a), (b), \ldots\}$ will denote the convex hull of the feasible solutions to (a) and (b) and so forth. $A \otimes B$ represents the Cartesian product of the two sets A and B. Given a constraint set $(*)$, $\Delta(*)$ denotes the polyhedron defined by $(*)$. We shall occasionally use "decomposition" for Lagrangean decomposition and "relaxation" for Lagrangean relaxation.

Lagrangean relaxation is a partial constraint dualization. Full dualization is equivalent to the dual of the continuous relaxation most commonly used in commercial mixed-integer programming codes. Through partial dualization, one can exploit exactly one of possibly many special structures embedded in the original problem. Partial dualization has two advantages over full dualization: (1) a possible bound improvement in the absence of the Integrality Property, (2) the non-dominated nature of the restricted dual space compared with the full dual space. Simultaneous exploitation of several special structures by Lagrangean decomposition may yield even further bound improvement. This section studies bound improvement in Lagrangean decomposition.

Lagrangean decomposition artificially constructs a block angular form in order to force decomposition. It replaces the original variables with different copies in different subsets of the constraints, each subset having a special structure, and dualizes the equivalence conditions between the original and the copied variables. The resulting subproblems may be tractable, while conventional Lagrangean relaxation could keep no more than one of these substructures to maintain tractability.

We shall briefly review some of the theory of Lagrangean relaxation before introducing Lagrangean decomposition.

Consider the following integer programming problem:

$$(P) \qquad \mathrm{Max}\{fx \mid Ax \leqq b,\ Cx \leqq d,\ x \in X\}.$$

where f, b, d, A and C are vectors and matrices of comformable dimensions. X is a special structure including integer requirements on a subset of variables.

One of the possible Lagrangean relaxations is to dualize $Ax \leqq b$:

$$(LR_v) \quad \operatorname{Max}\{fx+v(b-Ax) \mid Cx \leqq d,\ x \in X\}$$
$$=\operatorname{Max}\{fx+v(b-Ax) \mid x \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\}\}.$$

The corresponding Lagrangean dual is defined as:

$$(R) \qquad \operatorname*{Min}_{v \geqq 0} V(LR_v).$$

Consider the primal relaxation of (P):

$$(S) \qquad \operatorname{Max}\{fx \mid Ax \leqq b,\ x \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\}\}.$$

The following lemma [Geoffrion (1974)] holds since (R) is a partial dual of the LP problem defined on the convex hull of the feasible solutions of $Cx \leqq d$ and $x \in X$ intersected by the polyhedron $\{x \mid Ax \leqq b\}$.

LEMMA 1 [*Geoffrion* (1974)]: *The optimal value of the Lagrangean dual* (R) *is equal to the optimal value of the LP problem* (S).

In addition to Geoffrion's characterization of the Lagrangean dual given by lemma 1, the following result describes more specifically the relationship between Lagrangean solutions of (LR_{v^0}), with $v^0 \in OS(R)$, and optimal solutions of (S). We shall show that it is possible to construct an optimal solution of (S) if one knows all optimal solutions of (LR_{v^0}).

THEOREM 2: *There exists a convex combination of optimal solutions of* (LR_{v^0}) *which is an optimal solution of* (S).

Proof. — Let $x^k(v^0) \in OS(LR_{v^0})$ for every $k \in K$, where K is the index set of all multiple optimal solutions of (LR_{v^0}). Let us prove that there exists a set of multipliers μ_k, such that $x(\mu)=\sum_{k \in K} \mu_k x^k(v^0)$ is an optimal solution of (S), where $\sum_{k \in K} \mu_k = 1$ and $\mu_k \geqq 0$ for all $k \in K$.

Let us first show that there exists a set of convex multipliers μ_k such that

$$x(\mu)=\sum_{k \in K} \mu_k x^k(v^0)$$

is feasible for S. Obviously $x(\mu) \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\}$.

$$Ax(\mu)-b=\sum_{k \in K} \mu_k (Ax^k(v^0)-b)=-\sum_{k \in K} \mu_k g^k$$

where $g^k = -A x^k(v^0) + b$ is a subgradient of $V(LR_v)$ at $v=v^0$. The subdifferential of $V(LR_v)$ at $v=v^0$ is equal to the convex hull of g^k, $k \in K$. Since v^0 is a minimizer, there exists a subgradient at $v=v^0$ which belongs to the positive polar cone of the feasible set at $v=v^0$; therefore in this case, a subgradient belonging to the nonnegative orthant exists. Thus there exists a set of convex multipliers μ_k such that $\sum_{k \in K} \mu_k g^k \geqq 0$ at $v=v^0$, and $x(\mu)$ is a feasible solution of (S).

Now we show that $fx(\mu)$ is equal to $V(S)$:

$$\begin{aligned} fx(\mu) &= fx(\mu) - v^0 \sum_{k \in K} \mu_k g^k \quad \text{(by complementary slackness)} \\ &= \sum_{k \in K} \mu_k (fx^k(v^0) - v^0 g^k) \\ &= \sum_{k \in K} \mu_k V(R) \\ &= V(R) \\ &= V(S) \quad \text{(by lemma 1).} \ \blacksquare \end{aligned}$$

If we knew every optimal solution corresponding to the optimal Lagrangean multiplier, we could find an optimal solution of (S), even though the domain of (S) is defined only implicitly.

Let us now introduce Lagrangean decomposition. We treat the case of two explicit constraint sets for expository simplicity (some examples in section 3 deal with multiple special structures). It is another Lagrangean relaxation of (P) obtained by (1) introducing problem (P'):

$$(P') \quad \text{Max}\{ fx \mid Ay \leqq b,\ Cx \leqq d,\ x \in X,\ y=x,\ y \in Y \},$$

which is equivalent to (P) for any set Y containing X, and (2) relaxing the "copy" constraint $y=x$ in (P'). This yields a decomposable problem, (LD_u). justifying the name "Lagrangean decomposition":

$$\begin{aligned} (LD_u) \quad & \text{Max}\{ fx + u(y-x) \mid Cx \leqq d,\ x \in X,\ Ay \leqq b,\ y \in Y \} \\ & = \text{Max}\{ (f-u)x \mid Cx \leqq d,\ x \in X \} + \text{Max}\{ uy \mid Ay \leqq b,\ y \in Y \}. \end{aligned}$$

Let (D) denote the Lagrangean decomposition dual:

$$(D) \qquad \underset{u}{\text{Min}}\, V(LD_u).$$

Let us now compare Lagrangean decomposition bounds with Lagrangean relaxation bounds. Let v^0 be an optimal Lagrangean multiplier, then we can show that $V(LD_{u^0})$ with u^0 defined as $v^0 A$ is at least as good a bound as $V(LR_{v^0})$:

THEOREM 3: *Consider* $v^0 \in OS(R)$, $u^0 = v^0 A$, *and let* $(x^0, y^0) \in OS(LD_{u^0})$. *Then*

(i) $V(LR_{v^0}) - V(LD_{u^0}) = v^0 (b - A y^0)$, *and*

(ii) $V(D) \leqq V(R)$.

Proof. — Let $v^0 \in OS(R)$ and $u^0 = v^0 A$, then

$$\begin{aligned} V(LD_{u^0}) &= \operatorname{Max}\{(f - v^0 A)x \mid Cx \leqq d,\ x \in X\} + \operatorname{Max}\{v^0 A y \mid Ay \leqq b,\ y \in Y\} \\ &= (f - v^0 A)x^0 + v^0 A y^0 \quad (\text{as } (x^0, y^0) \in OS(LD_{u^0})) \\ &= (f - v^0 A)x^0 + v^0 b + v^0 (A y^0 - b) \\ &= V(LR_{v^0}) + v^0 (A y^0 - b) \quad (\text{as } x^0 \in OS(LR_{v^0}) \text{ as well}) \end{aligned}$$

therefore $V(D) \leqq V(R)$. ■

This result was also proved independently by [Glover and Klingman (1984)] and [Jörnsten, Näsberg and Smeds (1985)].

Notice that the potential bound improvement of $V(D)$ over $V(R)$ may come from two different directions:

(1) $V(LD_u) \leqq V(LR_v)$ for $u = vA$: in particular if it is impossible for y solution of $V(LD_u)$ to satisfy $Ay = b$, the slack $(b - Ay)$ may create a gap between $V(LD_u)$ and $V(LR_v)$.

(2) $V(D) = \operatorname{Min} V(LD_u)$: there may be better values for u than those in the subspace spanned by the rows of A, i. e. those of the form $u = vA$.

Lagrangean decomposition is unique among all possible Lagrangean relaxations in that it can capture multiple special structures, unlike standard Lagrangean relaxation which is limited to keeping at most one such structure. When there are more than two constraint sets, decomposition can easily be shown to be at least as good as any standard Lagrangean relaxation. One might also consider mixing relaxation and decomposition to achieve computational efficiency.

The set X might actually contain a third set of constraints. One may choose to keep these constraints in both subproblems (the only requirement on Y is that it must contain X), and this will usually yield a stronger bound, at the expense of having possibly more difficult problems to solve. The resource constrained arborescence problem of section 3.2 presents such an example.

Another way to further strengthen the Lagrangean decomposition bound is to add to the x-problem a surrogate constraint of $Ax \leqq b$ and/or to the

y-problem a surrogate constraint of $Cy \leqq d$. The capacitated plant location problem of section 3.3 presents such an example.

The following lemma gives a sufficient optimality condition for the Lagrangean solution:

LEMMA 4: *Let $(x(\hat{u}), y(\hat{u}))$ be an optimal solution of $(LD_{\hat{u}})$. If $x(\hat{u})$ and $y(\hat{u})$ are identical, then $x(\hat{u})$ is an optimal solution of (P), $\hat{u}$ is an optimal solution of (D) and there is no duality gap.*

Proof. — If $x(\hat{u}) = y(\hat{u})$, the weak optimality conditions hold ($x(\hat{u})$ is feasible for (P) and complementary slackness holds), thus $x(\hat{u})$ is optimal for (P). Since $V(D)$ is bounded from below by $V(P)$, $\hat{u}$ is optimal for (D). ■

It is then natural to ask when Lagrangean decomposition indeed yields a bound improvement; theorem 5 provides the necessary background to answer that question.

Consider problem (Q):

$$(Q) \quad \operatorname{Max}\{fx \mid x \in \operatorname{Co}\{x \mid Ax \leqq b,\ x \in Y\},\quad x \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\}\}.$$

Now we show that problems (Q) and (D) are equivalent:

THEOREM 5: *The optimal value of the Lagrangean dual (D) is equal to the optimal value of the LP problem (Q).*

Proof. — Consider the following primal relaxation of (P'):

$$(Q') \qquad \operatorname{Max}\left[fx \,\middle|\, \begin{array}{c} (x, y) \in \operatorname{Co}\left\{ (x, y) \,\middle|\, \begin{array}{l} Ay \leqq b,\ y \in Y, \\ Cx \leqq d,\ x \in X \end{array} \right\}. \\ x - y = 0 \end{array}\right]$$

By applying lemma 1 for problems (D) and (Q'), $V(D) = V(Q')$. Then

$$\begin{aligned} V(Q') &= \operatorname{Max}\left[fx \,\middle|\, \begin{array}{c} (x, y) \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\} \otimes \operatorname{Co}\{y \mid Ay \leqq b,\ y \in Y\} \\ x = y \end{array}\right] \\ &= \operatorname{Max}\left[fx \,\middle|\, \begin{array}{c} x \in \operatorname{Co}\{x \mid Cx \leqq d,\ x \in X\},\ y \in \operatorname{Co}\{y \mid Ay \leqq b,\ y \in Y\} \\ x = y \end{array}\right] \\ &= \operatorname{Max}\{fx \mid x \in \operatorname{Co}\{x \mid Ax \leqq b,\ x \in X\} \cap \operatorname{Co}\{x \mid Cx \leqq d,\ x \in Y\}\} \\ &= V(Q). \end{aligned}$$

Therefore, $V(D) = V(Q)$. ■

Thus optimizing the Lagrangean decomposition dual is equivalent to optimizing the primal objective function on the intersection of the convex hulls of the constraint sets. This provides a means to compare bound strength of alternative Lagrangean relaxation/decomposition schemes by comparing inclusion relationships among the respective *LP* polytopes in the original (x-) variable space even though the dual is complicated by the copied (y-) variables. This is illustrated on figure 1.

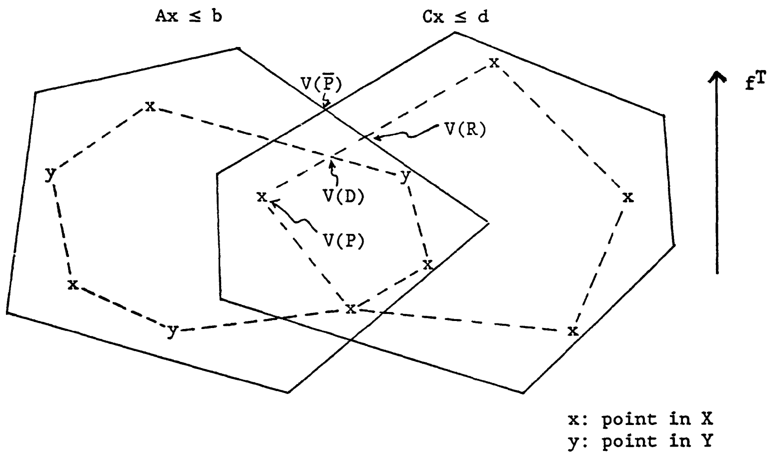

Figure 1

Finally, we introduce, in a negative way, a necessary condition for bound improvement.

THEOREM 6: *If either the x- or y-problem has the Integrality Property, $V(D)$ is equal to the stronger of the two Lagrangean bounds obtained by relaxing one or the other of the two sets of constraints.*

Proof. — Let $R1$ and $R2$ be the two Lagrangean relaxations obtained by dualizing $Ax \leqq b$ and $Cx \leqq d$ respectively. Without loss of generality, assume that the y-problem has the Integrality Property. Then

$$\begin{aligned}
V(D) &= \operatorname{Max}\{ fx \mid x \in \operatorname{Co}\{ x \mid Ax \leqq b,\ x \in X\} \cap \operatorname{Co}\{ x \mid Cx \leqq d,\ x \in X\} \\
&= \operatorname{Max}\{ fx \mid Ax \leqq b,\ x \in \operatorname{Co}(X),\ x \in \operatorname{Co}\{ x \mid Cx \leqq d,\ x \in X\}\} \quad \text{(by I.P.)} \\
&= \operatorname{Max}\{ fx \mid Ax \leqq b,\ x \in \operatorname{Co}\{ x \mid Cx \leqq d,\ x \in X\}\} \\
&= V(R1) \\
&\leqq \operatorname{Max}\{ fx \mid Ax \leqq b,\ Cx \leqq d,\ x \in \operatorname{Co}(X)\} \\
&= \operatorname{Max}\{ fx \mid Cx \leqq d,\ x \in \operatorname{Co}\{ x \mid Ax \leqq b,\ x \in X\}\} \quad \text{(by I.P.)} \\
&= V(R2)\,(= V(\bar{P})) \quad \blacksquare
\end{aligned}$$

Conversely, if neither problem has the Integrality Property, then the decomposition bound can be strictly better than the relaxation bounds.

Theorem 7 is the counterpart of theorem 2 for Lagrangean decomposition. It characterizes the relationship between optimal solutions of (LD_{u^0}), with $u^0 \in OS(D)$, and optimal solutions of (Q). We show that it is possible to construct an optimal solution of (Q) if one knows all optimal solutions of (LD_{u^0}).

THEOREM 7: *There exist two sets of convex multipliers* α_m, $m \in M$ *and* β_n, $n \in N$ *such that* $\sum_{m \in M} \alpha_m x^m(u^0) = \sum_{n \in N} \beta_n y^n(u^0) \in OS(Q)$ *where* $\{x^m(u^0) \mid m \in M\}$ *is the set of distinct optimal x-solutions and* $\{y^n(u^0) \mid n \in N\}$ *is the set of distinct optimal y-solutions to* (LD_{u^0}).

Proof. — Let $(x^k(u^0), y^k(u^0)) \in OS(LD_{u^0})$ for every $k \in K$ where K is the index set of all optimal solutions of (LD_{u^0}). By theorem 2, there exist convex multipliers λ_k such that

$$(x, y) = \sum_{k \in K} \lambda_k (x^k(u^0), y^k(u^0)) \in OS(Q').$$

Since (x, y) is feasible for (Q'),

$$x = y = \sum_{k \in K} \lambda_k x^k(u^0) = \sum_{k \in K} \lambda_k y^k(u^0) \in OS(Q).$$

Notice that some $x^k(u^0)'s$ and/or $y^k(u^0)'s$ may appear more than once, for $k \in K$. Adding those $\lambda_k's$ yields the two convex multiplier sets α_m and β_n, and the index sets M and N. ■

The above discussion is centered around the integrality property of each constraint set. Recently, however, in [Guignard (1986)], the potential usefulness of Lagrangean decomposition has been further enhanced by recognizing special classes of structures. The structures are those on the borderline between linear programming and integer programming; the undirected and directed spanning tree problems and the various matching problems are of this nature. The common characteristics of these problems are that a suitable LP description leads to a complete characterization of the IP polytope and there exists a polynomial time algorithm taking advantage of the special facet structure. Most importantly, the number of these implicit constraints (facets) would be too large to allow their explicit dualization.

Consider an integer programming problem whose implicit constraint set is denoted by I:

$$(P) \qquad \operatorname{Max}\{fx \mid Ax \leqq b,\ x \in I,\ x \in X\}.$$

Suppose that the problem $\max\{gx \mid x \in I \cap X\}$ can be solved (relatively) easily without writing a specific set of constraints to represent I. Let $\{x \mid Gx \leqq g, x \in X\} = I \cap X$ and $\{x \mid Hx \leqq h\} = \mathrm{Co}\{I \cap X\}$. In other words, $Hx \leqq h$ represents all the facets of $I \cap X$ and $Gx \leqq g$ defines a polyhedron whose integer points constitute $I \cap X$. An important distinction between $Hx \leqq h$ and $Gx \leqq g$ is that the former may consist of at least an exponential number of constraints and the latter a polynomial number of constraints. Then we construct two Lagrangean relaxations and one Lagrangean decomposition:

$(LR1_u)$ $\max\{fx + u(h - Hx) \mid Ax \leqq b,\ x \in X\}$ with dual $(R1)$,

$(LR2_v)$ $\max\{fx + v(g - Gx) \mid Ax \leqq b,\ x \in X\}$ with dual $(R2)$

and

(LD_w) $\max\{(f - w)x \mid Ax \leqq b,\ x \in X\} + \max\{wy \mid y \in I \cap X\}$ with dual (D).

By lemma 1 and theorem 5, $V(D) = V(R1) \leqq V(R2)$. Therefore the Lagrangean decomposition (LD_w) is a practical way of obtaining the same bound as the one given by $(LR1_u)$; $V(LR1_u)$ is only of theroretical interest due to the large number of constraints. The resource constrained arborescence problem of section 3.2 and the symmetric traveling salesman problem of section 3.4 present such examples.

3. APPLICATION PROBLEMS

The problems in this section illustrate how a careful analysis of the feasible sets corresponding to the primal equivalents of relaxations or decompositions enables us to compare their relative bound strength without any computational study. This type of analysis will either save unnecessary efforts or justify computational studies *a priori.*

3.1. The generalized assignment problem

The generalized assignment problem consists of disjoint knapsack constraints and multiple choice constraints. The problem concerns the assignment of jobs to agents such that each job is assigned to exactly one agent without violating an agent's capacity:

$$\text{(GAP)} \qquad \begin{aligned} &\text{Min } \sum_{i,j} c_{ij} x_{ij} && && \\ &\text{s. t. } \sum_j a_{ij} x_{ij} \geqq b_i, && \forall i, && (3.1) \\ &\sum_i x_{ij} = 1, && \forall j, && (3.2) \\ &x_{ij} = 0 \quad \text{or} \quad 1, && \forall i, j && (3.3) \end{aligned}$$

The common relaxations found in the literature dualize either set of constraints [see Ross and Soland (1975), Martello and Toth (1981) and Fisher *et al.* (1986)].

We will analyze the decomposition scheme of [Jörnsten and Näsberg (1986)] and another complicated decomposition scheme and compare them to the conventional Lagrangean relaxation where (3.2) is dualized [Fisher *et al.* (1986)].

Let $(R1)$ denote the Lagrangean dual where (3.2) is dualized and disjoint 0-1 knapsack problems are solved. Then, by lemma 1, $(R1)$ is equivalent to the LP problem (S) whose feasible space is defined as

$$FS(S) = \Delta\{(3.2)\} \cap \text{Co}\{(3.1), (3.3)\}.$$

Under the decomposition scheme found in [Jörnsten and Näsberg (1986)], one solves disjoint 0-1 knapsack problems defined by (3.1) and (3.3) and a multiple choice problem defined by (3.2) and (3.3). By theorem 5, the corresponding Lagrangean dual $(D1)$ is equivalent to the LP problem $(Q1)$ whose polytope is

$$\begin{aligned} FS(Q1) &= \text{Co}\{(3.2), (3.3)\} \cap \text{Co}\{(3.1), (3.3)\} \\ &= \Delta\{(3.2)\} \cap \text{Co}\{(3.1), (3.3)\} \quad \text{(by I.P. of (3.2))} \\ &= FS(S). \end{aligned}$$

Therefore $V(D1)$ is equal to $V(R1)$. .

Now consider a more complex form of decomposition where multiple copies of the original variables are created. Each subproblem is a 0-1 multiple choice knapsack problem defined by $\{(3.1.\text{i}), (3.2), \text{ and } (3.3)\}$. By theorem 5, the corresponding Lagrangean dual $(D2)$ is equivalent to the LP problem $(Q2)$ whose feasible space is

$$\begin{aligned} FS(Q2) &= \bigcap_i \mathrm{Co}\{(3.1.\mathrm{i}), (3.2), (3.3)\} \\ &= \bigcap_i \{[\mathrm{Co}\{(3.1.\mathrm{i}), (3.3)] \cap [\mathrm{Co}\{(3.2), (3.3)\}]\} \\ &= [\bigcap_i \mathrm{Co}\{(3.1.\mathrm{i}), (3.3)\}] \cap [\mathrm{Co}\{(3.2), (3.3)\}] \\ &= \mathrm{Co}\{(3.1), (3.3)\} \cap \mathrm{Co}\{(3.2), (3.3)\} \\ &= FS(Q1) \\ &= FS(S). \end{aligned}$$

Therefore $V(D2)$ is equal to $V(D1)$ and $V(R1)$. The second equality of the above derivation is valid because the intersection of Co $\{(3.1.\mathrm{i}), (3.3)\}$ and Co $\{(3.2), (3.3)\}$ does not create fractional vertices.

If $(R2)$ denotes the dual obtained by dualizing (3.1), the following relationships hold:

$$V(\mathrm{GAP}) \geqq V(D2) = V(D1) = V(R1) \geqq V(R2) = V(\overline{\mathrm{GAP}}).$$

Since bounds are the same for $(R1)$, $(D1)$ and $(D2)$, the increase in computational complexity for $(D1)$ and $(D2)$ would not pay off in terms of overall performance in branch and bound algorithms.

3.2. The resource constrained arborescence problem

The resource constrained arborescence problem treated in [Rosenwein (1986)] and [Guignard and Rosenwein (1987)] is formally described below. The problem is to find the minimum cost arborescence for a given root node, while satisfying the generalized degree constraints (3.6):

$$\begin{array}{llll} (\mathrm{RCAP}) & \mathrm{Min} \; \sum_{i,j} c_{ij} x_{ij} & & \\ & \text{s.t.} \; \sum_i x_{ij} = 1, & \forall j \neq \mathrm{root}, & (3.4) \\ & \sum_{i,j \in S} x_{ij} \leqq |S| - 1, & \forall S \subset \{1, 2, \ldots n\}, & (3.5) \\ & \sum_j a_{ij} x_{ij} \leqq b_i, & \forall i, & (3.6) \\ & x_{ij} = 0 \;\text{ or }\; 1, & \forall i, j. & (3.7) \end{array}$$

An obvious relaxation dualizes constraints (3.6) and solves the resulting arborescence problem defined by (3.4), (3.5) and (3.7). Let (R) denote the corresponding Lagrangean dual. Then, by lemma 1, (R) is equivalent to the LP problem (S) defined on the feasible space

$$FS(S)=\Delta\{(3.6)\}\cap \mathrm{Co}\{(3.4),\ (3.5) \text{ and } (3.7)\}.$$

[Rosenwein (1986)] treated the problem by decomposing it into one arborescence problem and a set of disjoint knapsack problem. Let $(D1)$ denote the corresponding Lagrangean dual. By theorem 5, $(D1)$ is equivalent to the LP problem $(Q1)$ defined on the feasible space

$$\begin{aligned} FS(Q_1)=&\mathrm{Co}\{(3.6) \text{ and } (3.7)\}\cap \mathrm{Co}\{(3.4), (3.5) \text{ and } (3.7)\}\\ &\subseteqq FS(S). \end{aligned}$$

Therefore $V(D_1)\geqq V(R)$.

Rosenwein also suggested to treat this problem by decomposing it into one arborescence problem defined by (3.4), (3.5) and (3.7) and one generalized assignment problem defined by (3.4), (3.6) and (3.7), sharing (3.4) as part of the special structure X. Let $(D2)$ denote the corresponding dual. By theorem 5, $(D2)$ is equivalent to the LP problem $(Q2)$ defined on the feasible space

$$\begin{aligned} FS(Q2)=&\mathrm{Co}\{(3.4),\ (3.6) \text{ and } (3.7)\}\cap \mathrm{Co}\{(3.4),\ (3.5) \text{ and } (3.7)\}\\ &\subseteqq FS(Q1). \end{aligned}$$

Therefore this decomposition provides a closer approximation of the IP polytope of (RCAP) than (R) or $(D1)$.

Then the following relationships hold:

$$V(\mathrm{RCAP}) \geqq V(D2) \geqq V(D1) \geqq V(R)=V(\overline{\mathrm{RCAP}}).$$

It should be pointed out that solving such hard problems as the generalized assignment problem as a Lagrangean subproblem is rarely found in the literature.

3.3. The capacitated plant location problem

The capacitated plant location problem is a classic mixed-integer programming problem and has been studied by too many researchers to name them all here. Works related to the following discussion are by [Nauss (1978)] and

[Van Roy (1980)]. The strong formulation of the problem is

(CPLP) $\quad \text{Min} \sum_{i,j} c_{ij} x_{ij} + \sum_i f_i y_i$

$$\text{s. t.} \quad \sum_i x_{ij} = 1, \qquad \forall j, \tag{3.8}$$

$$x_{ij} \leqq y_i, \qquad \forall i, j, \tag{3.9}$$

$$\sum_j d_j x_{ij} \leqq s_i y_i, \qquad \forall i, \tag{3.10}$$

$$0 \leqq x_{ij} \leqq 1, \qquad \forall i, j, \tag{3.11}$$

$$y_i = 0 \quad \text{or} \quad 1, \qquad \forall i. \tag{3.12}$$

One possible decomposition consists in combining the relaxations used by [Nauss (1978)] and [Van Roy (1980)]. The actual work of [Nauss (1978)] is based on the weak formulation [i. e., without (3.9)]. We consider a similar relaxation based on the strong formulation. Dualizing constraint (3.8) and (3.9) and adding an implied constraint

$$\sum_i s_i y_i \geqq \sum_j d_j, \tag{3.13}$$

Nauss transformed the resulting subproblem into a 0-1 knapsack problem by projecting problems associated with (3.10) on the y-space. Van Roy relaxed constraint (3.10) and solved the simple plant location problem defined by (3.8), (3.9), (3.11) and (3.12).

The decomposition introduced here consists in one 0-1 knapsack problem and one simple plant location problem. Let (D) and (Q) denote the corresponding Lagrangean dual and the equivalent LP problem respectively. Let $(R1)$ and $(S1)$ denote the Lagrangean dual defined by [Nauss (1978)] and the equivalent LP problem respectively. Let $(R2)$ and $(S2)$ be defined in the same manner for the case of [Van Roy (1980)]. First the feasible space of (Q) is

$$FS(Q) = \text{Co}\{(3.8), (3.9), (3.11) \text{ and } (3.12)\}$$
$$\cap \text{Co}\{(3.10), (3.11), (3.12) \text{ and } (3.13)\}.$$

The feasible space of $(S1)$ is

$$FS(\mathrm{S1})=\Delta\{(3.8) \text{ and } (3.9)\} \cap \mathrm{Co}\{(3.10, (3.11), (3.12) \text{ and } (3.13)\} \supseteq FS(Q).$$

The feasible space of $(S2)$ is

$$FS(S2)=\Delta\{(3.10)\} \cap \mathrm{Co}\{(3.8), (3.9), (3.11) \text{ and } (3.12)\} \supseteq FS(Q).$$

Therefore the following relationships hold:

$$V(\mathrm{CPLP}) \geq V(D) \geq V(R1) \geq V(\overline{\mathrm{CPLP}})$$

and

$$V(\mathrm{CPLP}) \geq V(D) \geq V(R2) \geq V(\overline{\mathrm{CPLP}}).$$

3.4. The symmetric traveling salesman problem

We discuss the symmetric traveling salesman problem to demonstrate the potential of the decomposition approach for problems with implicit constraints [for other examples see Guignard (1986)]. [Held and Karp (1970)] solve the 1-tree problem repeatedly to optimize the corresponding Lagrangean dual. The 1-tree is a spanning tree together with two edges incident to node 1, forming a single cycle. By applying lemma 1, their dual (R) is equivalent to the LP problem (S) defined on the 1-tree polytope (denoted Co $\{1\mathrm{T}\}$) intersected by the polyhedron [denoted $\Delta(d)$] generated by the degree constraints (d):

$$FS(S)=\Delta(d) \cap \mathrm{Co}\{1T\}.$$

Even though there always exits one cycle in a 1-tree solution, the primal solution defined by theorem 2 satisfies all subtour elimination constraints.

By the polyhedral theory of Edmonds (1965), the 2-matching polytope (denoted Co $\{2M\}$) is completely characterized by the degree constraints and the 2-matching inequalities. It has been shown that 2-matching inequalities are facets of the symmetric traveling salesman problem polytope. One possible decomposition consists in solving the 1-tree problem and the 2-matching problem. Let (D) denote the corresponding Lagrangean dual and (Q) the equivalent LP problem. By theorem 5, the feasible space of (Q) is

$$FS(Q)=\mathrm{Co}\{1T\} \cap \mathrm{Co}\{2M\} \subset FS(S) \quad (\text{since } \Delta(d) \supset \mathrm{Co}\{2M\}).$$

The primal solution defined by theorem 7 satisfies all 2-matching inequalities as well as the subtour elimination constraints.

The approximation of the symmetric TSP polytope given by Lagrangean decomposition is very close to that of [Crowder and Padberg (1980)], since the portion of comb inequalities generated compared to all the facets generated is only 1.1%, the rest being either subtour elimination constraints or 2-matching inequalities.

4. CONCLUSION

The decomposition of an integer programming problem into many subproblems which share the constraints of the original problem can yield bounds substantially better than "standard" Lagrangean relaxation bounds. Lagrangean decomposition introduces many additional ways of decomposing and/or relaxing a given problem. Careful analysis of the geometric structure of candidate decompositions and/or relaxations is required for successful development of branch and bound algorithms.

Optimization of the Lagrangean function associated with a particular decomposition scheme is mostly an unexplored area. Lagrangean dual ascent methods appear to be particularly well suited, as one can probably predict with enough accuracy the implications of multiplier changes on the two subproblems and their solutions. Further research is required to clarify the structure of the Lagrangean dual especially as the number of copies increases.

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