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ON INTERVAL GRAPHS AND MATRICE PROFILES (*)

by Alain BILLIONNET [1]

Abstract. — *If an undirected graph is the intersection graph of a set of intervals of the real line, it is called an interval graph and the set of intervals is called an interval representation of the graph. In this paper, we recall a characterization of an interval graph given by Tarjan. This characterization allows us to show that the problem of minimizing the envelope size of a sparse symmetric matrix is NP-complete. Then we give a short proof of a known result about a Turan type problem for interval graphs. We prove also a new result on the decomposition of a graph in an intersection of interval graphs. The end of the paper is concerned by the chronological orderings of interval graphs. We give an* $O(|E|)$ *method to determine whether an interval graph has a representation satisfying relative positions of the intervals.*

Keywords: Interval graphs; matrice profiles; optimization; NP-complete.

Résumé. — *Un graphe d'intervalles est le graphe d'intersection d'un ensemble d'intervalles de la droite réelle. Dans cet article nous rappelons une caractérisation des graphes d'intervalles donnée par Tarjan. Cette caractérisation nous permet de démontrer que le problème de la minimisation du profil d'une matrice creuse et symétrique est NP-complet. Nous donnons ensuite une preuve très courte d'un résultat connu concernant un problème de type problème de Turan sur un graphe d'intervalles. Nous démontrons également un résultat nouveau sur la décomposition d'un graphe en intersections de graphes d'intervalles. La fin de l'article concerne les ordres chronologiques que l'on peut associer à un graphe d'intervalles. Nous proposons une méthode de complexité* $O(|E|)$ *pour déterminer s'il existe, pour un graphe d'intervalles donné, un ensemble d'intervalles associé qui respectent un ordre fixé des extrémités de ces intervalles.*

Mots clés : Graphe d'intervalles; profil de matrice; optimisation; NP-complet.

1. INTRODUCTION

$G=(V, E)$ is an interval graph if there exists a set $\{I_1, \ldots, I_n\}$ of intervals of the real line such that, for $i \neq j$, $\{v_i, v_j\} \in E$ iff $I_i \cap I_j \neq \varnothing$. We recall in section 2 the definition of an interval graph given by Tarjan.

The envelope size of a n by n symmetric matrix A with entries $a_{ij}(a_{ii} \neq 0)$ is equal to $\sum_{i=1}^{n} [i - f_i(A)]$ where $f_i(A) = \min \{j \mid a_{ij} \neq 0\}$. In section 3 we consider the problem of reducing the envelope size of a sparse symmetric matrix. The definition of section 2 allows us to show that this problem is equivalent to the

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minimum completion of an interval graph. So we prove that the problem of minimizing the envelope size of a symmetric matrix is NP-complete.

In section 4 we deal with the connections which exist between, on the one hand, envelope of a symmetrix matrix and interval graphs and, on the other hand, between fill in gaussian elimination process and triangulated graphs.

We show in section 5 how the definition of an interval graph presented in section 2 allows us to easily prove a known result about a Turan type problem for interval graphs. This result concerns the largest integer c such that any interval graph with n vertices and at least m edges contains a complete subgraph on c vertices.

The section 6 is concerned by the intersection of interval graphs. (The intersection of several graphs on the same vertex set V is the graph in which two vertices in V are joined by an edge just when they are so joined in all the given "factor" graphs). We give a new result on the decomposition of a graph in an intersection of interval graphs. As a consequence of this result we give an original proof of the known following property:

every graph on v vertices is the intersection of $\left\lfloor \frac{1}{2} v \right\rfloor$ or fewer interval graphs.

The section 7 is concerned by the chronological ordering of interval graphs. Let $\{I_1, \ldots, I_n\}$ denote an interval representation of G in which the left [resp., right] endpoint of interval I_i is l_i [resp., r_i]. Let V_R denote the set $\{r_1, \ldots, r_n\}$ and V_L the set $\{l_1, \ldots, l_n\}$. The question we discuss in this section is: given an interval graph G, which linear orderings of V_R and V_L, respectively, give chronological orderings of G. We present a theorem which is an improvement of the known results about this question. This theorem is a consequence of the definition of interval graphs that we give in section 2.

2. INTERVAL GRAPHS

2.1. Definitions

Let an undirected graph $G = (V, E)$ have vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The graph G is called an *interval graph* if there exists a set $\{I_1, \ldots, I_n\}$ of intervals of the real line such that, for $i \neq j$:

$$\{v_i, v_j\} \in E \Leftrightarrow I_i \cap I_j \neq \varnothing.$$

The set $\{I_1, \ldots, I_n\}$ is called an interval representation of G (G has many different interval representations, differing not only in the lengths of the intervals, but also in the relative positions of these intervals).

An ordering (labelling) of $G = (V, E)$ is a mapping of $\{1, 2, \ldots, n\}$ onto V. Let A be the adjacency matrix associated with the labelled graph G. It is the n by n boolean symmetric matrix with entries a_{ij} such that $a_{ij} = 1$ if and only if $\{v_i, v_j\} \in E$ or $i = j$. (Here v_i denotes the node of V with label i.)

2.2. A characterization of an interval graph

THEOREM 1 [11 *bis*]: *$G = (V, E)$ is an interval graph if and only if there exists an ordering of G such that the associated adjacency matrix A verifies:*

$$(\mathscr{P}): \quad \forall i \in \{1, 2, \ldots, n\}, \quad a_{ij} = 1 \qquad \textit{for} \qquad j = f_i(A), f_i(A) + 1, \ldots, i.$$

Proof: The condition is sufficient.

For each i let us consider on the real line the i-th interval $I_i =]f_i(A) - 1, i[$. Let I_j and I_k be two intervals with $j < k$.

$$I_j \cap I_k \neq \varnothing \Leftrightarrow]f_j(A) - 1, j[\cap]f_k(A) - 1, k[\neq \varnothing$$
$$\Leftrightarrow j > f_k(A) - 1 \Leftrightarrow j \geqslant f_k(A) \Leftrightarrow a_{kj} = 1.$$

Therefore $(I_1, I_2, \ldots, I_n)$ is an interval representation of G.

The condition is necessary.

Let $G = (V, E)$ be an interval graph and $I_i =]l_i, r_i[$ $(i = 1, 2, \ldots, n)$ an interval representation of G. Suppose that the intervals are numbered in such a way that $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n$. Let us prove that if $a_{ij} = 1$ $(j < i)$ then $a_{ik} = 1$ for each k such that $j < k < i$.

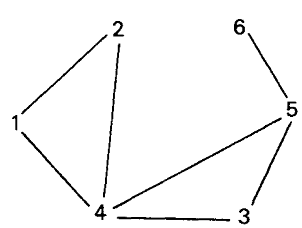
$$a_{ij} = 1 \Rightarrow I_i \cap I_j \neq \varnothing \Rightarrow]l_i, r_i[\cap]l_j, r_j[\neq \varnothing$$
$$\Rightarrow l_i < r_j (\text{since } r_j \leqslant r_i) \Rightarrow l_i < r_k.$$

$l_i < r_k$ and $r_k \leqslant r_i$ implies :

$$]l_i, r_i[\cap]l_k, r_k[\neq \varnothing \Rightarrow a_{ik} = 1.$$

EXAMPLE:

G



$$A \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 1. — **G is an interval graph since there exists a numbering of its nodes such that the adjacency matrix A verifies the property ($\mathscr{P}$).**

COROLLARY 1: *Let A be the adjacency matrix of an interval graph G which satisfies the property $(\mathscr{P})$ of theorem 1. Then for each $j \in \{1, \ldots, n\}$ the set $C_j = \{v_i \mid i \geqslant j$ and $a_{ij} = 1\}$ is a complete subgraph of G.*

Proof: Let us suppose that $\{v_k, v_l\} \subset C_j$ with $k < l$.

$$v_l \in C_j \Rightarrow a_{lj} = 1 \Rightarrow a_{lk} = 1 \qquad \text{since} \qquad j \leqslant k < l.$$

COROLLARY 2: *The maximum complete subgraph including v_j and some vertices v_k such that $k > j$ is C_j.*

Proof: Let us denote $\Gamma(v_j)$ the set of vertices adjacent to v_j:

$$\Gamma(v_j) \cap \{v_{j+1}, \ldots, v_n\} = C_j - \{v_j\}.$$

COROLLARY 3: *A maximum complete subgraph of G is C_{j_0} with $|C_{j_0}| = \max_{j=1,\ldots,n} |C_j|$.*

The proof is obvious after corollary 2.

3. REDUCING THE PROFILE OF A SPARSE MATRIX

3.1. Cholesky's method for sparse matrix factorization ([5], chap. 2)

Suppose the given system of equations to be solved is:

$$Ax = b$$

where A is an n by n symmetric, positive definite coefficient matrix, b is a vector of length n and x is the solution vector of length n. Applying Cholesky's method to A yields the triangular factorization:

$$A = LL^T$$

where L is lower triangular with positive diagonal elements. If A is symmetric and positive definite then such a factorization always exists.

The system of equations becomes:

$$LL^T x = b$$

and by substituting $y = L^T x$ we obtain x by solving the triangular systems:

$$Ly = b \qquad \text{and} \qquad L^T x = y.$$

The most important fact about applying Cholesky's method to a sparse matrix A is that the matrix usually suffers fill-in. That is L has nonzeros in positions which are zeros in the lower triangular part of A. However for most sparse matrix problems a judicious reordering of the rows and columns of the coefficient matrix can lead to enormous reductions in fill-in, and hence savings in computer execution time and storage. This task of finding a good ordering is central to the study of the solution of sparse positive definite systems.

3.2. The envelope method

3.2.1. *Definitions*

One of the simplest methods for solving sparse systems is the band scheme and the closely related envelope or profile method. Loosely speaking the objective is to order the matrix so that the nonzeros in the obtained matrix are clustered near the main diagonal because this property is retained in the corresponding Cholesky's factor L. We consider here the envelope method.

Let be an n by n symmetric positive definite matrix, with entries a_{ij}. For the i-th row of A let:

$$f_i(A) = \min \{j \mid a_{ij} \neq 0\}$$

the envelope of A, denoted by Env (A) is defined by:

$$\text{Env}\,(A) = \{\{i, j\} \mid f_i(A) \leqslant j < i\}$$

the quantity $|\text{Env}\,(A)|$ is called the profile of envelope size of A and is given by:

$$|\text{Env}\,(A)| = \sum_{i=1}^{n} [i - f_i(A)].$$

EXAMPLE:

A :

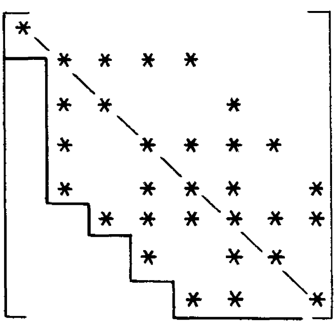


Figure 2. — A matrix A whose the envelope size is 15 (nonzeros are depicted by $*$).

The envelop method consists in ignoring the zeros outside Env (A) because Env (A) = Env (L). Although the orderings obtained by this method are often far from optimal in the least arithmetic or least-fill senses, they are often an attractive practical compromise because the programs and data structures needed to exploit the sparsity that these orderings provide are relatively simple.

3.2.2. *Minimization of the envelope size*

The problem that we consider here is to find, for a symmetric and positive definite matrix A a reordering of the rows and columns of A which minimizes the envelope size of the obtained matrix.

For a matrix A, if $a_{ij} \neq 0 \, \forall \{ i, j \} \in \text{Env}(A)$ then we say that the envelope of A is full.

PROPERTY 1. — Let A be a symmetric positive definite matrix A, there exists a reordering $\tilde{A}$ of A with a full envelope if and only if the associated graph G is an interval graph. (The graph $G = (V, E)$ associated to a n by n symmetric matrix is one for which the n vertices are numbered from 1 to n and $\{ x_i, x_j \} \in E$ if and only if $a_{ij} = a_{ji} \neq 0$, $i \neq j$.)

The proof is obvious after theorem 1.

The problem of minimizing the envelope size of a matrix A is equivalent to that of minimizing the number of zeros inside the envelope. In terms of graph this problem is equivalent to that of finding the minimum number of edges which must be added to the associated graph G^A to obtain an interval graph.

THÉORÈME 2: *Let A be a symmetric matrix with entries a_{ij} ($a_{ii} \neq 0 \; \forall i$) and K a non negative integer.*

Consider the question: is there a reordering $\tilde{A}$ of A such that:

$$\left| \{ a_{ij} \mid \{ i, j \} \in \text{Env}(\tilde{A}) \text{ and } a_{ij} = 0 \} \right| \leqslant K?$$

This decision problem is NP-complete.

Proof: It follows immediately the result on interval graph completion (Garey and Johnson [4]):

Let $G = (V, E)$ be a graph and K be a non negative integer. The problem "is there a superset E' containing E such that $|E' - E| \leqslant K$ and the graph $G' = (V, E')$ is an interval graph?" is NP-complete.

Since it is clear that a graph $G = (V, E)$ admits a superset $E' \supseteq E$ such that $|E' - E| \leqslant K$ and $G' = (V, E')$ is an interval graph if and only if there exists a reordering $\tilde{A}$ of the adjacency matrix A associated to G such that $\left| \{ a_{ij} \mid \{ i, j \} \in \text{Env}(\tilde{A}) \text{ and } a_{ij} = 0 \} \right| \leqslant K$.

4. TRIANGULATED GRAPHS, INTERVAL GRAPHS AND GAUSS ELIMINATION

4.1. Triangulated graphs and interval graphs

A vertex x of $G = (X, E)$ is called simplicial if its adjacency set $\Gamma(x)$ induces a complete subraph of G.

Let $\tau = (x_1, x_2, \ldots, x_n)$ be an ordering of the vertices. We say that τ is a perfect elimination scheme if each x_i is a simplicial vertex of the subgraph induced by $\{ x_i, x_{i+1}, \ldots, x_n \}$.

Let us recall that G is triangulated if and only if G has a perfect elimination scheme (Fulkerson and Gross [3]). Let us recall also that a graph G is an interval graph if and only if G is a triangulated graph and its complement $\overline{G}$ is a comparability graph (Gilmore and Hoffman [8]). For more details about interval and triangulated graphs the reader can see: Golumbic [9], chap. 4 and 8.

4.2. Gauss elimination

Let G_0 be the graph associated with a symmetric matrix A. The process of Gauss elimination applied to A can be interpreted as a sequence of graph transformations on G_0.

Let $G=(X, E)$ be a graph and y be a node in G: $X=\{x_1, x_2, \ldots, x_n\}$. The elimination graph of G by y, denoted by G_y, is the graph:

$$(X-\{y\},\ E(X-\{y\}) \cup \{\{u, v\} \mid u, v \in \Gamma(y)\}).$$

With this definition, the process of Gaussian elimination on A can be viewed as a sequence of elimination graphs $G_0, G_1, \ldots, G_{n-1}$ where $G_i=(G_{i-1})_{x_i}$ for $i=1, 2, \ldots, n-1$.

The graph G_i precisely reflects the structure of the matrix after the i-th step of the Gaussian elimination. Let us denote E_i the set of edges of G_i. The fill can be expressed as $\sum_{i=1}^{n-1} |E_i - E_{i-1}|$. A judicious numbering of the nodes can drastically reduce fill. (A heuristic algorithm which experience has shown to be extremely effective in finding low-fill orderings is the so-called minimum degree algorithm [5] [6].)

THEOREM 3 [10 *bis*]: *Let A be a matrix and G_0 the associated graph. The minimum fill is equal to the minimum number of edges which must be added to G_0 to form a triangulated graph.*

The proof is a direct consequence of the definition of the fill and of the previous characterization of a triangulated graph.

REMARK: This problem of triangulated graph completion is NP-complete [13].

We can now formulate the theorem 4 which gives a connection between fill in the envelope method and fill in the general Gaussian elimination process. (We call fill in the envelope method the number of zero elements which belong to the envelope.)

THEOREM 4: *Given a matrix A and its associated graph G_0, the additional fill in the optimal envelope method with regard to the fill in the optimal Gaussian elimination process is equal to the minimum number of edges which must be added to*

G_0 to form an interval graph minus the minimum number of edges which must be added to G_0 to form a triangulated graph.

Proof: It is a direct consequence of section 3.2.2 and of theorem 3 associated with the fact that an interval graph is a triangulated graph (the reverse is false).

5. A TURAN TYPE PROBLEM FOR INTERVAL GRAPHS

Consider the following analogue of a problem of Turan for interval graphs: let $c = c(n, m)$ be the largest integer such that any interval graph with n vertices and at least m edges contains a complete subgraph on c vertices, determine the value of $c(n, m)$ explicitely. H. Abbott and M. Katchalski [1] have proved that:

$$c(n, m) = \left\lceil n + \frac{3}{2} - \sqrt{\left(n - \frac{1}{2}\right)^2 - 2m} \right\rceil - 1$$

we give here a simple proof of this result which is a direct consequence of theorem 1 and of its first corollary.

LEMMA 1: *If L is a lower triangular matrix with no more than d nonzeros in each column then the total number of nonzeros in L is not greater than* $-\frac{d^2}{2} + \frac{d}{2} + nd.$

The proof is obvious.

THEOREM 5: *Let $c = c(n, m)$ be the largest integer such that any interval graph with n vectices and at least m edges contains a complete subgraph on c vertices, then:*

$$c(n, m) = \lceil n + 3/2 - \sqrt{(n - 1/2)^2 - 2m} \rceil - 1.$$

Proof: Let $G = (V, E)$ be an interval graph with n vertices and m edges. Let us consider a numbering of the nodes of G which verifies the condition of theorem 1.

After corollary 1 of theorem 1 and previous lemma if $m + n > -\frac{d^2}{2} + \frac{d}{2} + nd$ then there exists a complete subgraph with at least $d + 1$ vertices.

The greatest value of d which verifies this last inequality must be $< d_{\max} = \left(n + \frac{1}{2}\right) - \sqrt{\left(n - \frac{1}{2}\right)^2 - 2m}$. Hence $c(n, m) = \lceil d_{\max} + 1 \rceil - 1$.

6. ON INTERSECTIONS OF INTERVAL GRAPHS

Given several undirected simple graphs on the same vertex set V, their edge product or "intersection" is the graph in which two vertices in V are joined by an edge just when they are so joined in all of the given ("factor") graphs.

Let $G = (V, E)$ be an undirected simple graph. If vertices a and b are not adjacent let us consider an ordering of G which begins by a and ends by b with all the vertices adjacent to b immediately before b. Let us note $G(a, b)$ the obtained graph, $A(a, b)$ its adjacency matrix and $\hat{G}(a, b) = (V, \hat{E}(a, b))$ a new graph such that for each $\{r, s\}$ $r \neq s$, $\{v_r, v_s\} \in \hat{E}(a, b)$ if and only if $\{r, s\} \in \mathrm{Env}\,(A(a, b))$.

After theorem 1 it is clear that $\hat{G}(a, b)$ is an interval graph.

THEOREM 6 : *Let $G = (V, E)$ be an undirected simple graph, a and b two non adjacent vertices. Then G is the edge product of $\hat{G}(a, b)$ with p or fewer interval graphs where p is the number of edges in a maximal matching on the graph $(V, \hat{E}(a, b) - E)$.*

Proof: Let $(e_1, e_2. \ldots, e_p)$ such a maximal matching. Let us note $e_i = \{a_i, b_i\}$.

Let us prove that G is the edge product of $\hat{G}(a, b)$ with $\hat{G}(a_i, b_i)$ $(i = 1, \ldots, p)$.

It is clear that each edge of G is an edge of $\hat{G}(a, b)$ and an edge of $\hat{G}(a_i, b_i)$ for each $i \in \{1, \ldots, p\}$.

Let $\{x, y\}$ be an edge of $\hat{E}(a, b) - E$. x or y is an endpoint of an edge of the matching. Let us suppose (without loss of generality) that $\{x, z\}$ is an edge of the matching, then $\hat{G}(x, z)$ does not contain the edge $\{x, y\}$.

COROLLARY : *Every graph on v vertices is the edge product of $\lfloor v/2 \rfloor$ or fewer interval graphs. ($\lfloor \alpha \rfloor$ = greatest integer not exceeding α.)*

Proof : First let us remark that a and b are isolated vertices in the graph $(V, \hat{E}(a, b) - E)$. Therefore a maximal matching in this graph contains no more than $(v - 2)/2$ edges i.e. $p \leqslant (v - 2)/2$ which implies $p + 1 \leqslant \lfloor v/2 \rfloor$.

The upper bound $\lfloor v/2 \rfloor$ is due to F. Roberts [10]. This result has been also given by H. S. Witsenhausen [12]. His proof is based on investigations of finite families of finite sets with the Helly property.

7. CHRONOLOGICAL ORDERINGS OF INTERVAL GRAPHS

Let $G(V, E)$ be an interval graph with n vertices, and let $\{I_1, \ldots, I_n\}$ denote an interval representation of G in which the endpoints of the intervals are all distinct. Let P denote the set $\{l_1, \ldots, l_n, r_1, \ldots, r_n\}$. If we associate the left [resp., right] endpoint of interval I_i with l_i [resp. r_i] from P, for $i = 1, \ldots, n$ then the linear order of the endpoints of the intervals along the real line induces a linear ordering of the elements of P.

We study here those linear orderings of P induced by an interval representation of an interval graph G. We call such linear orderings of P chronological orderings of G.

D. Skrien [11] has proved that to completely describe a chronological ordering of a graph, all that is needed is the linear order of the subset $V_R = \{ r_1, \ldots, r_n \}$ of P and of the subset $V_L = \{ l_1, \ldots, l_n \}$.

The question we discuss here is: given an interval graph G, which linear orderings T_R and T_L of V_R and V_L, respectively, give chronological orderings of G. The theorem 7 gives a condition on T_R necessary and sufficient for there exists T_L such that T_R and T_L give a chronological ordering of G.

In fact, this theorem is stronger since it characterizes interval graphs, in that G is an interval graph iff there exists linear ordering T_R with the stated property.

The theorem 8 gives two conditions on T_R and T_L necessary and sufficient for them to give a chronological ordering of G. In fact this theorem is also slightly stronger since it characterizes interval graphs, in that G is an interval graph iff there exists linear orderings T_R and T_L with the two stated properties.

THEOREM 7: *Let $G = (V, E)$ be a graph and T_R a linear ordering of V_R. Then G is an interval graph for which there exist a chronological ordering which respects T_R if and only if T_R have the following property:*

If $(r_i, r_j) \in T_R$ and $(v_i, v_j) \in E$ then *$(r_i, r_k) \in T_R$ and $(r_k, r_j) \in T_R$ implies $\{ v_k, v_j \} \in E$.*

Proof: The condition is necessary.

Let us consider the adjacency matrix of G obtained by numbering the nodes of G with respect to T_R. Then the property of T_R follows immediately the proof of theorem 1 in the section 2.2.

The condition is sufficient.

T_R has the property of the theorem. Let us number the n nodes of G according to the linear ordering T_R. The property of T_R implies that the adjacency matrix A of G verifies: $\forall i \in \{ 1, \ldots, n \}$ $a_{ij} = 1$ for $j = f_i(A) + 1, \ldots, i$ and after theorem 1 of section 2.2, G is an interval graph.

THEOREM 8: *Let $G = (V, E)$ be a graph and T_R and T_L linear orderings of V_R and V_L respectively. Then G is an interval graph for which T_R and T_L give a chronological ordering if and only if T_R and T_L have the following properties:*

(a) *the adjacency matrix A obtained by numbering the nodes of G according to T_R verifies the property of theorem* 1

(b) $(l_i, l_j) \in T_L \Rightarrow f_i(A) \leqslant f_j(A)$.

Proof: The condition is necessary.

The property (a) follows immediately theorem 1.

Let us suppose $f_i(A) > f_j(A)$ and let us note $k = f_j(A)$. (First let us suppose $k < j$.) That implies $(r_k, r_j) \in T_R$, $(r_k, r_i) \in T_R$, $I_i \cap I_k = \varnothing$ and $I_j \cap I_k \neq \varnothing$.

Case 1: $(r_i, r_j) \in T_R$ (see fig. 3)

$$\left.\begin{array}{l} I_i \cap I_k = \varnothing \Rightarrow l_i > r_k \\ I_j \cap I_k \neq \varnothing \Rightarrow l_j < r_k \end{array}\right\} \Rightarrow (l_i, l_j) \notin T_L$$

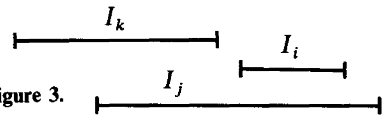


Figure 3.

Case 2: $(r_j, r_i) \in T_R$ (see fig. 4)

$$\left.\begin{array}{l} I_i \cap I_k = \varnothing \Rightarrow l_i > r_k \\ I_j \cap I_k \neq \varnothing \Rightarrow l_j < r_k \end{array}\right\} \Rightarrow (l_i, l_j) \notin T_L.$$

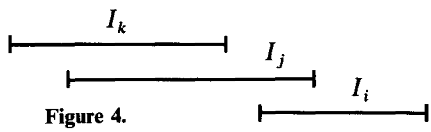


Figure 4.

Now let us suppose $f_j(A) = j$. That implies $(r_j, r_i) \in T_R$, $I_i \cap I_j = \varnothing$ and hence $l_i > r_j > l_j$.

The condition is sufficient.

If the property (a) is true then G is an interval graph after theorem 1. As in the proof of this theorem let us consider the interval representation of G constituted by the following intervals on the real line: $I_i =]\, f_i(A) - 1, i\,[$ for $i = 1, \ldots, n$. Let us note, $\sigma(i)$ the position of l_i in the linear order T_L. The interval representation of G, $I_i (i = 1, \ldots, n)$ induces another one: $I'_i = [f_i(A) - 1 + \sigma(i) \,.\, \varepsilon, i]$ $(i = 1, \ldots, n)$ with $0 < \varepsilon < 1/n$. (All the endpoints of these closed intervals are distinct).

It is clear that if T_R and T_L verifies the propoerties (a) and (b) they give a chronological ordering of G since it is the chronological ordering which corresponds to the set of intervals $I'_i (i = 1, \ldots, n)$.

Now let us compare this result with that of D. Skrien [11]: for each vertex v of $G = (V, E)$ we define the closed neighborhood:

$$N(v) = \{ w \in V : \{ v, w \} \in E \text{ or } v = w \}.$$

THEOREM 9: *Let $G = (V, E)$ be a graph and T_R and T_L linear orderings of V_R and V_L respectively. Then G is an interval graph for which T_R and T_L give a chronological ordering if and only if T_R and T_L have the following properties: For all i, j, k.*

(i) if $(r_i, r_j) \in T_R$ and $v_k \in N(v_i) - N(v_j)$, then $(l_k, l_j) \in T_L$, and

(ii) if $(l_i, l_j) \in T_L$ and $v_k \in N(v_j) - N(v_i)$ then $(r_i, r_k) \in T_R$.

Clearly this theorem gives us an algorithm for determining whether T_R and T_L give chronological orderings of G in time $0(|V|^3)$.

The theorem 8 allows us to resolve the same problem with an algorithm in time $0(|E| + |V|)$.

Suppose that we represent the graph G by its adjacency lists. For each node $v \in V$ we record the set of nodes adjacent to it (the nodes are supposed to be numbered according to T_R). Let us first calculate $f_i(A)$ for $i = 1, \ldots, |V|$. That can be done in $0(|E|)$ time. Then the property (*a*) can be verified in time $0(|E|)$ and the property (*b*) in time $0(|V|)$.

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