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A NOTE ON MINIMUM-DUMMY-ACTIVITIES PERT NETWORKS (*)

by Marian MROZEK [1]

Abstract. — *In the paper we present a polynomial-time method of verification if solutions to the minimum-dummy-activities problem in PERT networks produced by some suboptimal algorithms are optimal.*

Keywords: Network construction, network analysis, PERT networks, arc-dual digraph.

Résumé. — *Dans cet article, nous présentons une méthode à temps polynomial pour vérifier si les solutions au problème d'activités fictives minimum dans les réseaux de Pert données par des algorithmes suboptimaux sont en réalité suboptimales.*

1. INTRODUCTION

The problem of the construction of an event-node PERT network which minimizes the number of vertices and dummy activities has been studied by many authors (the detailed bibliography can be found in [7]). The complete solution to the minimum-vertices problem was given by Cantor and Dimsdale [2] in 1969. In the same year Hayes [8] observed that the number of vertices and the number of dummy arcs cannot be minimized simultaneously in general. In 1979 Krishnamoorthy and Deo [4] proved that the minimum-dummy-activities problem is NP-complete. According to their result Sysło suggested searching for a polynomial approximate algorithm and presented one in [7].

In the paper we consider the problem of the construction of a minimum-dummy-activities event-node PERT network in the class of all minimum-event

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networks. We prove that the problem has exactly one solution in a certain subclass of solutions and that the solution may be found in polynomial time on the base of algorithms presented by Cantor and Dimsdale [2], Sterboul and Wertheimer [6] and Mrozek [5]. Relatively often the above solution is also optimal in the general case, which may be verified in polynomial time too.

2. NOTATION

Let (G, S) be a directed finite graph (or simply a digraph), where G is the set of its vertices and a relation $S \subset G \times G$ is the set of its arcs. For an arc $s \in S$ its initial and terminal vertices will be denoted s^- and s^+ respectively, i. e. $s=(s^-, s^+)$. Let $a, b \in G$ and let $\{s_i\}_{i=1}^n$, $n \geqq 1$, be a sequence of arcs in S such that $a=s_1^-$, $s_n^+=b$. If $s_i^+=s_{i+1}^-$ for $i=1, 2, \ldots, n-1$ then the sequence will be called a path in S from a to b. We will say that the path is trivial if $\{s_i^-\} \cup \{s_i^+\} \subset \{a, b\}$. The digraph (G, S) will be called acircuit if for any vertex $a \in G$ there is no path from a to a.

The relation :

$$tc\, S := \{(a, b) \in G^2 : \quad \text{there exists a path in } S \text{ from } a \text{ to } b\},$$

will be called the transitive closure of S.

The relation:

$$tr\, S := \{(a, b) \in S : \quad \text{every path in } S \text{ from } a \text{ to } b \text{ is trivial}\},$$

will be called the transitive reduction of S. We will also use the notation:

$$tc_0\, S := tc\, S \cup \{(a, a) : a \in G\}.$$

REMARK 2.1: The operations tr and tc satisfy the following conditions

$$tr\, S \subset S \subset tc\, S, \tag{2.1}$$

$$\text{for any } S' \subset S \quad tc\,(tr\, S \cup S') = tc\, S, \tag{2.2}$$

$$tr\,(tc\, S) = tr\, S. \tag{2.3}$$

For any set A, its cardinality will be denoted by card A.

3. THE PROBLEM

Let (H, T) be an acircuit digraph. We recall (*see* [2]) that the triple (G, S, k) is called an arc-dual digraph of (H, T) if (G, S) is an acircuit digraph and

$k: H \to S$ is a mapping such that:

$$\forall\, h_1,\, h_2 \in H\ (h_1,\, h_2) \in tc\, T \quad \Leftrightarrow \quad (k\,(h_1)^+,\, k\,(h_2)^-) \in tc_0\, S.$$

Denote $S^r := \{s \in S : \exists\, h \in H : k\,(h) = s\}$, $S^f := S \setminus S^r$. Notice that in application to PERT networks (H, T) may be considered as an activity network and (G, S) as the corresponding event network, in which S^r and S^f represent real and dummy activities respectively.

Let $AD\,(H, T)$ denote the class of all arc-dual digraphs of (H, T). The following remark follows immediately from remark 2.1.

REMARK 3.1: If $(G, S, k) \in AD\,(H, T)$ then $(G, tc\,S, k) \in AD\,(H, T)$ and $(G, tr\,S \cup S^r, k) \in AD\,(H, T)$.

DEFINITION 3.1: Digraphs $(G_i, S_i, k_i) \in AD\,(H, T)$, $(i = 1, 2)$ are said to be weakly isomorphic if there exists a bijection $f: G_1 \to G_2$ such that the following diagram

$$\begin{tikzcd} & G_1 \times G_1 \arrow[dd, "f \times f"] \\ H \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & \\ & G_2 \times G_2 \end{tikzcd} \tag{3.1}$$

is commutative, i. e. for every $h \in H$:

$$f\,(k_1\,(h)^-) = k_2\,(h)^- \qquad \text{and} \qquad f\,(k_1\,(h)^+) = k_2\,(h)^+.$$

We distinguish the following two subclasses of $AD\,(H, T)$:

$$AD_0\,(H, T) := \{(G, S, k) \in AD\,(H, T) : G = G^+ \cup G^-\},$$
$$AD_1\,(H, T) := \{(G, S, k) \in AD_0\,(H, T) : \forall s \in S^f\ s^- \in G^+,\ s^+ \in G^-\}.$$

where:

$$G^+ := \{g \in G : \exists\, h \in H\ g = k\,(h)^+\},$$
$$G^- := \{g \in G : \exists\, h \in H\ g = k\,(h)^-\}.$$

DEFINITION 3.2: An arc-dual digraph $(G, S, k) \in AD\,(H, T)$ will be called vertex minimal if for every $(G', S', k') \in AD\,(H, T)$:

$$\text{card}\ G \leqq \text{card}\ G'.$$

A vertex minimal digraph $(G, S, k) \in AD(H, T)$ (or $AD_1(H, T)$) will be called arc-vertex minimal in $AD(H, T)$ (or in $AD_1(H, T)$) if for any other vertex minimal digraph $(G', S', k') \in AD(H, T)$ (or $AD_1(H, T)$):

$$\operatorname{card} S \leqq \operatorname{card} S'.$$

The following remark is obvious:

REMARK 3.2: Any vertex minimal digraph $(G, S, k) \in AD(H, T)$ belongs to $AD_0(H, T)$.

The following two theorems will be basic in the sequel. Since they are implicitly proved in [2], [5] and [6], we omit their proofs.

THEOREM 3.1: *Any two vertex minimal digraphs belonging to $AD(H, T)$ are weakly isomorphic.*

THEOREM 3.2: *There exists a vertex minimal digraph belonging to $AD_1(H, T)$, in particular $AD_1(H, T) \neq \emptyset$.*

According to theorem 3.1, further on we may assume that all vertex minimal graphs in $AD(H, T)$ have the same set of vertices, which we will denote by $G_{H,T}$. Obviously they have also the same mapping $k: H \to S$ and consequently the same set of real arcs S^r. Thus we may simply write S instead of $(G_{H,T}, S, k)$ in case of a vertex minimal digraph in $AD(H, T)$.

The following remark is an immediate consequence of the definition of $AD(H, T)$:

REMARK 3.3: Let S_1, S_2 be two vertex minimal digraphs in $AD(H, T)$. Assume $(a, b) \in G_{H,T}^+ \times G_{H,T}^-$. Then:

$$(a, b) \in tc\, S_1 \quad \Leftrightarrow \quad (a, b) \in tc\, S_2.$$

4. MAIN RESULT

LEMMA 4.1: *If $S \in AD(H, T)$ and $S_1 \in AD_1(H, T)$ are vertex minimal then:*

$$tc\, S \supset tc\, S_1. \tag{4.1}$$

Proof: To prove (4.1) it is enough to show that $S_1 \subset tc\, S$. Let $s \in S_1$. Since $S^r = S_1^r$, we may assume that $s \in S_1^f$. By the definition of $AD_1(H, T)$, there exist $h_1, h_2 \in H$ such that $k(h_1)^+ = s^-$ and $k(h_2)^- = s^+$. Hence $(h_1, h_2) \in tc\, T$ and consequently $(k(h_1)^+, k(h_2)^-) = s \in tc_0\, S$. Since $s^- \neq s^+$, we obtain $s \in tc\, S$. $\square$

LEMMA 4.2: *If* $S_1, S_2 \in AD_1(H, T)$ *are vertex minimal then:*

$$tc\, S_1 = tc\, S_2, \tag{4.2}$$

$$tr\, S_1 = tr\, S_2. \tag{4.3}$$

Proof: (4.2) follows immediately from lemma 4.1. To prove (4.3) observe that by (2.3) and (4.2):

$$tr\, S_1 = tr\,(tc\, S_1) = tr\,(tc\, S_2) = tr\, S_2. \quad \square$$

The following theorem follows immediately from the above lemma and remark 3.1.

THEOREM 4.1: *There exists exactly one arc-vertex minimal digraph in* $AD_1(H, T)$. *For any vertex minimal digraph* $S \in AD_1(H, T)$ *it equals* $tr\, S \cup S^r$.

Further on we will denote this unique in $AD_1(H, T)$ arc-vertex minimal digraph in $AD_1(H, T)$ by $S_{H, T}$.

The following relation in $G^2_{H, T}$ is important in the study of arc-vertex minimal digraphs in $AD(H, T)$:

$$A_{H, T} := \{(a, b) \in G^2_{H, T} : \forall (a', b') \in G^+_{H, T} \times G^-_{H, T}$$
$$(a', a) \in tc_0\, S_{H, T},\ (b, b') \in tc_0\, S_{H, T} \Rightarrow (a', b') \in tc_0\, S_{H, T}\}.$$

Its importance explains the following:

THEOREM 4.2: *For any vertex minimal digraph S in* $AD(H, T)$:

$$S \subset A_{H, T}.$$

Proof: Let $s \in S$. First assume that $s^- \in G^+_{H, T}$. If $s^+ \in G^-_{H, T}$ then it follows from remark 3.3 that $s \in tc_0\, S_{H, T}$ and consequently $s \in A_{H, T}$. Assume $s^+ \in G^+_{H, T}$ and let $b' \in G^-_{H, T}$, $(s^+, b') \in tc_0\, S_{H, T}$. Again by remark 3.3 $(s^+, b') \in tc_0\, S_{H, T}$, thus $(s^-, b') \in tc_0\, S$. Consequently $(s^-, b') \in tc_0\, S_{H, T}$ and $s \in A_{H, T}$. The remaining cases can be proved in a similar way. $\square$

THEOREM 4.3: *If* $A_{H, T} \subset tc\, S_{H, T}$ *then* $S_{H, T}$ *is the unique arc-vertex minimal digraph in* $AD(H, T)$.

Proof: Assume S is an arc-vertex minimal digraph in $AD(H, T)$. By theorem 4.2 $S \subset A_{H, T} \subset tc\, S_{H, T}$. Hence $S \in AD_1(H, T)$. It follows from theorem 4.1 that $S = S_{H, T}$. $\square$

THEOREM 4.4: *The verification of the assumptions of theorem* 4.3 *can be done in polynomial time.*

Proof: From the construction presented in [5] and [6] it follows that at least one vertex minimal digraph in $AD_1(H, T)$ can be found in polynomial time. In order to compute $S_{H,T}$ it is enough to construct the transitive reduction of any vertex minimal graph in $AD_1(H, T)$, which also may be done in polynomial time (*see* [1]). To verify the assumptions of theorem 4.3 it is now necessary to construct $tc\,S_{H,T}$ and $A_{H,T}$. It is well known that the transitive closure may be found in polynomial time. What concerns $A_{H,T}$ one can easily construct an algorithm, which analyses all possible quadruplets $(a', a, b, b') \in G^4_{H,T}$, i.e. needs $O(n^4)$ time, where n stands for the number of vertices in $G_{H,T}$. The verification of the inclusion $A_{H,T} \subset tc\,S_{H,T}$ can be obviously done in polynomial time. □

For a subset $B \subset A_{H,T} \setminus tc\,S_{H,T}$ define:

$$rd\,B := \{s \in S^f_{H,T} : \text{there exists a non-trivial path in } B \cup S_{H,T} \text{ from } s^- \text{ to } s^+\}.$$

Let $A_0 := \{s \in A_{H,T} \setminus tc\,S_{H,T} : rd\ s \neq \emptyset\}$.

THEOREM 4.5: *Assume that:*

$$A_0 \subset tr(A_0 \cup S_{H,T}) \tag{4.4}$$

and

$$\forall s, s' \in A_0 \qquad s \neq s' \quad \Rightarrow \quad rd\,s \cap rd\,s' = \emptyset. \tag{4.5}$$

Then $tr(A_0 \cup S_{H,T})$ is an arc-vertex minimal digraph in $AD(H, T)$. If $A_0 = \emptyset$ or there is only one element in A_0 then the assumptions (4.4) *and* (4.5) *are obviously satisfied. Additionally $A_0 = \emptyset$ is a necessary and sufficient condition for $S_{H,T}$ to be the unique arc-vertex minimal digraph in $AD(H, T)$. The verification of the assumptions* (4.4) *and* (4.5) *as well as the construction of $tr(A_0 \cup S_{H,T})$ can be done in polynomial time.*

Since the proof of the above theorem is mainly technical, we omit it.

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