

GILBERT LAPORTE

HÉLÈNE MERCURE

YVES NORBERT

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## OPTIMAL TOUR PLANNING WITH SPECIFIED NODES (\*)

by Gilbert LAFORTE <sup>(1)</sup>, Hélène MERCURE <sup>(1)</sup>  
and Yves NOBERT <sup>(2)</sup>

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**Abstract.** — *This paper considers the problem of determining the shortest circuit or cycle in a graph containing  $n$  nodes and such that (i) each of  $k$  nodes ( $k \leq n$ ) is visited exactly once; (ii) each of the remaining  $n-k$  nodes is visited at most once. A branch and bound algorithm for this problem is described. Results are presented for problems involving up to 200 nodes in the asymmetrical case and up to 80 nodes in the symmetrical case.*

**Keywords:** Travelling salesman problem; specified nodes; branch and bound.

**Résumé.** — *On considère le problème consistant à déterminer le circuit ou cycle le plus court dans un graphe contenant  $n$  nœuds et de telle sorte que (i)  $k$  nœuds ( $k \leq n$ ) soient visités chacun une et une seule fois; (ii) chacun des  $n-k$  nœuds restants soit visité au plus une fois. On décrit un algorithme de « branch and bound » pour ce problème et on présente des résultats numériques pour des problèmes asymétriques contenant jusqu'à 200 nœuds et des problèmes symétriques contenant jusqu'à 80 nœuds.*

**Mots clés :** Problème du voyageur de commerce; nœuds spécifiés; « branch and bound ».

### 1. INTRODUCTION

In its most common interpretation, the travelling salesman problem (TSP) consists of determining the shortest route for a salesman wishing to visit each of  $n$  cities once and only once [3]. Over the last three decades, this problem has attracted the attention of several operational researchers and has led to some significant developments in the O.R. field [6]. It is well known however that in its pure form, the TSP seldom fits the routing problems really

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<sup>(1)</sup> École des Hautes Études Commerciales de Montréal, 5255, avenue Decelles, Montréal H3T 1V6, Canada.

<sup>(2)</sup> Département des Sciences administratives, Université du Québec à Montréal, 1495, rue Saint-Denis, Montréal H3C 3P8, Canada.

encountered by salesmen since in practice, some of the restrictions of the TSP may be lifted (*see* for example [11]) while extra constraints may have to be incorporated (*see* for example [8, 10]).

The following problem studied by Saksena and Kumar [13], Dreyfus [4] and Ibaraki [7] applies to situations where only a subset of the  $n$  cities have to be visited by the salesman. More specifically, let  $N = \{1, \dots, n\}$  be a set of nodes (cities),  $K = \{1, \dots, k\}$  ( $k \leq n$ ) a set of "specified" nodes (i. e. those requiring a visit) and  $C = (c_{ij})$  a non-negative distance matrix defined on  $N^2$ . Each pair  $(i, j)$  defines an arc from  $i$  to  $j$ .

We wish to determine the shortest circuit passing through each node of  $K$  exactly once; we shall refer to this problem as the STSP (TSP with specified nodes). As was shown in [9], the difficulty of the problem depends largely on the nature of  $C$  and on the degree imposed on nodes in  $N - K$ :

(i) if  $C$  satisfies the triangle inequality (i. e. if  $c_{ij} \leq c_{ik} + c_{kj}$  ( $i, j, k \in N$ )), the STSP always reduces to:

— a TSP on  $N$  if the degree of nodes in  $N - K$  equals 2 (trivially) or if it is greater than or equal to 2. This last case can be explained as follows. Consider a node  $j$  in  $N - K$  and assume the degree of  $j$  must be greater than or equal to 2 in the optimal solution. Any feasible solution can be represented by a sequence of nodes in which  $j$  appears  $t$  times ( $t \geq 1$ ), i. e. it contains  $t$  subsequences of the form  $(i_l, j, k_l)$  where  $l = 1, \dots, t$ . All but one of these subsequences can be replaced by  $(i_l, k_l)$  since  $c_{i_l j} + c_{j k_l} \geq c_{i_l k_l}$ . Thus, all degrees will be equal to 2 in the optimal solution;

— a TSP on  $K$  if the degree of nodes in  $N - K$  is less than or equal to 2 or unspecified. Indeed, all nodes  $j$  of  $N - K$  can be eliminated by using the same argument as above. Therefore the problem presents little interest in this case.

(ii) if  $C$  does not satisfy the triangle inequality, three cases can be disregarded:

— the case where all nodes of  $N$  have a degree of 2 corresponds to a TSP on  $N$  and need not be considered;

— the case where all nodes of  $N - K$  must have a degree at least equal to 2 is similar to the shortest complete cycle problem treated in [11];

— when the degrees of nodes in  $N - K$  are left unspecified, it suffices to solve a TSP on  $K$  where each  $c_{ij}$  is replaced by the shortest distance between  $i$  and  $j$  (*see* [4]).

The only remaining case is that where  $C$  does not satisfy the triangle inequality and where the degrees of nodes in  $N - K$  must be at most 2. This short note presents an efficient algorithm for the solution of this problem.

## 2. IBARAKI'S APPROACH

The problem considered by Saksena and Kumar [13], Dreyfus [4] and Ibaraki [7] is that of determining the shortest path between a source and a sink, passing through a set of specified nodes exactly once and through the other nodes at most once. It can be shown that this problem is very similar to the STSP treated in this paper.

The following formulation for the STSP can be derived from Ibaraki's paper: let us define a binary variable  $x_{ij}$  as follows:

- (i) if  $i \neq j$ ,  $x_{ij}$  indicates the presence ( $x_{ij}=1$ ) or the absence ( $x_{ij}=0$ ) of an arc from node  $i$  to node  $j$  in the optimal solution;
- (ii) if  $i=j$ ,  $x_{ii}$  indicates whether node  $i$  is used ( $x_{ii}=0$ ) or not ( $x_{ii}=1$ ) in the optimal solution.

The problem is then to:

$$(P) \quad \text{minimize} \quad \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij},$$

subject to:

- (1)  $\sum_{i=1}^n x_{ij} = 1 \quad (j \in N),$
- (2)  $\sum_{j=1}^n x_{ij} = 1 \quad (i \in N),$
- (3)  $\sum_{i, j \in S} x_{ij} \leq |S| - 1 \quad (S \subset N, S \cap K \neq K, S \cap K \neq \emptyset),$
- (4)  $x_{ij} = 0 \text{ or } 1 \quad (i, j \in N).$

In this formulation,  $c_{ii}$  is set equal to:

- (i) 0 if  $i \in N - K$ ;
- (ii) an arbitrarily large number  $M$  if  $i \in K$ .

This ensures that  $x_{ii}$  will be equal to zero if  $i \in K$ ; therefore all nodes belonging to  $K$  will be used in the optimal solution.

Constraints (1), (2) and (4) require no explanations. Constraints (3) are imposed in order to eliminate illegal subtours. As in the case of the TSP [3], illegal subtours involving  $|S|$  nodes are eliminated by specifying that there may be at most  $|S| - 1$  arcs linking nodes of  $S$  in the optimal solution. In the STSP, illegal subtours are those which contain some but not all nodes of

$K$  since the optimal tour must contain all nodes of  $K$ . On the other hand, it is not necessary to consider cases where  $S \cap K = \emptyset$  since it is never advantageous (in terms of the objective function) to produce subtours disconnected from  $K$ . Thus, the optimal solution will contain only one subtour involving all nodes of  $K$  and possibly some of  $N - K$ .

It can be seen that:

- (i) if  $K = \emptyset$ , (i. e. constraints (3) are removed), (P) reduces to an assignment problem for which there exist efficient algorithms [1];
- (ii) if  $K = N$ , this formulation is identical to that of the asymmetrical TSP (see [3] for example);
- (iii) otherwise, the difficulty of the STSP should lie somewhere between that of the assignment problem and that of the TSP.

It is easy to solve (P) without resorting to the simplex method. Indeed, the relaxed problem containing only constraints (1) and (2) is an assignment problem. Constraints (3) and (4) can be handled by fixing some  $x_{ij}$ 's at 0 or 1 in a branch and bound tree. This is the essence of Shapiro's method for the TSP [14] later used by Ibaraki for the STSP. It can be summarized as follows.

- (i) At each node  $h$  of the search tree, we define  $E_h$ , the set of arcs excluded from the solution (at the first node of the tree,  $E_h = \emptyset$ ). An assignment problem constrained by  $E_h$  is solved.
- (ii) Consider the subtours contained in the solution at node  $h$ . If there is only one subtour, it constitutes a feasible solution and a backtracking procedure is applied. Otherwise, consider the subtour with the minimum number of arcs. This subtour is characterized by a set of nodes  $\{r_1, \dots, r_m\}$  and a set of arcs  $\{(r_1, r_2), \dots, (r_m, r_1)\}$ . To the descendant nodes  $j$  from node  $h$  are associated the following sets of excluded arcs:

$$E_j = E_h \cup \{(r_j, r_{j+1})\}$$

where  $r_{m+1} = r_1$ .

- (iii) Branching is always made on the pending node having the least lower bound for the problem.

- (iv) The procedure ends when all branches of the tree have been explored, according to the usual branch and bound rules.

Ibaraki reports computational results for a limited number of randomly generated problems involving 21 and 31 nodes, on the Kyoto University FACOM 230-60 computer. The assignment problems were solved by means of Munkres' algorithm [12].

### 3. AN IMPROVED ALGORITHM FOR THE STSP

The main attraction of this algorithm lies in the fact that at each node of the search tree, the problem solved is an assignment problem, and therefore, the solution remains integer during the whole process. Furthermore, when applied to the STSP, the algorithm really exploits the fact that the problem is a relaxation of the TSP, as fewer subtours than in the TSP need to be branched upon.

However, Ibaraki's results can be improved by taking advantage of recent developments in the construction of algorithms for the assignment problem [1] and for the TSP (see for example [2]). Further, we feel that additional computational tests are required to validate the suggested approach.

In their paper on the TSP, Carpaneto and Toth demonstrate that Shapiro's algorithm can be vastly improved by using a more efficient algorithm for the assignment problem [1] and by modifying the rule for generating subproblems. They use the partitioning scheme proposed by Garfinkel [5] to which they add a refinement. At each node  $h$  of the tree,  $E_h$  is defined as above and  $I_h$  is the set of all arcs included in the solution (at the first node of the tree,  $I_h = \emptyset$ ). Branching is made from the subtour with the minimum number of arcs not included in  $I_h$  (as opposed to the subtour with the minimum number of arcs as in [14] and in [5]).  $E_j$  is defined as in [14] and  $I_j$  as in [5]:

$$I_j = \begin{cases} I_h & \text{if } j=1 \\ I_h \cup \{(r_u, r_{u+1}) : u=1, \dots, j-1\} & \text{if } j>1 \end{cases}$$

The overall effect of this strategy is to drastically reduce the number of nodes in the search tree. The authors report results for TSP's ranging from 40 to 240 nodes.

We therefore suggest applying a similar approach to the STSP: i.e. we use the same relaxation as Ibaraki but the approach developed by Carpaneto and Toth for the solution of the assignment problems and for generation of the subproblems.

### 4. COMPUTATIONAL RESULTS

It is now well known that the computational performance of subtour elimination algorithms such as those described by Shapiro [14] and by Carpaneto and Toth [2] varies greatly according to whether  $C$  is symmetrical (i.e.  $c_{ij} = c_{ji}$  for all  $i, j \in N$ ) or not. In the first case, this type of approach is less efficient since a large number of subtours involving only two nodes are

generated and have to be eliminated; this phenomenon is far less frequent in asymmetrical problems. We have tested the algorithm on both types of problems.

The algorithm was first tested on a series of asymmetrical problems ranging from 80 to 200 nodes. The  $c_{ij}$ 's for these problems were randomly generated from a uniform distribution on  $[0, 100]$ . For each value of  $n$ ,  $k$  was successively set at  $n/4$ ,  $n/2$ ,  $3n/4$  and  $n$ . Five problems of each type were solved; table I reports average values. All problems were solved on the University of Montreal Cyber 173 computer, using an FTN4 compiler. Memory appears to be the main factor limiting the size of the largest problems which could be solved by the algorithm: all problems involving no more than 200 nodes could be solved within 20 seconds; however, problems containing more than 200 nodes required more than the maximum memory allowed (200000<sub>8</sub> words).

For a given value  $n$ , we observe that the time required to solve the problem is not monotonic with respect to  $k$ : there appears to be a peak at about  $n/2$  and a trough at  $3n/4$ . (The peak was noted by Ibaraki [7], but not the

TABLE I  
Results for asymmetrical problems  
(average values on 5 random problems)

$n$		$k = n/4$	$k = n/2$	$k = 3n/4$	$k = n$
80 . . . . .	Time	1.65	3.52	3.34	3.35
	AP <sup>(1)</sup>	9.0	28.0	33.8	41.0
	Q <sup>(2)</sup>	5.6	10.8	15.2	15.6
	Nodes <sup>(3)</sup>	3.0	4.4	5.8	6.6
120 . . . . .	Time	5.96	6.69	3.71	5.79
	AP	25.0	30.6	20.8	36.0
	Q	11.4	15.4	11.0	27.6
	Nodes	5.2	5.6	5.2	7.4
160 . . . . .	Time	8.52	10.9	6.66	6.59
	AP	24.8	39.8	56.4	57.2
	Q	10.0	22.4	11.4	16.2
	Nodes	6.2	8.6	6.2	5.6
200 . . . . .	Time	8.70	18.19	12.16	15.31
	AP	15.4	53.0	60.0	128.0
	Q	8.6	26.8	23.8	35.8
	Nodes	2.6	7.4	7.8	13.4

<sup>(1)</sup> AP: number of assignment problems solved.

<sup>(2)</sup> Q: number of subproblems inserted in the queue (see [2]).

<sup>(3)</sup> Nodes: number of nodes explored in the branch and bound tree.

trough.) These results were confirmed by solving several small problems ( $n=20$  and  $30$ ) with all values of  $k$  from  $1$  to  $n$ .

The algorithm was also tested on symmetrical problems not satisfying the triangle inequality. These were generated as in the asymmetrical case with the additional requirement that  $c_{ij}=c_{ji}$ . As expected, a very large number of subtours involving only two nodes were observed, leading to excessive memory requirements; computation times were also larger than in the asymmetrical cases but in general better than those reported in an earlier paper on the symmetrical STSP [9]. Results for problems ranging from  $20$  to  $80$  cities are reported in table II.

TABLE II  
Results for symmetrical problems  
[average values on the number of successful problems out of  $5^{(1)}$ ]

$n$		$k=n/5$	$k=2n/5$	$k=3n/5$	$k=4n/5$	$k=n$
20	Time	0.09	0.19	0.54	1.81	4.56
	AP <sup>(2)</sup>	6.8	10.2	25.2	77.2	179.6
	Q <sup>(3)</sup>	4.0	6.4	19.0	45.0	106.4
	Nodes <sup>(4)</sup>	3.8	5.8	13.4	38.6	88.6
40	Time	0.50	4.07	23.72	21.65(4)	(0)
	AP	18.2	79.2	390.6	315.2	
	Q	10.6	53.0	226.8	256.5	
	Nodes	10.4	39.2	197.8	162.0	
60	Time	2.91	23.62	(0)	(0)	(0)
	AP	49.6	203.0			
	Q	27.2	140.0			
	Nodes	25.4	103.2			
80	Time	14.8	(0)	(0)	(0)	(0)
	AP	145.2				
	Q	85.6				
	Nodes	77.0				

<sup>(1)</sup> Whenever the number of successful problems is less than  $5$ , this is indicated in brackets.

<sup>(2)</sup> AP: number of assignment problems solved.

<sup>(3)</sup> Q: number of subproblems inserted in the queue (see [2]).

<sup>(4)</sup> Nodes: number of nodes explored in the branch and bound tree.

## REFERENCES

1. G. CARPANETO and P. TOTH, *Solution of the Assignment Problem*, ACM Transactions on Mathematical Software, Vol. 6, No. 1, 1980, pp. 104-111.
2. G. CARPANETO and P. TOTH, *Some New Branching and Bounding Criteria for the Asymmetric Travelling Salesman Problem*, Management Science, Vol. 26, No. 7, 1980, pp. 736-743.



3. G. B. DANTZIG, D. R. FULKERSON and S. M. JOHNSON, *Solution of a Large Scale Travelling Salesman Problem*, Operations Research, Vol. 2, 1954, pp. 393-410.
4. S. E. DREYFUS, *An Appraisal of Some Shortest-Path Algorithms*, Operations Research, Vol. 17, 1969, pp. 395-412.
5. R. S. GARFINKEL, *On Partitioning the Feasible Set in a Branch-and-Bound Algorithm for the Asymmetric Travelling Salesman Problem*, Operations Research, Vol. 21, 1973, pp. 340-343.
6. M. HELD, *Influences of the Travelling Salesman Problem*, Paper presented at the XI. International Symposium on Mathematical Programming, Bonn, 1982.
7. T. IBARAKI, *Shortest Path Problems Visiting Specified Nodes*, Electronics and Communications in Japan, Vol. 53-A, 1970, pp. 10-18.
8. G. LAPORTE and Y. NOBERT, *A Branch and Bound Algorithm for the Capacitated Vehicle Routing Problem*, Operations Research Spektrum, Vol. 5, 1983, pp. 77-85.
9. G. LAPORTE and Y. NOBERT, *Finding the Shortest Cycle through k Specified Nodes*, Cahiers du GERAD G-83-07, École des Hautes Études Commerciales de Montréal, 1983, 11 pages.
10. G. LAPORTE, M. DESROCHERS and Y. NOBERT, *Two Exact Algorithms for the Distance Constrained Vehicle Routing Problem*, Networks, Vol. 14, 1984, pp. 161-172.
11. P. MILIOTIS, G. LAPORTE and Y. NOBERT, *Computational Comparison of Two Methods for Finding the Shortest Complete Cycle or Circuit in a Graph*, R.A.I.R.O. (recherche opérationnelle), Vol. 15, 1981, pp. 233-239.
12. J. MUNKRES, *Algorithms for the Assignment and Transportation Problems*, Journal of S.I.A.M., Vol. 5, No. 1, 1957, p. 32.
13. J. P. SAKSENA and S. KUMAR, *The Routing Problem with "K" Specified Nodes*, Operations Research, Vol. 14, 1966, pp. 909-913.
14. D. M. SHAPIRO, *Algorithms for the Solution of the Optimal Cost and Bottleneck Travelling Salesman Problems*, Ph. D. Thesis, Washington University, 1966, 157 pp.