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CRITERION OF MAXIMIZING THE EXPECTED QUIETNESS (INVARIANT BY HOMOTETHIES IN RELATION TO THE UTILITIES) (*)

by M. A. GIL ALVAREZ ([1])

Abstract. — *In this article we consider the general decision problem: we need information about certain parameter space. For the problem resolution we can perform some experiments or random variables.*

If we cannot perform two random variables simultaneously, we must make a comparison between them.

In the present papaer, we formule a criterion for the comparison of experiments, which take in account the quietness that the knowledge of a performance provides about the parameter space.

Keywords: unquietness; quietness; decision problem; comparison of experiments.

Résumé. — *Dans cet article nous considérons le problème général de la décision : on a besoin d'information sur l'ensemble des valeurs possibles de certain paramètre. Pour résoudre le problème nous pouvons effectuer des observations sur quelques expérimentations ou variables aléatoires.*

Si on ne peut pas observer deux variables aléatoires en même temps, on doit faire une comparaison parmi ceux-là.

On trouvera dans cet article l'établissement d'un critère pour la comparaison des expérimentations qui tient compte la quiétude que la connaissance d'une observation apporte sur l'ensemble des valeurs du paramètre.

Mots clés : inquiétude; quiétude; problème de décision; comparaison des expérimentations.

1. INTRODUCTION

Let $(\mathscr{E}, \mathscr{X}, A, \Theta)$ be a general decision problem, where $\mathscr{E}$ is the experiments space, $\mathscr{X}$ is the outcomes space, A is the actions space and, finally, Θ is the state of nature space (or parameter space).

An experiment $\mathscr{X}$ over Θ consists of a measurable space $(X, \mathscr{A})$ and a class of distributions $\{\mathscr{P}_\theta, \theta \in \Theta\}$ over it. We will denote this experiment by $\mathscr{X} = \{X; \mathscr{A}; \mathscr{P}_\theta, \theta \in \Theta\}$.

From here on, we will admit that there is an *a priori* distribution $p(\theta)$ over a σ-field $\mathscr{F}$ of Θ, absolutely continuous with respect to a certain measure α over $\mathscr{F}$, and that for each experiment $\mathscr{X} = \{X; \mathscr{A}; \mathscr{P}_\theta, \theta \in \Theta\}$ the conditioned distributions $\mathscr{P}_\theta$ are absolutely continuous with respect to a certain σ-finite measure μ over Θ, where this class of distributions along with $p(\theta)$ determines

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a distribution over the measurable space $(\Theta \times X; \mathscr{F} \times \mathscr{A})$. Finally, so that the development of the subsequent study will be easier, we will assume that Θ and X are euclid spaces with the Borel σ-field and that α and μ are the corresponding Lebesgue measures.

If over the space $A \times \Theta$ the decision maker is able to build a positive utility function u to evaluate the interest that the election of an action before a state of nature has for him and to confront it with the interests of the remaining pairs (action, state of nature), we can introduce the following notions:

1.1. By a_0^* we designate the action of A such that:

$$-\mathscr{E}_\theta\left(\log \frac{u(a_0^*, \theta)}{\mathscr{E}_\theta[u(a_0^*, \theta)]}\right) = \min_{a \in A}\left\{-\mathscr{E}_\theta\left(\log \frac{u(a, \theta)}{\mathscr{E}_\theta[u(a, \theta)]}\right)\right\}.$$

1.2. Let $\mathscr{X}$ be an experiment over Θ and let us assume that χ has been the outcome of the performance of $\mathscr{X}$. We designate by a_χ^* the action of A such that:

$$-\mathscr{E}_{\theta/\chi}\left(\log \frac{u(a_\chi^*, \theta)}{\mathscr{E}_{\theta/\chi}[u(a_\chi^*, \theta)]}\right) = \min_{a \in A}\left\{-\mathscr{E}_{\theta/\chi}\left(\log \frac{u(a, \theta)}{\mathscr{E}_{\theta/\chi}[u(a, \theta)]}\right)\right\}$$

1.3. The "value of the expected quietness (invariant by homotethies in relation to the utilities) which the experiment $\mathscr{X}$ provides to the decision maker about the parameter space Θ" may be defined by the espression:

$$\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}; p(\theta)) = \mathscr{E}_{\chi,\theta}\left(\log \frac{u(a_\chi^*, \theta) \cdot \mathscr{E}_\theta[u(a_0^*, \theta)]}{u(a_0^*, \theta) \cdot \mathscr{E}_{\theta/\chi}[u(a_\chi^*, \theta)]}\right).$$

if it exists.

The preceeding expression and the adoption of the actions a_0^* and a_χ^* may be interpreted and justified in terms of the unquietness measure propounded in [6] and axiomatically characterized in [7]. So the former value can be interpreted as the expected loss of unquietness that brings the knowledge of the outcome for $\mathscr{X}$.

It is advisable to outline that the action a_0^* is the Bayes action for the distribution $p(\theta)$ over Θ and the loss function:

$$\mathscr{L}(a, \theta) = -\log \frac{u(a, \theta)}{\mathscr{E}_\theta[u(a, \theta)]},$$

and a^*_χ is the Bayes action for the distribution $p(\theta/\chi)$ over Θ and the loss function:

$$\mathscr{L}(a, \theta) = -\log \frac{u(a, \theta)}{\mathscr{E}_{\theta/\chi}[u(a, \theta)]}.$$

Note: When it is necessary for the existence problems of actions recently defined, we may substitute $\min_{a \in A}$ *by* $\inf_{a \in A}$.

Because the mentioned unquietness measure is invariant by homotethies, we can infer that this measure makes the diversity of utilities evident considering the ratios between them or, as is done in such a function, the ratios between each one of the utilities and their expected utility. In consequence, so that $\mathscr{I}\mathscr{U}^*$ will measure with more effectiveness, it is advisable for the decision maker to build the utility function paying attention to the ratios between the values which he allocates to each two elements of $A \times \Theta$. If he does not do this or does not know how he has built it, he could, nevertheless, apply the criterion which we will go on to see eventhough in that circunstance the obtained results would not be as coherent. Particularly, if the decision maker established his utility function payin attention only to the differences between each two of the values granted to the different pairs (action, state of nature) he would get appreciable results if he assumed this criterion, previously performing the variable change in the utility function given by $v(a, \theta) = 2^{u(a, \theta)}$, but the use of criterion propounded and studied in [8] in this case is more operative.

2. CRITERION OF EXPECTED QUIETNESS (INVARIANT BY HOMOTETHIES IN RELATION TO THE UTILITIES)

For the preceeding decision problem, we can say that the experiment $\mathscr{X}$ belonging to $\mathscr{E}$ is preferred to the experiment $\mathscr{Y}$ belonging to $\mathscr{E}$ with the criterion of expected quietness (invariant by homotethies in relation to the utilities), and we will denote it by $\mathscr{X} \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{Y}$, if and only if:

$$\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}; p(\theta)) \geqq \mathscr{I}\mathscr{U}^*(\Theta; \mathscr{Y}; p(\theta)),$$

and we say that $\mathscr{X}$ and $\mathscr{Y}$ are equivalent with this criterion, and it is represented by $\mathscr{X} \overset{\mathscr{I}\mathscr{U}^*}{\sim} \mathscr{Y}$, if and only if $\mathscr{X} \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{Y}$ and $\mathscr{Y} \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{X}$, for which it is a necessary and sufficient condition that:

$$\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}; p(\theta)) = \mathscr{I}\mathscr{U}^*(\Theta; \mathscr{Y}; p(\theta)).$$

The definitions which we have just given only make sense if the expressions $\mathcal{IU}^*(\Theta;\ \mathcal{X};\ p(\theta))$ (which we will write as $\mathcal{IU}^*(\Theta;\ \mathcal{X})$ when the specification of the *a priori* distribution over Θ is not necessary) are well defined and the different expectations which appear in it, exist. From here on, we will admit that we can apply the criterion to experiments with which we will deal.

Obviously, the experiments could also be compared in an absolute way and with respect to the same characteristic which in this criterion, not making reference to a concrete *a priori* distribution, but in such a situation we would establish a partial preordering and for non-comparable experiments we would have to use previously fixed distributions.

Finally and before going on with the next section, we must outline the fact that the definitions and same basic conclusions of other criteria for the comparison of experiments with which we will relate the one that motivates this paper are detailed in [5], and the analysis that we make has a similar structure to which are developed there.

3. PROPERTIES

3.1. The relation $\overset{\mathcal{IU}^*}{\gtrsim}$ is a complete preordering.

In effect:

This relation is reflexive, transitive and a total order relation, as it can be identified with the relation $\geqq$ over a subset of real numbers.

3.2. If $\mathcal{N}$ is the null experiment, $\mathcal{N}=\{Y;\ \mathcal{B};\ \mathcal{Q}_\theta,\ \theta\in\Theta\}$ such that $\mathcal{Q}_\theta$ does not depend on θ, whatever the experiment $\mathcal{X}$ over Θ may be, $\mathcal{X}\overset{\mathcal{IU}^*}{\gtrsim}\mathcal{N}$.

In effect:

As $\mathcal{N}$ is the null experiment, the distribution $\mathcal{Q}_\theta$ is independent of θ and, in consequence, $q(\theta/y)$ is independent of y as well, whence whatever y may be the action a_y^* coincides with a_0^*. Thus, $\mathcal{IU}^*(\Theta;\ \mathcal{N})=0$.

But considering the definition 1.2 whatever $\chi\in X$ may be (where $\mathcal{X}=\{X;\ \mathcal{A};\ \mathcal{P}_\theta,\ \theta\in\Theta\}$) :

$$-\mathcal{E}_{\theta/\chi}\left(\log\frac{u(a_0^*,\ \theta)}{\mathcal{E}_{\theta/\chi}[u(a_0^*,\ \theta)]}\right)\geqq-\mathcal{E}_{\theta/\chi}\left(\log\frac{u(a_\chi^*,\ \theta)}{\mathcal{E}_{\theta/\chi}[u(a_\chi^*,\ \theta)]}\right)$$

and this implies that:

$$\mathcal{E}_\chi\{\log\mathcal{E}_{\theta/\chi}[u(a_0^*,\ \theta)]\}-\mathcal{E}_{\chi/\theta}\{\log u(a_0^*,\ \theta)\}\geqq-\mathcal{E}_{\chi,\theta}\left(\log\frac{u(a_\chi^*,\ \theta)}{\mathcal{E}_{\theta/\chi}[u(a_\chi^*,\ \theta)]}\right).$$

As $\mathscr{E}_\theta[u(a_0^*, \theta)] = \mathscr{E}_\chi\{\mathscr{E}_{\theta/\chi}[u(a_0^*, \theta)]\}$, in virtue of the convexity of the logarithmic function and because of Jensen's inequality, we have:

$$\log \mathscr{E}_\theta[u(a_0^*, \theta)] - \mathscr{E}_{\chi,\theta}\{\log u(a^*, \theta)\} \geqq -\mathscr{E}_{\chi,\theta}\left(\log \frac{u(a_\chi^*, \theta)}{\mathscr{E}_{\theta/\chi}[u(a_\chi^*, \theta)]}\right).$$

Then:

$$\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}) = \mathscr{E}_{\chi,\theta}\left(\log \frac{u(a_\chi^*, \theta) \cdot \mathscr{E}_\theta[u(a_0^*, \theta)]}{u(a_0^*, \theta) \cdot \mathscr{E}_{\theta/\chi}[u(a_\chi^*, \theta)]}\right) \geqq 0.$$

3.3. Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two experiments over Θ and let $(\mathscr{X}_1, \mathscr{X}_2)$ be the compound experiment. Then, it can be verified that:

$$(\mathscr{X}_1, \mathscr{X}_2) \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{X}_1.$$

In effect:

$$\mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_1, \mathscr{X}_2)) = \mathscr{E}_{\chi_1\chi_2\theta}\left(\log \frac{u(a_{\chi_1\chi_2}^*, \theta) \cdot \mathscr{E}_\theta[u(a_0^*, \theta)]}{u(a_0^*, \theta) \cdot \mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1\chi_2}^*, \theta)]}\right)$$

$$= \mathscr{E}_{\chi_1,\theta}\left(\log \frac{u(a_{\chi_1}^*, \theta) \cdot \mathscr{E}_\theta[u(a_0^*, \theta)]}{u(a_0^*, \theta) \cdot \mathscr{E}_{\theta/\chi_1}[u(a_{\chi_1}^*, \theta)]}\right)$$

$$+ \mathscr{E}_{\chi_1\chi_2\theta}\left(\log \frac{u(a_{\chi_1\chi_2}^*, \theta) \cdot \mathscr{E}_{\theta/\chi_1}[u(a_{\chi_1}^*, \theta)]}{u(a_0^*, \theta) \cdot \mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1\chi_2}^*, \theta)]}\right)$$

being the first addend $\mathscr{I}\mathscr{U}(\Theta; \mathscr{X}_1)$, and thus it will be enough to check the non-negativity of the second addend to prove the present property.

By the definition for $a_{\chi_1\chi_2}^*$ we may verify that:

$$-\mathscr{E}_{\theta/\chi_1\chi_2}\left(\log \frac{u(a_{\chi_1}^*, \theta)}{\mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1}^*, \theta)]}\right) \geqq -\mathscr{E}_{\theta/\chi_1\chi_2}\left(\log \frac{u(a_{\chi_1\chi_2}^*, \theta)}{\mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1\chi_2}^*, \theta)]}\right)$$

and, since $\mathscr{E}_{\theta/\chi_1}[u(a_{\chi_1}^*, \theta)] = \mathscr{E}_{\chi_2/\chi_1}\{\mathscr{E}_{\theta/\chi_1\chi_2} u(a_{\chi_1}^*, \theta)\}$in virtue of the convexity of the logarithmic function and because of Jensen's inequality, whatever $\chi_1 \in X_1$ may be:

$$\log \mathscr{E}_{\theta/\chi_1}[u(a_{\chi_1}^*, \theta)] - \mathscr{E}_{\chi_2\theta/\chi_1}\{\log u(a_{\chi_1}^*, \theta)\}$$

$$\geqq \mathscr{E}_{\chi_2/\chi_1}\{\log \mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1}^*, \theta)]\} +$$

$$-\mathscr{E}_{\chi_2\theta/\chi_1}\{\log u(a_{\chi_1}^*, \theta)\} \geqq -\mathscr{E}_{\chi_2\theta/\chi_1}\left(\log \frac{u(a_{\chi_1\chi_2}^*, \theta)}{\mathscr{E}_{\theta/\chi_1\chi_2}[u(a_{\chi_1\chi_2}^*, \theta)]}\right),$$

whence:

$$\mathscr{E}_{\chi_2\theta/\chi_1}\left(\log\frac{u(a^*_{\chi_1\chi_2},\theta)\,.\,\mathscr{E}_{\theta/\chi_1}[u(a^*_{\chi_1},\theta)]}{u(a^*_{\chi_1},\theta)\,.\,\mathscr{E}_{\theta/\chi_1\chi_2}[u(a^*_{\chi_1\chi_2},\theta)]}\right)\geqq 0$$

and, because of this, the expected value, with respect to χ, of the last expression will also be non-negative. In consequence:

$$\mathscr{I}\mathscr{U}^*(\Theta;(\mathscr{X}_1,\mathscr{X}_2))\geqq\mathscr{I}\mathscr{U}^*(\Theta;\mathscr{X}_1).$$

3.4. Let $\mathscr{X}^{(n)}$ be an experiment corresponding to the simple random sample of size n from $\mathscr{X}$. Then $\forall n\in\mathbb{N}$, $n\geqq 1$, it can be verified that:

$$\mathscr{X}^{(n+1)}\overset{\mathscr{I}\mathscr{U}^*}{\gtrsim}\mathscr{X}(n).$$

In effect:

To prove it we will follow a reasoning similar to that expounded in 3.3, and from it the proof is inmediate.

3.5. Let $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ be three experiments over Θ such that $\mathscr{X}_1\overset{\mathscr{I}\mathscr{U}^*}{\gtrsim}\mathscr{X}_3$ for all *a priori* distribution over Θ and $\mathscr{X}_2$ independent of $\mathscr{X}_1$ and $\mathscr{X}_3$. It can be verified, then, that for all *a priori* distribution:

$$(\mathscr{X}_1,\mathscr{X}_2)\overset{\mathscr{I}\mathscr{U}^*}{\gtrsim}(\mathscr{X}_3,\mathscr{X}_2).$$

In effect:

We will previously establish a supporting lemme.

3.5.1. Let $\mathscr{Y}_1$ and $\mathscr{Y}_2$ be two independent experiments over Θ and $p(\theta)$ the *a priori* distribution over this parameter space. If $(\mathscr{Y}_1,\mathscr{Y}_2)$ is the compound experiment, it can be verified that:

$$\mathscr{I}\mathscr{U}^*(\Theta;(\mathscr{Y}_1,\mathscr{Y}_2);p(\theta))=\mathscr{I}\mathscr{U}^*(\Theta;\mathscr{Y}_1;p(\theta))+\mathscr{E}_{y_1}[\mathscr{I}\mathscr{U}^*(\Theta;\mathscr{Y}_2;p(\theta/y_1))].$$

Proof :

$$\mathscr{I}\mathscr{U}^*(\Theta;(\mathscr{Y}_1,\mathscr{Y}_2);p(\theta))=\mathscr{I}\mathscr{U}^*(\Theta;\mathscr{Y}_1;p(\theta))$$
$$+\mathscr{E}_{y_1}\left\{\mathscr{E}_{y_2\theta/y_1}\left(\log\frac{u(a^*_{y_1y_2},\theta)\,.\,\mathscr{E}_{\theta/y_1}[u(a^*_{y_1},\theta)]}{u(a^*_{y_1},\theta)\,.\,\mathscr{E}_{\theta/y_1y_2}[u(a^*_{y_1y_2},\theta)]}\right)\right\}.$$

If, having fixed an arbitrary y_1, we denote by $p'(\theta)$ the distribution over Θ, $p(\theta/y_1)$, the corresponding *a posteriori* distribution $p'(\theta/y_2)$ will be in virtue of the independence between $\mathcal{Y}_1$ and $\mathcal{Y}_2$, $p(\theta/y_1 y_2)$, whence the associated actions $a_0^{*\prime}$ and $a_{y_2}^{*\prime}$ will be, respectively, $a_{y_1}^*$ and $a_{y_1 y_2}^*$, from where the second addend of the last expression is precisely:

$$\mathscr{E}_{y_1}[\mathscr{I}\mathscr{U}(\Theta; \mathcal{Y}_2; p'(\theta))]$$

and with this the lemme has been proven.

Applying to preceeding lemme to the hypothesis of the property wich we are working with, and for any distribution $p(\theta)$ over Θ:

$$\mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_1, \mathscr{X}_2); p(\theta)) = \mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_2; p(\theta)) + \mathscr{E}_{\chi_2}[\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_1; p(\theta/\chi_2))],$$

$$\mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_3, \mathscr{X}_2); p(\theta)) = \mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_2; p(\theta)) + \mathscr{E}_{\chi_2}[\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_3; p(\theta/\chi_2))].$$

But, as $\mathscr{X}_1 \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{X}_3$ for all distribution over Θ, fixed an arbitrary χ_2:

$$\mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_1; p(\theta/\chi_2)) \geqq \mathscr{I}\mathscr{U}^*(\Theta; \mathscr{X}_3; p(\theta/\chi_2)),$$

we can infer in this way that:

$$\mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_1, \mathscr{X}_2); p(\theta)) \geqq \mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_3, \mathscr{X}_2); p(\theta)),$$

whatever the distribution $p(\theta)$ over Θ may be.

3.6. Let $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{X}_3$ and $\mathscr{X}_4$ be four experiments over Θ such that $\mathscr{X}_1 \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{X}_2$ and $\mathscr{X}_3 \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{X}_4$ for all *a priori* distribution. If $\mathscr{X}_1$ is independent of $\mathscr{X}_3$ and $\mathscr{X}_2$ is independent of $\mathscr{X}_4$, we can verify that:

$$(\mathscr{X}_1, \mathscr{X}_3) \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} (\mathscr{X}_2, \mathscr{X}_4),$$

for all *a priori* distribution.

In effect:

If we define four experiments $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ such that $\forall\,\theta \in \Theta$ it can be verified that $\mathcal{Y}_i$ has the same distribution as $\mathscr{X}_i$ (for $i = 1, 2, 3, 4$) and $\mathcal{Y}_1$ is independent of $\mathcal{Y}_3$, $\mathcal{Y}_2$ of $\mathcal{Y}_4$ and $\mathcal{Y}_2$ of $\mathcal{Y}_3$, in virtue of the property 3.5, and whatever the *a priori* distribution $p(\theta)$ may be:

$$\mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_1, \mathscr{X}_3); p(\theta)) = \mathscr{I}\mathscr{U}^*(\Theta; (\mathcal{Y}_1, \mathcal{Y}_3); p(\theta)) \geqq \mathscr{I}\mathscr{U}^*(\Theta; (\mathcal{Y}_2, \mathcal{Y}_3); p(\theta))$$
$$\geqq \mathscr{I}\mathscr{U}^*(\Theta; (\mathcal{Y}_2, \mathcal{Y}_4); p(\theta)) = \mathscr{I}\mathscr{U}^*(\Theta; (\mathscr{X}_2, \mathscr{X}_4); p(\theta)).$$

3.7. Let $\mathscr{X} = \{X; \mathscr{A}; \mathscr{P}_\theta, \theta \in \Theta\}$ be an experiment over Θ and let $\{\mathscr{E}_i\}_{i \in \mathbb{N}}$ a partition of X by elements of the σ-field $\mathscr{A}$. If we consider the experiment

$\mathscr{Y}=\{Y;\ \mathscr{B};\ \mathscr{Q}_\theta,\ \theta\in\Theta\}$ where $Y=\{\mathscr{E}_i/i\in\mathbb{N}\}$, $\mathscr{B}$ is the σ-field generated by the sets $\mathscr{E}_i$, and such that:

$$\mathscr{Q}_\theta(\mathscr{E}_i)=\mathscr{P}_\theta(\mathscr{E}_i), \qquad \forall\, i\in\mathbb{N},$$

we can verify that $\mathscr{X} \overset{\mathscr{I}\mathscr{U}^*}{\gtrsim} \mathscr{Y}$ for all *a priori* distribution over Θ.

In effect:

Let $p(\theta)$ be any distribution over Θ. Considering the definition 1.2, $\forall\, i\in\mathbb{N}$, $\forall\,\chi\in\mathscr{E}_i$, it can be verified that:

$$-\mathscr{E}_{\theta/\chi}\left(\log\frac{u(a_\chi^*,\theta)}{\mathscr{E}_{\theta/\chi}[u(a_\chi^*,\theta)]}\right)\leqq-\mathscr{E}_{\theta/\chi}\left(\log\frac{u(a_{\mathscr{E}_i}^*,\theta)}{\mathscr{E}_{\theta/\chi}[u(a_{\mathscr{E}_i}^*,\theta)]}\right),$$

whence:

$$\int_{\mathscr{E}_i}\mathscr{E}_{\theta/\chi}\left\{\log\frac{u(a_\chi^*,\theta)}{\mathscr{E}_{\theta/\chi}[u(a_\chi^*,\theta)]}\right\}\mathscr{P}(d\chi)\geqq\mathscr{P}(\mathscr{E}_i)\,.\,\mathscr{E}_{\theta/\mathscr{E}_i}\{\log u(a_{\mathscr{E}_i}^*,\theta)\}+$$

$$-\int_{\mathscr{E}_i}\log\mathscr{E}_{\theta/\chi}[u(a_{\mathscr{E}_i}^*,\theta)]\,\mathscr{P}(d\chi).$$

Taking into account that:

$$\mathscr{E}_{\theta/\mathscr{E}_i}[u(a_{\mathscr{E}_i}^*,\theta)]=\int_{\mathscr{E}_i}\frac{1}{\mathscr{P}(\mathscr{E}_i)}\mathscr{E}_{\theta/\chi}[u(a_{\mathscr{E}_i}^*,\theta)]\,\mathscr{P}(d\chi)..$$

and in virtue of the convexity of the logarithmic function and by Jensen's inequality:

$$\int_{\mathscr{E}_i}\mathscr{E}_{\theta/\chi}\left\{\log\frac{u(a_\chi^*,\theta)}{\mathscr{E}_{\theta/\chi}[u(a_\chi^*,\theta)]}\right\}\mathscr{P}(d\chi)\geqq\mathscr{P}(\mathscr{E}_i)\,.\,\mathscr{E}_{\theta/\mathscr{E}_i}[\log u(a_{\mathscr{E}_i}^*,\theta)]+$$

$$-\mathscr{P}(\mathscr{E}_i)\,.\,\log\{\mathscr{E}_{\theta/\mathscr{E}_i}[u(a_{\mathscr{E}_i}^*,\theta)]\}=\mathscr{P}(\mathscr{E}_i)\,.\,\mathscr{E}_{\theta/\mathscr{E}_i}\left\{\log\frac{u(a_{\mathscr{E}_i}^*,\theta)}{\mathscr{E}_{\theta/\mathscr{E}_i}[u(a_{\mathscr{E}_i}^*,\theta)]}\right\}.$$

Adding in i along ℕ and aggregating to both members of the last inequality, the value:

$$\mathscr{E}_{\chi,\theta}\left\{\log\frac{\mathscr{E}_\theta[u(a_0^*,\theta)]}{u(a_0^*,\theta)}\right\}=\mathscr{E}_{y,\theta}\left\{\log\frac{\mathscr{E}_\theta[u(a_0^*,\theta)]}{u(a_0^*,\theta)}\right\},$$

we obtain:

$$\mathcal{I}\mathcal{U}^*(\Theta;\ \mathcal{X};\ p(\theta)) \geqq \mathcal{I}\mathcal{U}^*(\Theta;\ \mathcal{Y};\ p(\theta)).$$

3.8. The expected quietness which a simple random sample $\mathcal{X}^{(n)}$ of an experiment $\mathcal{X}$ provides over the parameter space Θ is always greater or equal to the expected quietness which over this space provides a statistic $\mathcal{T}=\mathcal{T}(\mathcal{X}^{(n)})$ of this sample. What is more, if $\mathcal{T}$ is a sufficient statistic both values of quietness coincide.

In effect:

Let $p(\theta)$ be any *a priori* distribution over Θ. We will denote by

$$\mathcal{X}_t=\{z\in\mathbb{R}\mathcal{T}(z)=t\},$$

for $t\in\mathbb{R}$ and by

$$p(\theta,\ t)=\mathcal{P}\{(\theta,\ t)/(\theta,\ t)\in\Theta\times\mathcal{X}_t\}.$$

Then, $\forall\, t\in\mathbb{R}$ and $\forall\, z\in\mathcal{X}_t$, it can be verified that:

$$-\mathcal{E}_{\theta/z}\left(\log\frac{u(a_z^*,\ \theta)}{\mathcal{E}_{\theta/z}[u(a_z^*,\ \theta)]}\right)\leqq -\mathcal{E}_{\theta/z}\left(\log\frac{u(a_t^*,\ \theta)}{\mathcal{E}_{\theta/z}[u(a_t^*,\ \theta)]}\right),$$

whence:

$$\int_{\mathcal{X}_t}\mathcal{E}_{\theta/z}\left\{\log\frac{u(a_z^*,\ \theta)}{\mathcal{E}_{\theta/z}[u(a_z^*,\ \theta)]}\right\}\mathcal{P}(dz)$$
$$\geqq \mathcal{P}(\mathcal{T})\,.\,\mathcal{E}_{\theta/t}\{\log u(a_t^*,\ \theta)\}+$$
$$-\int_{\mathcal{X}_t}\log\mathcal{E}_{\theta/z}[u(a_t^*,\ \theta)]\,\mathcal{P}(dz).$$

As:

$$\mathcal{E}_{\theta/t}[u(a_t^*,\ \theta)]=\int_{\mathcal{X}_t}\frac{1}{\mathcal{P}(t)}\mathcal{E}_{\theta/z}[u(a_t^*,\ \theta)]\,\mathcal{P}(dz),\quad \text{whatever } t\in\mathbb{R},$$

may be, in virtue of the convexity of the logarithmic function and by Jensen's inequality:

$$\int_{\mathcal{X}_t}\mathcal{E}_{\theta/z}\left\{\log\frac{u(a_z^*,\ \theta)}{\mathcal{E}_{\theta/z}[u(a_z^*,\ \theta)]}\right\}\mathcal{P}(dz)\geqq\mathcal{P}(t)\,.\,\mathcal{E}_{\theta/t}\{\log u(a_t^*,\ \theta)\}+$$
$$-\mathcal{P}(t)\,.\,\log\mathcal{E}_{\theta/t}[u(a_t^*,\ \theta)]=\mathcal{P}(t)\,.\,\mathcal{E}_{\theta/t}\left(\log\frac{u(a_t^*,\ \theta)}{\mathcal{E}_{\theta/t}[u(a_t^*,\ \theta)]}\right)$$

and, because of this, following a similar reasoning to the one of property 3.7, we have:

$$\mathcal{IU}^*(\Theta;\ \mathcal{X}^{(n)};\ p(\theta)) \geqq \mathcal{IU}^*(\Theta;\ \mathcal{T};\ p(\theta)).$$

Particularly, if $\mathcal{T}$ is a sufficient statistic of $\mathcal{X}^{(n)}$, $\mathcal{P}_\theta(t) \overset{\text{a. s.}}{=} \mathcal{P}_\theta(z)\ \forall z \in \mathcal{Z}_t$ (according to the bayesian definition for the sufficient statistic) should occur. In this way, having fixed an arbitrary $t \in \mathbb{R}$, if $z \in \mathcal{Z}_t$ the action a_z^* coincides with the action a_t^* (a. s.). Thus, it is obvious that:

$$\mathcal{IU}^*(\Theta;\ \mathcal{X}^{(n)};\ p(\theta)) = \mathcal{IU}^*(\Theta;\ \mathcal{T};\ p(\theta)).$$

3.9. If $\mathcal{S}$ and $\mathcal{T}$ are two sufficient statistics of a simple random sample of size n, $\mathcal{X}^{(n)}$, and of a simple random sample of size m, $\mathcal{Y}^{(m)}$, respectively, we can verify that $\mathcal{X}^{(n)} \overset{\mathcal{IU}^*}{\gtrsim} \mathcal{Y}^{(m)}$ if and only if $\mathcal{S} \overset{\mathcal{IU}^*}{\gtrsim} \mathcal{T}$, whatever the distribution over the parameter space may be.

In effect:

Due to the former property, whatever the *a priori* distribution $p(\theta)$ may be:

$$\mathcal{IU}^*(\Theta;\ \mathcal{S};\ p(\theta)) = \mathcal{IU}^*(\Theta;\ \mathcal{X}^{(n)};\ p(\theta)),$$

$$\mathcal{IU}^*(\Theta;\ \mathcal{T};\ p(\theta)) = \mathcal{IU}^*(\Theta;\ \mathcal{Y}^{(m)};\ p(\theta)).$$

If an order relation is satisfied between the first members, in consequence, the same relation will be satisfied for the second members, and this proves the enunciated result.

3.10. Let $\mathcal{X} = \{\mathbb{R};\ \beta;\ \mathcal{P}_\theta,\ \theta \in \Theta\}$ an experiment over Θ, being β the Borel σ-field. If $\mathcal{T} = \mathcal{T}(\mathcal{X})$ and $\mathcal{T}$ is a strictly monotonic and derivable real mapping, we can verify that:

$$\mathcal{X} \overset{\mathcal{IU}^*}{\sim} \mathcal{T},$$

whatever the distribution over Θ may be.

In effect:

Let $Y = \mathcal{T}(\mathbb{R})$ be and let us consider the experiment $\mathcal{T} = \{Y;\ \beta;\ \mathcal{Q}_\theta,\ \theta \in \Theta\}$. Because $\mathcal{T}$ is strictly monotonic and derivable, there is a unique inverse mapping h, which is strictly monotonic and derivable, and we can verify that:

$$\forall t \in Y, \quad \mathcal{Q}_\theta(t) = \mathcal{P}_\theta[h(t)] \,.\, |h'(t)|$$

and thus:

$$\mathcal{Q}(t)=\mathcal{P}(h(t)).|h'(t)|.$$

As h is strictly monotonic and derivable, it is obvious that $|h'(t)|\neq 0$, whence:

$$q(\theta/t)=\frac{\mathcal{Q}_\theta(t).p(\theta)}{\mathcal{Q}(t)}=\frac{\mathcal{P}_\theta(h(t)).p(\theta)}{\mathcal{P}(h(t))}=p(\theta/h(t)).$$

Thus, and in virtue of the injectivity of the mappings h and $\mathcal{T}$ and for the measurability of $\mathcal{T}$, it is satisfied that $\mathcal{T}=\mathcal{T}(\mathcal{X})$ is a sufficient statistic of a simple random sample of size 1, $\mathcal{X}$. Applying 3.8, we then obtain:

$$\mathcal{X}\overset{\mathcal{I}\mathcal{U}^*}{\sim}\mathcal{T}.$$

3.11. Let $\mathcal{X}=\{\mathbb{R};\ \beta;\ \mathcal{P}_\theta,\ \theta\in\Theta\}$ be an experiment and $\mathcal{T}=t(\mathcal{X})$ such that t is a continuous and bijective mapping from $(\mathbb{R},\ \beta)$ in $(\mathbb{R},\ \beta)$, being β the Borel σ-field. Then, it can be verified that $\mathcal{X}\overset{\mathcal{I}\mathcal{U}^*}{\sim}\mathcal{T}$, whatever the *a priori* distribution over Θ may be.

In effect:

Because t is continuous and bijective, there is an inverse mapping t^{-1} which is continuous and bijective as well, and this indicates that t and t^{-1} are measurable, whence applying the property 3.8:

$$\mathcal{X}\overset{\mathcal{I}\mathcal{U}^*}{\gtrsim}t(\mathcal{X})=\mathcal{T}\qquad\text{and}\qquad\mathcal{T}\overset{\mathcal{I}\mathcal{U}^*}{\gtrsim}t^{-1}(\mathcal{T})=\mathcal{X}$$

and this implies that $\mathcal{T}\overset{\mathcal{I}\mathcal{U}^*}{\sim}\mathcal{X}$.

4. FINAL OBSERVATIONS

The criterion which we have studied is clearly applicable to those situations in which, in spite of unknowing the *a priori* distribution over we can dispose of information about Θ, $\mathcal{H}(\Theta)$, or punctual informations $\mathcal{I}(\{\theta\})$, $\theta\in\Theta$, such that from them we are able to build a probability distribution over Θ ([10] and [9], respectively).

It is advisable to outline that from the criterion recently described and from the criterion of maximizing Shannon's information [5], we can define different mixed criterion as, for example, those which we will go on to describe:

Mixed criterion of quietness and information: In this criterion we compare two experiments in relation to the quietness they provide about the parameter space and in the case that they coincide with respect to this characteristic, they are confronted with respect to the information which they provide about this space.

Mixed criterion of ε-quietness and information: It consists on admiting that the experiment is "better" than another one, if the difference between the quietness which are brought over a parameter space by the first and the second experiment is, at least, equal to a value ε (positive constant previously fixed by the decision maker, under which the difference between the quietnesses is considered non-significant in relation with the possibility of comparing with respect to information) and, in the case where this difference is inferior to ε, if the information that the first one provides about the parameter space is greater than that of the second one.

A similar criterion to this last one can be established interchanging the characteristics "quietness" and "information".

Finally, if we take into account the properties that go along with the criteria which constitute the mixed criteria which we have just described, we can conclude that these last ones also have the properties desirable in any criterion for comparison of experiments.

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