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BORDA EFFICIENCY OF CONSTANT SCORING RULES WITH LARGE ELECTORATES (*) (1)

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Abstract. — *The study examines the Borda efficiency of constant scoring rules for large electorates. The condition of impartial culture is assumed. It is shown that the Borda efficiency of the rule requiring individuals to vote for k candidates is identical to the Borda efficiency of the vote against k candidates rule. The most Borda efficient constant scoring rule is shown to be the rule requiring individuals to vote for half of the candidates on the ballot.*

Keywords: Voting models, Borda rule

Résumé. — *Cette étude examine l'efficacité Borda des règles constantes de marque que l'on emploie pour les grands électorats. On présume la condition d'une culture impartiale. On démontre que l'efficacité Borda de la règle qui exige les personnes de voter pour les candidats k est identique à l'efficacité Borda de la règle du vote contre les candidats k. La règle constante de marque la plus efficace est la règle qui exige les personnes de voter pour la moitié des candidats au scrutin.*

1. INTRODUCTION

Societies are frequently encountered with the problem of selecting some alternative from a set of feasible alternatives. This situation can take the form of electing a candidate to some public office, selecting a policy that is to be implemented, or any of a number of similar situations. For convenience, this study is developed in the context of electing a candidate from a set of candidates competing for election to office. The purpose of this study is to examine the actual process of selecting an election winner.

How should we go about selecting the winner of an election? In order to consider this question it is first necessary to identify factors that are relevant to

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the election process. Three major factors are economic cost of implementation, level of voter input that results, and specific properties of the election process being considered. Both cost of implementation and level of voter input can be associated with the degree of complexity of the voting process being used. Consider the implementation costs of single-stage and multi-stage election procedures. In a single-stage election the voters cast their ballots only once and the winner is selected on the basis of the information contained on the ballot. In a multi-stage election the voters must cast ballots more than once. In each intermediate stage of a multi-stage election the ballot information is used to drop candidates from consideration in later stages of the election. Losers are dropped in each intermediate stage and the winner is determined in the last stage.

A multi-stage election is obviously going to be more costly than a single-stage election both in terms of economic cost of implementation and level of voter input. Having to set up polling places for each stage, having to print up new ballots in each stage, and other costs cause the multi-stage election to be much more costly than single-stage elections. However, for elections of some importance the economic cost of a voting process may not be considered as a significant factor. The level of voter input is important and as election processes become more complicated fewer individuals in the society will participate by actually voting. Greater voter turnout would be expected to result in a winner that more accurately reflects the overall preference of the society. Thus an increased voter level of participation should be viewed as a positive factor. The use of multi-state elections is a definite complicating factor that could reduce the level of voter input.

Degrees of complication can also arise when considering single-stage elections by themselves. Of particular importance is the consideration of non-ranked voting procedures versus ranked voting procedures. For an election on m candidates a non-ranked voting procedure requires voters to select some number, say k , of candidates. Each voter can select his or her k most preferred candidates by doing something like checking off boxes next to the names of candidates on the ballot. The winner is then selected as the candidate receiving the most votes.

For a ranked voting procedure individuals must do more than report their k most preferred candidates. Voters must also rank these k most preferred candidates from most preferred to least preferred. Since non-ranked voting procedures require less effort on the part of voters we might expect a greater voter turnout if they are used instead of ranked voting procedures. To select a winner in a ranked voting procedure, weighted scoring rules are often used. For an m candidate election a weighted scoring rule consists of m weights $w_1, w_2, \dots, w_m$ with $w_i \geq w_{i+1}$ and $w_m \geq 0$. Each voter then gives a score of w_i

to his or her i -th most preferred candidate and the winner is selected as the candidate receiving the greatest total score from all voters.

In considering the third relevant factor of election processes, properties of the election process, a very important concern is how well an election procedure does at selecting a candidate that is, in some sense, the candidate most preferred by the society. There are a number of criteria that can be used to determine this overall most preferred candidate and we shall consider two of the most common ones, namely the Condorcet Criterion [5] and the Borda Criterion [3].

For a candidate to be the Condorcet winner voters' preferences must be such that this candidate would be able to defeat all other candidates by simple majority voting in a series of pairwise elections. That is, if there were only two candidates in an election and the Condorcet winner was one of them then it would be the majority rule winner. The Condorcet Criterion requires that the Condorcet winner should be selected as the winner when a Condorcet winner exists. It is well known that voters' preferences might be such that no Condorcet winner exists but if there is one it would be a desirable candidate for selection as the winner. Various properties of Condorcet voting rules are given in [1, 2, 6, 18, 20, 23]. To obtain the Condorcet winner, the minimum voter input required on a ballot would be the total preference ranking for each voter.

For a candidate to be the Borda winner voters' preferences must be such that this candidate maximizes the total number of instances in which a candidate is preferred to any other candidate, provided that voters are never indifferent between candidates. Borda rule is a member of the family of weighted scoring rules. For m candidates the scoring rule weights are $w_1, w_2, \dots, w_m$ and for Borda rule the difference in weights $w_i - w_j$ is proportional to $j - i$ for all i and j . An example of Borda weights is $w_i = m - i$ for all i . Any example of Borda weights, linearly decreasing w_i 's, must result in the same winner. Various properties of Borda rule are presented in [2, 6, 11, 12, 25, 27]. The minimum voter input required on a ballot would be the total preference ranking for each voter if Borda rule is used.

Justification can be made for using either the Condorcet Criterion or the Borda Criterion when deciding how to develop an election process. However, if either a Condorcet rule or Borda rule is used then each voter will be required to rank all candidates from most preferred to least preferred. An additional problem develops for the Condorcet Criterion since some other rule must be implemented to pick a winner when no Condorcet winner exists.

The most commonly used election processes are single-stage non-ranked voting procedures. These are also referred to as constant scoring rules since they can be thought of as trivial weighted scoring rules. The constant scoring rule vote

for k candidates, denoted by Rule C_k , results in the same winner as the weighted scoring rule with $w_i = 1$ for $i = 1, 2, \dots, k$ and $w_j = 0$ for $j = k+1, k+2, \dots, m$. These constant scoring rules are primarily used due to their simplicity of implementation.

The natural subject of consideration is how well these simple constant scoring rules do relative to the Condorcet Criterion and the Borda Criterion. If in fact these constant scoring rules are quite likely to obtain a winner that is identical to the Borda winner or Condorcet winner then it would be quite logical to use the simpler and lower cost constant scoring rules.

Some measure is needed to determine how likely constant scoring rules are to pick winners according to the Condorcet Criterion and Borda Criterion. One measure has been developed by Fishburn [8] and it is referred to as efficiency. The Condorcet efficiency of Rule C_k on m alternatives, denoted by E_k^m , is the probability that Rule C_k will pick the Condorcet winner given that a Condorcet winner exists. In order to make statements about these probabilities it is necessary to make some assumptions about voters' preferences. For an m candidate election there are $m!$ (m factorial) possible linear preference rankings on the candidates. The condition of impartial culture is usually assumed such that if a voter is selected at random his or her preference ranking on the candidates is equally likely to be any one of the $m!$ possible linear rankings. Impartial culture only considers linear preference rankings so voters' preferences are assumed to contain no indifference between candidates. It is also assumed in the current study and in all other studies mentioned that all individuals vote sincerely according to their preferences and that all individuals vote independently.

A number of studies have been conducted to determine both simulation estimates and exact values of E_k^m for various voting procedures under the assumption of impartial culture [7, 8, 9, 10, 19, 21]. Studies reported in [4] and [13] have considered Condorcet efficiency of constant scoring rules with assumptions on voters' preferences other than that of impartial culture. Analytical results which give information about the general behavior of E_k^m have not only made the assumption of impartial culture but have also assumed that the number of voters, n , is large ($n \rightarrow \infty$) [14, 15, 16, 17]. In all further discussion E_k^m will refer to the Condorcet efficiency of Rule C_k under the assumption of impartial culture as $n \rightarrow \infty$. In [15] the Condorcet efficiency of plurality rule, in general Rule C_1 , is considered for three candidate elections and it is shown that $E_1^3 = E_2^3 = .901189$. Thus, for impartial culture with large electorates either plurality ($k=1$) or negative plurality ($k=2$) can be used with equal Condorcet efficiency and the Condorcet winner will be elected over ninety percent of the time when there is a Condorcet winner.

While exact numerical values of E_k^m are not available for m greater than three some analytical results have been obtained in [17] which prove that $E_k^m = E_{m-k}^m$ for all k and that $E_k^m \leq E_{k+1}^m$ for all $k \leq (m-2)/2$. Therefore, the Condorcet efficiency of Rule C_k is the same as that of Rule C_{m-k} . It should be noted that Rule C_{m-k} is equivalent to requiring individuals to vote against k of the candidates. So the Condorcet efficiency of the vote for k rule is the same as the Condorcet efficiency of the vote against k rule for large electorates under impartial culture. These results also indicate that the most Condorcet efficient constant scoring rule requires individuals to vote for about half of the candidates on the ballot. If m is even Rule $C_{m/2}$ is most Condorcet efficient and if m is odd Rule $C_{(m-1)/2}$ and Rule $C_{(m+1)/2}$ are equally most Condorcet efficient.

Borda efficiency can be defined with a probability statement similar to the one defining Condorcet efficiency. However, since Borda rule must always result with some candidate as the Borda winner no conditional statement is needed in the probability definition. Let B_k^m be the Borda efficiency or probability that the Rule C_k winner coincides with the Borda winner under the assumption of impartial culture. Much less research has been done concerning Borda efficiency than Condorcet efficiency even though Borda rule has many positive properties and deserves consideration. Some estimates of Borda efficiencies for various voting rules have been obtained by computer simulation [8, 9]. Analytical results concerning B_k^m have been restricted to the three alternative case with the assumption of $n \rightarrow \infty$. We assume from this point on that B_k^m is the Borda efficiency of Rule C_k under impartial culture as $n \rightarrow \infty$. In [16] it was shown that $B_1^3 = .758338$ so that plurality rule will select the Borda winner more than seventy-five percent of the time under impartial culture with a large electorate.

The purpose of the current study is to examine the behavior of B_k^m . In the next section a representation of B_k^m is obtained for general m and k . It is then shown that $B_k^m = B_{m-k}^m$ for all k and that $B_k^m \leq B_{k+1}^m$ for all $k \leq (m-2)/2$. Thus, we find a behavior of B_k^m that is identical to the behavior of E_k^m . The Borda efficiency of the vote for k rule is identical to the Borda efficiency of the vote against k rule. The most Borda efficient constant scoring rule is Rule $C_{m/2}$ when m is even and when m is odd the equally most Borda efficient rules are Rule $C_{(m-1)/2}$ and Rule $C_{(m+1)/2}$. Conclusions are presented in the final section.

2. A REPRESENTATION FOR BORDA EFFICIENCY

We wish to obtain an analytical representation of B_k^m , the probability that the Rule C_k winner coincides with the Borda winner for m alternatives and n voters under impartial culture as $n \rightarrow \infty$. Individuals are assumed to vote sincerely,

according to their preferences, and they are assumed to vote independently. We begin by finding a representation for the probability of coincidence of the Rule C_k winner and the weighted scoring rule winner with general weights $w_1, w_2, \dots, w_m$. The results for Borda winner coincidence with the Rule C_k winner will then be treated as a special case.

Let the m candidates be denoted by $A_1, A_2, \dots, A_m$. Each voter is equally likely to have any one of the $m!$ linear preference rankings on the candidates under the impartial culture assumption. Let W_k^* be the probability that any specific candidate, say A_1 , is both the Rule C_k winner and the weighted scoring rule winner with weights $w_1, w_2, \dots, w_m$. By the symmetry of impartial culture $B_k^m = m W_k^*$ when the weights in the weighted scoring rule are Borda weights with $w_i = m - i$. Therefore, the behavior of W_k^* with Borda weights will be exactly the same as the behavior of B_k^m .

To obtain a general representation of W_k^* we define $2(m-1)$ discrete variables which describe the linear preference ranking for a given voter.

$$x_i = +1,$$

if A_1 is ranked among the k most preferred alternatives and A_{i+1} is ranked among the $m-k$ least preferred alternatives.

$$x_i = -1,$$

if A_{i+1} is ranked among the k most preferred alternatives and A_1 is ranked among the $m-k$ least preferred alternatives.

$$x_i = 0,$$

otherwise.

$$y_i = w_a - w_b$$

if A_1 is ranked a th and A_{i+1} is ranked b th where x_i and y_i are defined for $i = 1, 2, \dots, m-1$.

Let $\bar{x}_i$ and $\bar{y}_i$ denote the average of x_i and y_i over the n voters. An examination of the definitions will show that A_1 is the Rule C_k winner when $\bar{x}_i > 0$ for all i and that A_1 is the weighted scoring rule winner when $\bar{y}_i > 0$ for all i . Therefore, W_k^* is the probability that $\bar{y}_i > 0$ and $\bar{x}_i > 0$ for all i as $n \rightarrow \infty$ under impartial culture. As $n \rightarrow \infty$ the probability that $\bar{x}_i = 0$ or $\bar{y}_i = 0$ for any i goes to zero. W_k^* can thus be defined as the probability that $\bar{y}_i \geq 0$ and $\bar{x}_i \geq 0$ for all i . Since the right-hand sides of these inequalities is always zero it follows that W_k^* can also be defined as the probability that $\bar{x}_i \sqrt{n} \geq 0$ and $\bar{y}_i \sqrt{n} \geq 0$ for all i . We can obtain a

representation for W_k^* under this last definition as $n \rightarrow \infty$ by appealing to the multivariate extension of the central limit theorem [26]. As $n \rightarrow \infty$ the joint distribution of the $\bar{x}_i \sqrt{n}$ and $\bar{y}_i \sqrt{n}$ variables is multivariate normal.

Again by the symmetry of the impartial culture condition, $E(x_i) = E(y_i) = 0$ and thus $E(\bar{x}_i \sqrt{n}) = E(\bar{y}_i \sqrt{n}) = 0$ for all i where E denotes expected value. Our definition of W_k^* can now be stated as the probability that the $\bar{x}_i \sqrt{n}$ and $\bar{y}_i \sqrt{n}$ variables all exceed their respective means. By this definition W_k^* is the positive orthant probability of the multivariate normal distribution of the $\bar{x}_i \sqrt{n}$ and $\bar{y}_i \sqrt{n}$ variables. The fact that we are dealing with a positive orthant probability will greatly simplify things later. The positive orthant probability of a multivariate normal distribution can be expressed totally in terms of the correlation matrix of the distribution.

It has already been noted that $E(x_i) = E(y_i) = 0$ so to obtain the correlation matrix we need $E(x_i^2)$, $E(y_i^2)$, $E(y_i y_j)$, $E(x_i x_j)$, and $E(x_i y_j)$. From previous studies [14, 17]:

$$\begin{aligned} E(x_i^2) &= \frac{2k(m-k)}{m(m-1)}, \\ E(y_i^2) &= \frac{(m-2)!}{m!} \sum_{i=1}^m \sum_{j=1}^m (w_i - w_j)^2, \\ E(y_i y_j) &= \frac{(m-2)!}{2m!} \sum_{i=1}^m \sum_{j=1}^m (w_i - w_j)^2, \\ E(x_i x_j) &= \frac{k(m-k)}{m(m-1)}. \end{aligned}$$

We can obtain the correlation matrix after finding $E(x_i y_j)$. This is done for the case of $i=j$ first and then for $i \neq j$. To obtain $E(x_i y_i)$ we know that $x_i y_i$ is positive if A_1 is among the k most preferred and A_{i+1} is among the $m-k$ least preferred candidates. There are $(m-2)!$ rankings which allow this, and each ranking has a probability of $1/m!$. To generalize, this:

$$E(x_i y_i) = \left[\sum_{i=1}^k \sum_{j=k+1}^m (w_i - w_j) - \sum_{i=k+1}^m \sum_{j=1}^k (w_i - w_j) \right] \frac{(m-2)!}{m!}.$$

This reduces to:

$$E(x_i y_i) = \left[2(m-k) \sum_{i=1}^m w_i - 2m \sum_{j=k+1}^m w_j \right] \frac{(m-2)!}{m!}.$$

Using a similar development for $E(x_i y_j)$ we find:

$$E(x_i y_j) = \left\{ \sum_{i=1}^k \sum_{j=k+1}^m \left[\sum_{l=1}^m (w_i - w_l) - (w_i - w_j) \right] - \sum_{i=k+1}^m \sum_{j=1}^k \left[\sum_{l=1}^m (w_i - w_l) - (w_i - w_j) \right] \right\} \frac{(m-3)!}{m!}.$$

This reduces to:

$$E(x_i y_j) = \frac{1}{2} E(x_i y_i).$$

The correlation matrix ρ thus has:

$$\rho(x_i x_j) = \frac{1}{2},$$

$$\rho(y_i y_j) = \frac{1}{2},$$

$$\rho(x_i y_j) = \frac{1}{2} \rho(x_i y_i),$$

$$\rho(x_i y_i) = \frac{2m \sum_{j=1}^k w_j - 2kV}{\left[2k(m-k) \sum_{i=1}^m \sum_{j=1}^m (w_i - w_j)^2 \right]^{1/2}},$$

where $V = \sum_{i=1}^m w_i$.

For a fixed set of w_i values the correlation matrix is solely a function of the k value of the Rule C_k being used. Since W_k^* is a positive orthant probability the results of Slepian [24] apply and W_k^* is maximized by the k which maximizes z with:

$$z = \frac{2m \sum_{j=1}^k w_j - 2kV}{\left[2k(m-k) \sum_{i=1}^m \sum_{j=1}^m (w_i - w_j)^2 \right]^{1/2}}.$$

We could therefore define a weighted scoring rule efficiency, as was done for Condorcet efficiency and Borda efficiency. The Rule C_k that would maximize the weighted scoring rule efficiency would correspond to the k that maximized z .

We now turn our attention to Borda efficiency and recall that Borda rule is equivalent to the weighted scoring rule with $w_i = m - i$. If we substitute this relation into the equation for z and use known relations for sums of powers of integers [22] the specific z' that results is:

$$z' = \left[\frac{3 k (m - k)}{(m + 1) (m - 1)} \right]^{1/2}$$

By previous discussion B_k^m is only a function of the correlation matrix and thus only of z' . Since z' is symmetric in k around $m/2$ it follows that $B_k^m = B_{m-k}^m$. For integer valued k , the k closest to $m/2$ maximizes z' . Therefore B_k^m is maximized by Rule $C_{m/2}$ when m is even and when m is odd B_k^m is equally maximized by Rule $C_{(m-1)/2}$ and Rule $C_{(m+1)/2}$.

3. CONCLUSIONS

Constant scoring rules were examined on the basis of Condorcet efficiency in [17] and the current study considers the Borda efficiency of constant scoring rules. It is assumed in both studies that the number of voters is large and that voters' preferences meet the condition of impartial culture. It is seen by both efficiency measures that the vote for k rule is equivalent to the vote against k rule. Also, by both efficiency measures the most efficient rule is Rule $C_{m/2}$ when m is even or either of Rule $C_{(m-1)/2}$ or Rule $C_{(m+1)/2}$ when m is odd. It can be concluded that if the condition of impartial culture is reasonable for a large electorate then serious consideration should be given to the voting rule which requires individuals to vote for half of the candidates on the ballot.

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