RAIRO. RECHERCHE OPÉRATIONNELLE

GARY J. KOEHLER

A generalized Markov decision process

RAIRO. Recherche opérationnelle, tome 14, nº 4 (1980), p. 349-354

<http://www.numdam.org/item?id=RO_1980__14_4_349_0>

R.A.I.R.O. Recherche opérationnelle/Opérations Research
(vol. 14, n° 4, novembre 1980, p. 349 à 354)

A GENERALIZED MARKOV DECISION PROCESS (*)

by Gary J. KOEHLER [1]

Abstract. — *In this paper we present a generalized Markov decision process that subsumes the traditional discounted, infinite horizon, finite state and action Markov decision process, Veinott's discounted decision processes, and Koehler's generalization of these two problem classes.*

Résumé. — *Nous présentons dans cet article un processus de Markov généralisé qui englobe le processus de décision markovien actualisé à l'horizon infini, avec état et action finis; les processus de décision actualisés de Veinott; et la généralisation de Koehler de ces deux classes de problèmes.*

1. INTRODUCTION

In this paper we explore the extension of results obtained in [2] to a broader class of problems. The author's motivation is as follows. Many practical large-scale linear programs in energy and food allocation modelling can be easily partitioned to the form

$$\left.\begin{array}{ll}\text{Max} & c'x \\ \text{S.T.} & \\ & Bx = b \\ & Ex = d \\ & x \geqq 0\end{array}\right\} \qquad (1.1)$$

where $b \geqq 0$, B is essentially Leontief (to be defined later), and $Bx = b$ accounts for most of the constraints. Programs having Leontief constraint sets can be solved without inversion [2] and procedures taking advantage of this fact in solving (1.1) are of interest. In this paper we explore the results obtained in [2] on Leontief Systems and extend these to use on essentially Leontief Systems.

In the interest of brevity we borrow heavily from [2] in notation and preliminary results.

(*) Recevied September 1979.
([1]) School of Industrial Engineering, Purdue University, West Lafayette, Indiana, U.S.A.

2. PROBLEM STATEMENT

Consider the problem

$$\left.\begin{array}{ll} \text{Max} & c'x \\ \text{S.T.} & \\ & Bx = b \\ & x \geqq 0 \end{array}\right\} \qquad (2.1)$$

where B is an $m \times k$ essentially Leontief matrix and $b \geqq 0$. An essentially Leontief matrix is a pre-Leontief matrix [4]—a matrix having at most one positive element per column—where, additionally, there is an $x \geqq 0$ such that $A x > 0$. We assume the following:

Assumption A

Problem 2.1 has a bounded objective and the columns of B are scaled so that the positive elements of B are not greater than one.

Let $A_i \subseteq \{1, 2, \ldots, n\}$ for $i = 1, \ldots, m$ such that if $B_{ij} > 0$ then $j \in A_i$ and $\bigcup_i A_i = \{1, 2, \ldots, n\}$. Note that each $A_i \neq \emptyset$ since $Bx > 0$ for some $x \geqq 0$. Let $\Delta = \prod_{i=1}^{m} A_i$. For $\delta \in \Delta$ let B^δ be the corresponding submatrix of B and let $Q^\delta = I - B^\delta$ and $P_\delta = (Q^\delta)'$. Here $P_\delta \geqq 0$.

From [1,4] there is some $\delta^* \in \Delta$ such that:

1. $v^* = [(B^{\delta^*})']^{-1} c^{\delta^*}$ solves the dual to (2.1);
2. $\rho(P_{\delta^*}) < 1$, where $\rho(P)$ is the spectral radius of the square matrix P.

Let

$$\mathscr{L}_\delta(v) = P_\delta v + c^\delta$$

and

$$\mathscr{L}(v) = \max_{\delta \in \Delta} \mathscr{L}_\delta(v).$$

Both operators are isotone. Define

$$F = \{v : v = \mathscr{L}(v)\} \qquad \text{and} \qquad C = \{v : \lim_{n \to \infty} \mathscr{L}^n(v) = v^*\}.$$

Note that $v^* \in F$ and $v^* \in C$. Finally, denote the dual feasible set of (2.1) as

$$D = \{ v : (I - P_\delta) v \geqq c^\delta, \delta \in \Delta \}.$$

Consider the following four conditions:

1. $P_\delta \geqq 0, \quad \delta \in \Delta$; (2.2)

2. $\rho(P_\delta) < 1$ for all $\delta \in \Delta$;

3. $D \neq \emptyset$;

4. I-P_δ has a positive diagonal for each $\delta \in \Delta$.

These conditions hold for discounted (semi-) Markov decisions and the discounted processes of Veinott [3]. Under these conditions $C = R^m$. In Koehler [2] condition 2 was relaxed to yield

2. $\rho(P_\delta) < 1$ for some $\delta \in \Delta$. (2.3)

1, 3 and 4 as in (2.2).

Here $C \neq R^m$ in general. In this paper we further relax (2.2) to

1, 2 and 3 as in (2.3). (2.4)

Drop 4.

Some properties of $\mathscr{L}(.)$ and C which were discovered in [2] which also hold for (2.4) are summarized below. Refer to [2] for notation.

PROPOSITION 2.5

1. $\liminf_{n \to \infty} \mathscr{L}^n(v) \geqq v^*$ *for all* $v \in R^m$.

2. $L(C) \subseteq C$.

3. C *is convex.*

4. *If* $F = [v^*]$, *then* $L(D) \subseteq C$. *In addition, if for some* $d > 0$ $P_\delta d \leqq d$ *for all* $\delta \in \Delta$, *then* $C = R^m$.

5. *If* $\rho(P_\delta) \leqq 1$ *for all* $\delta \in \Delta$ *and* $c^\delta \notin$ *range* $(I - P_\delta)$ *whenever* $\rho(P_\delta) = 1$, *then* $F = \{ v^* \}$.

6. *Assume that* $C = R^m$. *Then for every* $\delta \in \Delta$, $\rho(P_\delta) \leqq 1$ *and for* δ *having* $\rho(P_\delta) = 1$ *there exists no* $v \in R^m$ *such that* $v \leqq P_\delta v + c^\delta$.

Notice that the isotonicity of $\mathscr{L}(.)$ and $L(v^*) \subseteq C$ give an easily satisfied sufficient condition for picking a point, v^0, of C. That is, let $v^0 = -Me$ where $M \gg 0$ and e is an m vector of ones.

3. SPECTRAL PROPERTIES OF THE P_δ's AND GEOMETRIC PROPERTIES OF D

In this section we extend almost all of the properties found in [2] which relate the spectral radii of the P_δ's to boundedness of D and the property that it has a non-empty interior. From [2] we have:

PROPOSITION 3.1: *Consider the following properties of the* $\rho(P_\delta)$'*s and* D:

(*a*) *For some* $d > 0$, $P_\delta d \leqq d$ *for all* $\delta \in \Delta$.

(*b*) $\rho(P_\delta) \leqq 1$ *for all* $\delta \in \Delta$.

(*c*) *D is unbounded from above.*

(*d*) *For some* $d \geqq 0$, $P_\delta d \leqq d$ *for all* $\delta \in \Delta$.

Then $(a) \rightarrow (b) \rightarrow (c) \rightleftarrows (d)$.

In [2] it was shown, under the conditions given in (2.3), that if D has a non-empty interior then $\rho(P_\delta) = 1$ implies $c^\delta \notin$ range $(I - P_\delta)$. This is no longer the case under the conditions in (2.4). For example, let

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c' = (-2 \quad 2 \quad 0)$$

D has a non-empty interior yet for $\delta = (3,2)$ we have $P_\delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\rho(P_\delta) = 1$ and $c^\delta = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \in$ range $(I - P_\delta)$.

The following results from [2] can be established under the conditions of (2.4).

THEOREM 3.2: *Suppose for some* $d > 0$, $P_\delta d \leqq d$ *for every* $\delta \in \Delta$ *and if* $\rho(P_\delta) = 1$ *then* $c^\delta \notin$ *range* $(I - P_\delta)$. *Then D has a non-empty interior.*

Proof: Suppose $B_j \leqq 0$. Then $B_j = e_i - (-B_j + e_i)$. For some δ the i-th row of P_δ looks like $(e_i - B_j)'$. Thus $(e_i - B_j)' d \leqq d_i$ by the first part of our premise for some $d > 0$. Thus $d_i - B_j' d \leqq d_i$ or $0 \leqq B_j' d$. However, $B_j' d < 0$.

Thus $B_j \leqq 0$ implies $B_j = 0$.

Let $B_j = 0$ and δ be a decision using δ^* except in the state where B_j may be used. It is readily demonstrated that $\rho(P_\delta) = 1$. So $c^\delta \notin$ range $(I - P_\delta)$ implies that $c_j < 0$ since otherwise v^* would give c^δ.

Thus we may restrict our attention to just those columns of B having a positive element and proceed to use the proof of Theorem 4.3 in [2] to get the desired result. □

Notice that under the conditions of Theorem 3.2 $F = \{ v^* \}$ [*see* Prop. 2.5, No. (5)].

4. THE IRREDUCIBLE CASE

Refer to [2] for appropriate definitions. In this section we assume:

Assumption B

B of problem (2.3) is not permutable to an essentially dynamic Leontief matrix.

Assumption B can be tested by discarding each non-positive column of B and using the procedures mentioned in [2] on the remaining matrix.

LEMMA 4.1: *If Assumption B holds,* $d \geqq 0$ *and* $P_\delta d \leqq d$ *for every* $\delta \in \Delta$, *then* $d > 0$ *and* $B_j \leqq 0$ *implies* $B_j = 0$.

Proof: Without loss of generality we may partition B as $B = (B_1, B_2)$ with $B_2 \leqq 0$ and B_1 Leontief. Then $d'(B_1, B_2) \geqq 0$ gives that $d' B_2 = 0$. From [2] (Lemma 5.1) $d' B_1 \geqq 0$ and B_1 not dynamic Leontief gives that $d > 0$. Hence, $d' B_2 = 0$ implies $B_2 = 0$. □

From the Proposition 3.1 and Lemma 4.1 we get the following.

PROPOSITION 4.2: *If Assumption B holds, then the following are equivalent:*

(*a*) *D is unbounded from above*;

(*b*) $P_\delta d \leqq d$ *for some* $d > 0$;

(*c*) $\rho(P_\delta) \leqq 1$ *for all* $\delta \in \Delta$.

Finally, pulling together several of the above results, we get:

THEOREM 4.3: *If Assumption B holds, then consider the following:*

(*a*) $C = R^m$;

(*b*) $\rho(P_\delta) \leqq 1$ *for all* $\delta \in \Delta$ *and* $c^\delta \notin$ *range* $(I - P_\delta)$ *whenever* $\rho(P_\delta) = 1$;

(*c*) *D is unbounded from above and has a non-empty interior.*

Then (*a*) $\rightleftarrows$ (*b*) $\rightarrow$ (*c*).

Proof: (*a*) $\rightarrow$ (*b*): This follows from Proposition 2.5 part (6).

(*b*) $\rightarrow$ (*a*): From Proposition 2.5 part (5) we have $F = \{ v^* \}$. From Proposition 3.1 and Lemma 4.1 there is a $d > 0$ such that $P_\delta d \leqq d$ for all $\delta \in \Delta$. Thus from Proposition 2.5 part (4) we have that $C = R^m$.

$(b) \to (c)$: As in the previous part of the proof we have some $d > 0$ such that $P_\delta d \leqq d$ for all $\delta \in \Delta$. By Proposition 3.1 D is unbounded from above. The rest follows from Proposition 4.2 and Theorem 3.2. □

We cannot complete the implication in this theorem as can be seen by the following example. Let

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c' = (1 \quad 1 \quad 0)$$

Here D has a non-empty interior and D has a positive direction of recession ($d' = (1,1)$) but

$$\left\{ \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \end{pmatrix} : \alpha \geqq 1 \right\} \subseteq F,$$

which implies $C \neq R^m$.

Theorem 4.3 can be relaxed to complete the implications. This is done below.

THEOREM 4.4: *If Assumption B holds and if $B_j = 0$ implies that $c_j < 0$, then the conditions (a), (b), and (c) of Theorem 4.3 are equivalent.*

Proof: From Theorem 4.3 we have $(a) \rightleftarrows (b) \to (c)$.

$(c) \to (a)$: If D is unbounded from above then from Proposition 3.1 and Lemma 4.1 there is a $d > 0$ such that $P_\delta d \leqq d$ for all $\delta \in \Delta$ and $B_j \leqq 0$ implies $B_j = 0$. By Proposition 4.2, $\rho(P_\delta) \leqq 1$ for all $\delta \in \Delta$. Suppose $\rho(P_\delta) = 1$ for some $\delta \in \Delta$. From Theorem 4.3 of [2] $c^\delta \notin$ range $(I - P_\delta)$ if some row has no positive element since then this row is a vector of zeroes and the corresponding element of c^δ is negative by assumption. Thus $c^\delta \notin$ range $(I - P_\delta)$. The rest follows from Proposition 2.5 parts (5) and (4). □

REFERENCES

1. R. W. COTTLE, and A. F. VEINOTT Jr., *Polyhedral Sets Harving a Least Element*, Mathematical Programming, 3, 1972, pp. 238-249.
2. G. J. KOEHLER, *Value Convergence in a Generalized Markov Decision Progress*, S.I.A.M. J. Optimization and Control, 17, 2, 1979, pp. 180-186.
3. A. F. VEINOTT Jr., *Discrete Dynamic Programming with Sensitive Discount Optimality Criteria*, Ann. Math. Stat., 40, 1969, pp. 1635-1660.
4. A. F. VEINOTT Jr., *Extreme Points of Leontief Substitution Systems*, Linear Algebra and Its Application, 1, 1968, pp. 181-194.