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OPTIMAL REPLACEMENT UNDER ADDITIVE DAMAGE AND SELF-RESTORATION (*)

by Dror ZUCKERMAN ([1]).

Abstract. — *In this article we examine a production system which is subject to random failure. The system accumulates damage through a "wear" process and the failure time depends on the accumulated damage in the system. As long as the system is still operating the accumulated damage decreases with time and the system recovers. Upon failure the system is replaced by a new identical one and a failure cost is incurred. If the system is replaced before failure a smaller cost is incurred. Our goal is to specify a replacement rule that is optimal under the long run average cost criterion.*

Résumé. — *Nous examinons dans cet article un système de production sujet à des défaillances aléatoires. Le système accumule les dommages à travers un processus d'usure et le temps de défaillance dépend des dommages accumulés dans le système. Aussi longtemps que le système continue à être opérationnel, les dommages accumulés décroissent avec le temps et le système se rétablit. A la défaillance, le système est remplacé par un nouveau système identique et subit un coût de défaillance. Si le système est remplacé avant défaillance, le coût est plus faible. Notre objet est de spécifier une règle de remplacement qui soit optimale pour le critère du coût moyen à long terme.*

1. INTRODUCTION AND SUMMARY

A production system is subject to a sequence of random shocks occurring in a Poisson stream at rate λ. Each shock causes a random amount of damage and these damages accumulate additively. The successive shock magnitudes Y_1, Y_2, ... are positive, independent and identically distributed random variables, having a known distribution function $F(y)$. A failure can occur only at the occurrence of a shock, and the probability of such a failure is a function of the accumulated damage in the system. As long as the system is still operating, the accumulated damage decreases with time and the system recovers. This phenomena is representative of certain physical systems which after being exposed to external interferences, reobtain their original properties after some

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time. The human body is an example of such a *self-restoring system*.

Let $X(t)$ be the *accumulated damage* at time t. There is an *instantaneous self-restoration rate* depending on the accumulated damage in the system, such that in the absence of shocks we have

$$\frac{dX(t)}{dt} = -e(X(t)),$$

where we assume $e(0)=0$ and $e(x)$ is positive and continuous on $(0, \infty)$.

If a shock of magnitude y occurs at time t, then the system fails with known probability $1-r(X(t-)+y)$. Clearly, $0 \leqq r(z) \leqq 1$ for every position z. We refer to $r(\cdot)$ as the *survival function. It will be assumed that* $r(\cdot)$ *is a nonincreasing function of the cumulative damage*. Upon failure the system is replaced by a new *identical* one and the replacement cycles are repeated indefinitely.

Each replacement costs C dollars and each failure adds a cost of K dollars. Thus there is an incentive to attempt to replace the system before failure occurs. A controller has the option to replace the system at any *Markov time* $T \leqq \delta$, where δ is the failure time of the system. *Throughout the paper, we will restrict our attention to those policies for which a decision can be taken only at shock points of time*. We consider the problem of specifying a *replacement rule* under a long run average cost criterion.

Assuming that the expected failure time is finite, we show that an optimal Markov time determines a *control limit policy*. The term *control limit policy* refers to a policy in which we replace either upon failure, or when the accumulated damage first exceeds a critical control level ξ^*, whichever occurs first. The main contribution of our article is the presentation of the deterministic self-restoration of the system into the breakdown model.

A predecessor model in which the system accumulates damage through a shock process without self-restoration, has been considered by a number of researchers. Taylor [6] and Zuckerman [10] assume that the damage process is a shock process and that failure may occur only at shock point of time. In Taylor [6] the shock process is a compound Poisson process and in Zuckerman [10] the shock process is a one-sided Lévy process. In a recent paper, Zuckerman [9] derived the optimal replacement rule under a discounted cost criterion for a breakdown model in which the shock rate is monotonically non-decreasing over the state space of the damage process.

Feldman [4] derived an optimal replacement rule for the case in which the cumulative damage process is a semi-Markov process, where only policies within the class of control limit policies were considered. The above paper by Feldman was generalized by Zuckerman [8]. An additional semi-Markovian breakdown model under a discounted cost criteria was examined by Feldman in [5]. Esary, Marshall and Proschan [3] investigated the property of a breakdown model for which the damage process is determined by a Poisson process. Buckland [1] reviewed the case in which the system fails when the accumulated damage first exceeds a fixed threshold. It is interesting to note that in the threshold situation, our damage process appears to be identical with that of a content of a dam, where the total input into the dam is assumed to be a compound Poisson process and the instantaneous output rate is given by the function $e(x)$.

In Section 2 we will consider the breakdown model under the long run average cost criterion. Section 3 treats the problem of how to determine the optimal critical level ξ^*. An example will be presented illustrating computational procedures.

The following will be standard notation used through the paper:

$$E_x[\cdot]=E[\cdot \mid X(0)=x], \qquad P_x(\cdot)=P(\cdot \mid X(0)=x),$$

and reserve $E(P)$ without affixes for expectation (probability) conditional on $X(0)=0$. The notation $E[Y; A]$, where Y is a random variable and A is an event, refers to the expectation $E[I_A Y]=E[Y \mid I_A=1]\ P(A)$, where I_A is the set characteristic function of A.

2. OPTIMAL REPLACEMENT UNDER THE LONG RUN AVERAGE COST CRITERION

For $t<\delta$, $X(t)$ represents the cumulative damage attributed to shocks occurring during $[0, t]$. Let Δ be a distinct point not in $R_+ = [0, \infty)$ and define $X(t)=\Delta$ for $t \geqq \delta$.

Throughout, we assume that $E[\delta]$ *is finite.* To obtain the long term expected cost per unit time, we consider the renewal process formed by successive replacements of identical systems. By the law of large numbers, the average cost associated with a Markov time T is given by

$$\psi_T=\frac{C+KP\{T=\delta\}}{E[T]}. \tag{2.1}$$

We will restrict our attention to the following subset of Markov times

$$\mathscr{T}=\{T: T \leqq \delta,\ T=t_i \text{ for some } i \geqq 1\},$$

where $t_1, t_2, t_3, \ldots$ are the shock points of time. Note that the time interval between two successive shocks is exponentially distributed with parameter λ. Since the exponential distribution is memoryless it is intuitively clear that an optimal Markov time is in $\mathscr{T}$. Let $\psi^* = \inf_{T \in \mathscr{T}} \psi_T$ be the optimal average cost. In this section we show that an optimal Markov time is determined by a single critical number ξ^*. The optimal strategy is to replace either upon failure or when the accumulated damage first exceeds ξ^*, whichever occurs first.

Let G be the *infinitesimal operator* of the damage process $\{X(t); t \geqq 0\}$. For a function φ in the domain of G, the infinitesimal operator is defined as follows

$$G_\varphi(x) = \lim_{t \downarrow 0} t^{-1} \{ E_x[\varphi(X(t))] - \varphi(x) \}. \tag{2.2}$$

Of great importance to us is *Dynkin's formula*

$$E_x[\varphi(X(T))] - \varphi(x) = E_x\left[\int_0^T G_\varphi(X(s))\, ds \right], \tag{2.3}$$

valid for any φ in the domain of G and any Markov time T having finite expectation (theorem 5.1 and its corollary in Dynkin [2]).

We proceed with the following result.

THEOREM 1: *T is an optimal Markov time if and only if it maximizes*

$$E\left[\int_0^T \{ \psi^* - \lambda K [1 - R(X(s))] \}\, ds \right], \tag{2.4}$$

where

$$R(x) = \int_0^\infty r(x+y)\, dF(y). \tag{2.5}$$

Proof: For every Markov time $T \in \mathscr{T}$ the following inequality holds

$$\psi^* \leqq \frac{C + KP\{T = \delta\}}{E[T]}, \tag{2.6}$$

and a Markov time $T \in \mathscr{T}$ minimizes the long run average cost, if and only if it maximizes

$$\theta_T = \psi^* E[T] - C - KP\{T = \delta\} = -C + E\left[\int_0^T \psi^*\, ds \right] - KP\{T = \delta\} \tag{2.7}$$

and the maximum value of θ_T is zero. Next note that

$$P_x\{T = \delta\} = E_x[I_{(X(T) = \Delta)}]. \tag{2.8}$$

Let $f(x) = I_{(x = \Delta)}$, for $x \neq \Delta$ we have

$$\begin{aligned} G_f(x) &= \lim_{t \downarrow 0} t^{-1} \{ E_x[I_{(X(t) = \Delta)}] - f(x) \} \\ &= \lim_{t \downarrow 0} t^{-1} \left\{ \lambda t \int \{ 1 - r((x + o(1))^+ + y) \}\ dF(y) \right. \\ &\qquad \left. + o(t) \right\} = \lambda [1 - R(x)]. \end{aligned} \tag{2.9}$$

In order to clarify equation (2.9) we note that if the initial damage is x and a shock occurs at time αt for $0<\alpha<1$, then $X(\alpha t-)$ tends to $(x+o(1))^{+}$ as t tends to zero, where $Y^{+}=\max(Y, 0)$. Further, the number of shocks in $(0, t)$ equals one with probability $\lambda t+o(t)$ and it exceeds one with probability $o(t)$.

Using Dynkin's formula we obtain:

$$E_x[f(X(T))]-f(x)=P_x[T=\delta]=E_x\left[\lambda\int_0^T\{1-R(X(s))\}\,ds\right]. \quad (2.10)$$

Using equation (2.10), θ_T can be expressed as follows

$$\theta_T=-C+E\left[\int_0^T\{\psi^*-\lambda K\{1-R(X(s))\}\}\,ds\right]. \quad (2.11)$$

This concludes the proof.

In what follows we shall denote by S the state space of the stochastic process $\{X(t);\ 0\leqq t<\delta\}$. Let

$$A_x=\sup_{T\in\mathscr{T}} E_x\left[\int_0^T g(X(s))\,ds\right], \quad (2.12)$$

where

$$g(x)=\psi^*-\lambda K[1-R(x)]. \quad (2.13)$$

Note that $g(\cdot)$ is bounded and nonincreasing over S. Let

$$S_1=\{x\in S;\ A_x>0\}, \quad (2.14)$$

$$S_2=\{x\in S;\ A_x\leqq 0\}. \quad (2.15)$$

Now let us consider the following Markov time

$$T^*=\min\{\inf_{i\geqq 1} t_i;\ X(t_i)\in S_2\},\ \delta\}, \quad (2.16)$$

where $t_1, t_2, \ldots$ are the shock points of time. If S_2 is empty then $T^*=\delta$. We will show that T^* is an optimal replacement time. Furthermore, as will be proved, T^* is a control limit policy. We proceed with the following proposition.

PROPOSITION 1: *S_1 is nonempty, and for every point x in S_1:*

$$E_x\left[\int_0^{T^*} g(X(s))\,ds\right]>0. \quad (2.17)$$

Proof: As we know, $\sup_{T\in\mathscr{T}}\theta_T=0$. Therefore there exists a Markov time $\overline{T}\in\mathscr{T}$, such that

$$\theta_{\bar{T}} = -C + E\left[\int_0^{\bar{T}} g(X(s))\,ds\right] > -\frac{C}{2}.$$

Hence

$$E\left[\int_0^{\bar{T}} g(X(s))\,ds\right] > \frac{C}{2} > 0. \tag{2.18}$$

Therefore $0 \in S_1$. For a given point x in S_1, let

$$\mathscr{T}(x) = \left\{ T \in \mathscr{T};\ E_x\left[\int_0^T g(X(s))\,ds\right] > 0 \right\}. \tag{2.19}$$

Let T_x be an element in $\mathscr{T}(x)$. Consider the following Markov time

$$T_x^1 = \min\{T_x, T^*\}.$$

First we show that $T_x^1 \in \mathscr{T}(x)$.

If S_2 is empty then $T^* = \delta$, and from the definition of T_x^1 it follows that $T_x^1 = T_x$ a. s. Hence $T_x^1 \in \mathscr{T}(x)$ (*see* 2.19). We proceed to examine the case when S_2 is nonempty. Observe that

$$E_x\left[\int_0^{T_x} g(X(s))\,ds\right] - E_x\left[\int_0^{T_x^1} g(X(s))\,ds\right]$$

$$= E_x\left[\int_{T_x^1}^{T_x} g(X(s))\,ds;\ T_x^1 < T_x\right]$$

$$= \int_{\{T_x^1 < T_x\}} \left\{ E_x\left[\int_{T_x^1}^{T_x} g(X(s))\,ds \,\middle|\, X(t),\ 0 \leqq t \leqq T_x^1\right]\right\} dP_{(x,\,T_x^1)}, \tag{2.20}$$

where $P_{(x,\,T_x^1)}$ is the probability on sample paths $X(t)$ $(0 \leqq t \leqq T_x^1)$, given $X(0) = x$.

Clearly $X(T_x^1) = X(T^*) \in S_2$ on the set $\{T_x^1 < T_x\}$. According to the definition of S_2 and by the strong Markov property we have

$$\int_{\{T_x^1 < T_x\}} \left\{ E_x\left[\int_{T_x^1}^{T_x} g(X(s))\,ds \,\middle|\, X(t), 0 \leqq t \leqq T_x^1\right]\right\} dP_{(x,\,T_x^1)} \leqq \sup_{y \in S_2} A_y \leqq 0. \tag{2.21}$$

Using (2.20) and (2.21) we obtain:

$$E_x\left[\int_0^{T_x^1} g(X(s))\,ds\right] > 0.$$

Hence $T_x^1 \in \mathscr{T}(x)$. Now, let us define a new Markov time T_x^2 as follows:

$$T_x^2 = \begin{cases} \min\{T_x^1 + T_{X(T_x^1)}, T^*\} & \text{if } T_x^1 < T^*, \\ T^* & \text{if } T_x^1 = T^*, \end{cases}$$

where

$$T_{X(T_x^1)} \in \mathscr{T}(X(T_x^1)).$$

Obviously $T_x^2 \geqq T_x^1$ and $E_x\left[\int_0^{T_x^2} g(X(s))\,ds\right] > 0$.

Generally, we will define an increasing sequence of Markov times $\{T_x^n\}_{n \geqq 1}$, such that

I $$E_x\left[\int_0^{T_x^n} g(X(s))\,ds\right] > 0 \quad \text{for every} \quad n \geqq 1,$$

II $$T_x^n \leqq T^* \quad \text{for every} \quad n \geqq 1,$$

in the following way

$$T_x^n = \begin{cases} \min\{T_x^{n-1} + T_{X(T_x^{n-1})}, T^*\} & \text{if } T_x^{n-1} < T^*, \\ T^* & \text{if } T_x^{n-1} = T^*, \end{cases}$$

where $T_{X(T_x^{n-1})} \in \mathscr{T}(X(T_x^{n-1}))$.

The sequence $\{T_x^n\}$ converges to a limit, say V. Clearly, $V \leqq T^*$; we show that $V = T^*$ a.s. We do so by contradiction. Let us suppose that $P\{V < T^*\} = \varepsilon > 0$. Recalling that action may be taken only at shock points of time and that the time interval between two successive shocks is a random variable exponentially distributed with parameter λ, it follows that for every $n \geqq 1$:

$$\begin{aligned} \int_{T_x^n < T^*} I_{(V - T_x^n \geqq 1)}\,dP &= E[I_{(V - T_x^n \geqq 1)};\, T_x^n < T^*] \\ &\geqq E[I_{(T_x^{n+1} - T_x^n \geqq 1)};\, T_x^n < T^*] \\ &= P\{T_x^n < T^*\}\, E[I_{(T_x^{n+1} - T_x^n \geqq 1)} \mid T_x^n < T^*] \\ &\geqq P\{V < T^*\} e^{-\lambda} = \varepsilon e^{-\lambda} > 0. \end{aligned}$$

This contradicts the assumption that T_x^n converges to V. Hence $P\{V < T^*\} = 0$. Thus T_x^n converges to T^* a.s. Since $g(\cdot)$ is a bounded function and

$$E_x\left[\int_0^{T_x^n} g(X(s))\,ds\right] > 0 \quad \text{for every} \quad n \geqq 1,$$

it follows that

$$E_x\left[\int_0^{T^*} g(X(s))\,ds\right] > 0.$$

This completes the proof of proposition 1.

We are now in a position to prove the main result.

THEOREM 2: *T^* minimizes the long run average cost.*

Proof: It suffices to show that for every Markov time $T \in \mathscr{T}$:

$$\theta_{T^*} - \theta_T \geqq 0.$$

For a given Markov time $T \in \mathscr{T}$ we have

$$\begin{aligned}\theta_{T^*} - \theta_T &= E\left[\int_0^{T^*} g(X(s))\,ds\right] - E\left[\int_0^{T} g(X(s))\,ds\right] \\ &= E\left[\int_T^{T^*} g(X(s))\,ds;\ T < T^*\right] \\ &\quad - E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T\right]. \end{aligned} \qquad (2.22)$$

First note that

$$\begin{aligned} E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T\right] &= E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T,\ T^* < \delta\right] \\ &\quad + E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T,\ T^* = \delta\right]. \end{aligned} \qquad (2.23)$$

Given that $T^* = \delta$, it follows that $T^* = T = \delta$ on the set $\{T^* \leqq T\}$, therefore

$$E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T,\ T^* = \delta\right] = 0. \qquad (2.24)$$

On the other hand, given that $T^* < \delta$, it follows from the definition of T^* that $X(T^*) \in S_2$. By the strong Markov property we find that for every Markov time $T \in \mathscr{T}$:

$$\begin{aligned} E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T,\ T^* < \delta\right] &= \int_{\{T^* \leqq T,\ T^* < \delta\}} \left\{ E\left[\int_{T^*}^{T} g(X(s))\,ds \,\middle|\, X(t),\ 0 \leqq t \leqq T^*\right]\right\} dP_{(0,T^*)} \\ &\leqq \sup_{y \in S_2} A_y \leqq 0, \end{aligned} \qquad (2.25)$$

where $P_{(x, T^*)}$ is the probability on sample paths $X(t)$ $(0 \leqq t \leqq T^*)$, given $X(0)=x$.

Using (2.23), (2.24) and (2.25) we obtain:

$$E\left[\int_{T^*}^{T} g(X(s))\,ds;\ T^* \leqq T\right] \leqq 0.$$

In order to establish the optimality of T^* it suffices to show that

$$E\left[\int_{T}^{T^*} g(X(s))\,ds;\ T < T^*\right] \geqq 0 \qquad \text{for every} \quad T \in \mathscr{T}. \qquad (2.26)$$

According to the definition of T^*, $T < T^*$ implies that $X(T) \in S_1$. By the strong Markov property and by proposition 1, the non-negativity of (2.26) follows immediately.

Whence $\theta_{T^*} - \theta_T \geqq 0$ for every $T \in \mathscr{T}$ and this establishes the optimality of T^*, as desired.

Finally, we show that T^* is a control limit policy.

THEOREM 3: *T^* is a control limit policy.*

Proof: Let $y \in S_1$ and x be an arbitrary state such that $0 \leqq x < y$. It suffices to show that $x \in S_1$. We will construct a Markov time $T(x)$ such that

$$E_x\left[\int_0^{T(x)} g(X(s))\,ds\right] \geqq E_y\left[\int_0^{T^*} g(X(s))\,ds\right] > 0. \qquad (2.27)$$

Let $\{\Omega_j, \mathscr{F}_j, P_j\}$; $j \geqq 1$, be the probability space associated with the random vector $(\bar{t}_j, Y_j)$, where $\bar{t}_j$ is exponentially distributed with parameter λ and Y_j is distributed according to the distribution function $F(y)$; for $j \geqq 1$, the random variables $\bar{t}_j$ and Y_j are independent. Furthermore the random vectors $(\bar{t}_j, Y_j)$ for $j \geqq 1$ are also independent. Let $(\Omega, \mathscr{F}, P)$ be the probability space defined by the infinite product $\left(\prod_{j=1}^{\infty} \Omega_j, \prod_{j=1}^{\infty} \mathscr{F}_j, \prod_{j=1}^{\infty} P_j\right)$. Every sample point $\omega \in \Omega$ describes a sequence

$$\{t_i, Y_i\}_{i=1}^{\infty} \qquad \text{where} \quad t_i = \sum_{j=1}^{i} \bar{t}_j \qquad \text{and} \quad Y_i,$$

are the time of the *ith* stock and the magnitude of damage associated with the *ith* shock respectively.

Let

$$\delta_z = \inf\{t \geqq 0;\ X(t) = \Delta \mid X(0) = z\}.$$

For a given sample point ω let $\delta_z(\omega)$ be the failure time associated with the sample point ω, conditional on $X(0)=z$. Let $X(t \mid z, \omega)$, $t \in [0, \delta_z(\omega))$, be the accumulated damage at time t associated with the sample point ω, when the initial damage at time zero is z. Note that the triple $(z, \omega, \delta_z(\omega))$ determines completely the damage process along the time interval $[0, \delta)$.

Let $\overline{X}(t \mid z, \omega)$ $t \geqq 0$ be the accumulated damage at time t associated with the sample point ω, when the initial damage at time zero is z, under the assumption that the survival function is identically one on $[0, \infty)$. Clearly

$$\overline{X}(t \mid z, \omega) = X(t \mid z, \omega)$$

for every $t < \delta_z(\omega)$. For every sample point ω we define a sequence of Markov times $\{\tilde{T}_i(\omega)\}_{i \geqq 1}$ as follows

$$P\{\tilde{T}_i(\omega) = t_i\} = 1 - \frac{r(\overline{X}(t_i \mid y, \omega))}{r(\overline{X}(t_i \mid x, \omega))}$$

and

$$P\{\tilde{T}_i(\omega) = \infty\} = 1 - P\{\tilde{T}_i(\omega) = t_i\},$$

with the convention that $0/0 = 1$ above.

For every sample point $\omega \in \Omega$ the probability laws of the random variables $T_i(\omega)$ are known explicity for $i \geqq 1$. Furthermore, for every given i, the random variables $T_i(\omega)$ and $\delta_x(\omega)$ are stochastically independent.

Now consider the following Markov times:

$$\hat{T}_1(x)(\omega) = \inf_{i \geqq 1}\{\tilde{T}_i(\omega)\},$$

$$\hat{T}_2(x)(\omega) = \inf_{i \geqq 1}\{t_i;\ X(t_i \mid y, \omega) \in S_2\},$$

$$\hat{T}_3(x)(\omega) = \delta_x(\omega).$$

The representation of the above Markov times is just a mathematical tool in order to construct a Markov time $T(x)$ which will satisfy (2.27).

Using the definition of the random sequence $\{\tilde{T}_i(\omega)\}_{i \geqq 1}$ it can be seen that

$$P\{\hat{T}_1(x)(\omega) > t_i\} = \prod_{j=1}^{i} \frac{r(\overline{X}(t_j \mid y, \omega))}{r(\overline{X}(t_j \mid x, \omega))}, \tag{2.28}$$

on the other hand the probability measure of the failure time associated with a sample point ω and initial damage x can be expressed as follows

$$P\{\delta_x(\omega) > t_i\} = \prod_{j=1}^{i} r(\overline{X}(t_j \mid x, \omega)). \tag{2.29}$$

Using (2.28) and (2.29) we obtain:

$$\mathrm{P}_x\{\{\hat{T}_1(x)(\omega) > t_i\} \cap \{\delta_x(\omega) > t_i\}\} = P_y\{\delta_y(\omega) > t_i\}. \tag{2.30}$$

Let $T(x) = \min\{\hat{T}_1(x), \hat{T}_2(x), \hat{T}_3(x)\}$.

Recalling the definition of T^* and using (2.30), it can be seen that for every given $\omega \in \Omega$, $T(x)$ given $X(0)=x$ and T^* given $X(0)=y$ are *identically distributed*. As a result of the monotonicity of $g(\cdot)$ we obtain (recall that $x < y$):

$$E_y\left[\int_0^{T^*} g(X(s \mid y, \omega))\,ds\right] \leqq E_x\left[\int_0^{T(x)} g(X(s \mid x, \omega))\,ds\right]. \qquad (2.31)$$

applying the law of total probability and using (2.27) and (2.31) it follows that

$$E_x\left[\int_0^{T(x)} g(X(s))\,ds\right] = \int E_x\left[\int_0^{T(x)} g(X(s \mid x, \omega))\,ds\right] dP(\omega)$$

$$\geqq \int E_y\left[\int_0^{T^*} g(X(s \mid y, \omega))\,ds\right] dP(\omega) = E_y\left[\int_0^{T^*} g(X(s))\,ds\right] \geqq 0.$$

Therefore, $x \in S_1$, which implies that T^* is a control limit policy.

3. REMARKS ON THE DETERMINATION OF ξ^* UNDER THE LONG RUN AVERAGE COST CRITERION

We now investigate the problem of computing the optimal control level ξ^*. Let T_ξ be a control limit policy with critical control level ξ. In order to minimize ψ_{T_ξ} analytically, one has to express $E[T_\xi]$ and $P\{T_\xi = \delta\}$ as functions of ξ. Next we examine the special case where $e(x) = I_{(x>0)}$. Let us consider the sub-stochastic kernel

$$k(x, \xi) = E_x[r(X(t_1-)+Y_1);\ X(t_1-)+Y_1 \leqq \xi]$$
$$= P\{N > 1 \text{ and } X(t_1-)+Y_1 \leqq \xi \mid X(0)=x\},$$

where N is the index of shock at which failure occurs. Also, let K^0 be the identity kernel, namely

$$K^0(x, \xi) = \begin{cases} 1 & \text{if } x \leqq \xi, \\ 0 & \text{if } x > \xi. \end{cases}$$

For a given point x in S we have

$$K(x, \xi) = e^{-\lambda x} K(0, \xi) + \int_{\max(0, x-\xi)}^{x} \int_0^{(\xi - x + t)^+} \lambda e^{-\lambda t} r(x - t + y)\,dF(y)\,dt,$$

where

$$K(0, \xi) = \int_0^{\xi+} r(y)\,dF(y).$$

Finally define

$$K^n(x, \xi)=P\{N > n \text{ and } X(t_i-)+Y_i \leqq \xi \text{ for } i=1, 2, \ldots, n \mid X(0)=x\}=\int_0^\xi K(x, dz)K^{n-1}(z, \xi),$$

and

$$Q(x, \xi)=\sum_{n=0}^{\infty} K^n(x, \xi).$$

Clearly $Q(x, \xi)$ gives the mean number of n's satisfying $n < N$ and $X(t_i-)+Y_i \leqq \xi (i=1, 2, \ldots, n)$, conditional on $X(0)=x$. Let us abbreviate

$$Q(\xi)=Q(0, \xi).$$

Recalling that the intershock times are exponentially distributed, each having mean λ^{-1} it follows that

$$E[T_\xi]=\lambda^{-1} Q(\xi-).$$

Now we compute the probability for a planned replacement when a Markov time T_ξ is employed.

$$P\{T_\xi < \delta\} = \sum_{n=0}^{\infty} \int_0^\xi K^n(0, dz) \times \left\{\left\{\int_0^z \int_{\xi-(z-t)}^{\infty} r(z-t+y)\lambda e^{-\lambda t} dF(y)\, dt\right\}+e^{-\lambda z}\int_\xi^\infty r(y)\, dF(y)\right\}$$

$$= \int_0^\xi \left\{\int_0^z \int_{\xi-(z-t)}^{\infty} r(z-t+y)\lambda e^{-\lambda t} dF(y)\, dt+e^{-\lambda z}\int_\xi^\infty r(y)\, dF(y)\right\} dQ(z).$$

Thus $E[T_\xi]$ and $P\{T_\xi < \delta\}$ are expressible as functions of ξ.

The cumulative damage failure model that has received the most attention in the literature is the *threshold model* in which $r(x)$ is 1 or 0 according as $x < L$ or $x \geqq L$.

In the threshold situation the accumulated damage process appears to be identical with that of the content of a dam with finite capacity, where the total input to the dam is assumed to be a compound Poisson process. Consequently, many results of storage theory can be used to simplify computational procedures, as will be demonstrated in the following example.

Example: Let us consider the following model

$$r(z)=\begin{cases} 1 & \text{for } 0 \leqq z < L, \\ 0 & \text{for } z \geqq L. \end{cases}$$

In words, system failure occurs when the cumulative damage first exceeds a fixed threshold L. The self-restoration has a constant rate of one. That is $e(x)=1$ for every $x \in S$, $x > 0$. It will be assumed that the damage distribution F is exponentially distributed with parameter μ. Obviously it suffices to consider control levels ξ such that $0 < \xi \leqq L$. By using the memoryless property of the exponential distribution it follows that

$$P\{T_\xi = \delta\} = e^{-\mu(L-\xi)}.$$

On the other hand

$$E[T_\xi] = \frac{1}{\lambda} + \frac{\mu}{\lambda(\lambda-\mu)^2}(\mu e^{-(\lambda-\mu)\xi} - \mu + \lambda(\lambda-\mu)\xi).$$

The above can be obtained as a special case from Yeo [7]. Now in order to find the optimal control level ξ^* we have simply to minimize

$$\psi_{T_\xi} = \frac{C + K e^{-\mu(L-\xi)}}{(1/\lambda) + (\mu/\lambda(\lambda-\mu)^2)(\mu e^{-(\lambda-\mu)\xi} - \mu + \lambda(\lambda-\mu)\xi)}$$

as a function of ξ for $0 < \xi \leqq L$.

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