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ON THE BUSY PERIOD FOR THE $M|G|1$ QUEUE WITH FINITE WAITING ROOM

by E. J. VANDERPERRE ⁽¹⁾

Summary. — The Laplace-Stieltjes transform (L.S.T.) of the probability distribution function of the busy period for the $M|G|1$ queueing system with finite waiting room can be obtained by using a supplementary variable method (a Markov characterization).

In the present paper, it is shown that applying this method, it is rather easy to analyse queueing systems with a finite waiting room and with Poisson input.

1. INTRODUCTION

For the well known model of the $M|G|1$ queue with finite waiting room, Cohen [1] obtained the L.S.T. of the distribution function of the busy period in a rather simple way using a Markov renewal branching argument.

In this paper, we use a simple method based on the inclusion of a supplementary variable (Markov characterization). Moreover, the method is not restricted to queueing systems with Poisson input.

2. MATHEMATICAL ANALYSIS OF THE SYSTEM

Denote by λ^{-1} the mean interarrival time of customers and by $F(\cdot)$ the distribution function of the service times. Let θ_K denote the duration of a busy period of the $M|G|1$ queue with K waiting places. A customer who finds all waiting places occupied, cannot enter the system. He departs and never returns (overflow).

Moreover, let X_t be the number of customers waiting in the system at any instant of time t and x_t the past service time of the customer being served at time t . It is assumed that a customer arrives at time $t = 0$ and that he meets an empty system.

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The busy period θ_K is the time between $t = 0$ and the first instant at which the system becomes empty (the absorbing state).

For $i = 0, 1, \dots, K$ and $t, x > 0$ we define

$$p_i(t, x) dx \stackrel{\text{def}}{=} \Pr[\underline{X}_t = i, x \leq \underline{x}_t < x + dx \mid x_0 = 0]$$

By simple probabilistic arguments, we have for $i = 0, 1, \dots, K-1$

$$p_i(t, x) = \sum_{j=0}^{i-1} p_j(t-x, 0+) e^{-\lambda x} \frac{(\lambda x)^{i-j}}{(i-j)!} [1 - F(x)]$$

$$p_K(t, x) = \sum_{j=0}^{K-1} p_j(t-x, 0+) \left\{ 1 - \sum_{l=0}^{K-1-j} e^{-\lambda x} \frac{(\lambda x)^l}{l!} \right\} [1 - F(x)]$$

The Laplace transform with respect to t of $p_i(t, x)$ will be denoted by $p_i^*(s, x)$; $\text{Re } s \geq 0$.

If we consider the events in which $\underline{x}_t = 0+$, then we obtain the following set of recurrence relations for the boundary values of the functions $p_i^*(s, x)$; $i = 0, 1, \dots, K-1$,

$$(1) \quad \begin{cases} p_i^*(s, 0+) = \delta_{0i} + \sum_{j=0}^{i-1} p_j^*(s, 0+) \int_0^\infty e^{-sx} e^{-\lambda x} \frac{(\lambda x)^{i+1-j}}{(i+1-j)!} dF(x) \\ p_{K-1}^*(s, 0+) = \sum_{j=0}^{K-1} p_j^*(s, 0+) \int_0^\infty e^{-sx} \left\{ 1 - \sum_{l=0}^{K-1-j} e^{-\lambda x} \frac{(\lambda x)^l}{l!} \right\} dF(x) \\ p_K^*(s, 0+) = 0 \end{cases}$$

where δ_{0i} is the Kronecker symbol.

Let

$$f(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sx} dF(x) \quad \text{Re } s \geq 0$$

$$f_i(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-sx} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dF(x) \quad i \geq 0$$

The L.S.T. of the probability distribution function of the busy period θ_K will be denoted by $E \{ e^{-s\theta_K} \}$.

Clearly

$$(2) \quad E \{ e^{-s\theta_K} \} = f_0(s) p_0^*(s, 0+)$$

The following column K-vectors will be used.

$$\bar{p}^*(s, 0+) \stackrel{\text{def}}{=} [p_0^*(s, 0+), \dots, p_i^*(s, 0+), \dots, p_{K-1}^*(s, 0+)]^T$$

$$\bar{1} = [1, 0, \dots, 0, \dots, 0]^T$$

and the $K \times K$ matrix function

$$\mathbf{P}(s) \stackrel{\text{def}}{=} \begin{bmatrix} f_1(s) & f_0(s) & 0 & 0 & . & . & . & 0 & 0 & 0 \\ f_2(s) & f_1(s) & f_0(s) & 0 & . & . & . & 0 & 0 & 0 \\ f_3(s) & f_2(s) & f_1(s) & f_0(s) & 0 & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ f_{i+1}(s) & f_i(s) & f_{i-1}(s) & \dots & f_2(s) & f_1(s) & f_0(s) & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ f_{K-1}(s) & f_{K-2}(s) & f_{K-3}(s) & . & . & . & . & f_2(s) & f_1(s) & f_0(s) \\ a_{K-1}(s) & a_{K-2}(s) & a_{K-3}(s) & . & . & . & . & a_2(s) & a_1(s) & a_0(s) \end{bmatrix}$$

where

$$a_i \stackrel{\text{def}}{=} f(s) - \sum_{l=0}^{l=i} f_l(s) \quad i = 0, 1, \dots, K-1.$$

By (1) we obtain

$$[I - \mathbf{P}(s)]\bar{p}^*(s, 0+) = \bar{1}$$

where I is the unit matrix.

It can be shown that the matrix $[I - \mathbf{P}(s)]$, $0 \leq \text{Re } s < \infty$, is nonsingular. For $K \geq 0$ we define $\det [I - \mathbf{P}(s)] = \Delta_K(s)$ and $\Delta_0(s) = 1$. By (2) and Cramer's rule, we have

$$(3) \quad E \{ e^{-s_0 \pi} \} = f_0(s) \frac{\Delta_{K-1}(s)}{\Delta_K(s)} \quad K \geq 1.$$

But

$$(4) \quad \Delta_K(s) = \Delta_{K-1}(s) - \sum_{j=1}^{j=K-1} f_0^{j-1}(s) f_j(s) \Delta_{K-j}(s) - f_0^{K-1}(s) a_{K-1}(s).$$

Let

$$(5) \quad \gamma_{K+1-j}(s) \stackrel{\text{def}}{=} \frac{\Delta_{K-j}(s)}{f_0^{K-j}(s)} \quad j = 0, 1, \dots, K.$$

By (4) and (5) we obtain after some simplifications

$$(6) \quad \gamma_K(s) = \sum_{j=0}^{K-1} f_j(s) \gamma_{K+1-j}(s) + a_{K-1}(s) \quad K \geq 1$$

Let

$$(7) \quad \gamma(s, Z) \stackrel{\text{def}}{=} \sum_{K=1}^{\infty} \gamma_K(s) Z^K \quad |Z| < |\rho(s)|$$

where $\rho(s)$ is the smallest root of the functional equation $Z - f(s + \lambda - \lambda Z) = 0$ $\text{Re } s \geq 0$.

By (6), (7) and algebra, we find

$$(8) \quad \gamma(s, Z) = \frac{Z}{1-Z} \left\{ 1 - \frac{Z - Zf(s)}{Z - f(s + \lambda - \lambda Z)} \right\}$$

If $\odot$ denote a circle with center at the origin of the complex Z -plane and with radius $|Z| < |\rho(s)|$, $\text{Re } s \geq 0$, then by (3), (5) and (8) we obtain for $K = 1, 2, \dots$ $\text{Re } s \geq 0$,

$$E \{ e^{-s\theta_K} \} = \frac{1 - \{ 1 - f(s) \} \frac{1}{2\pi i} \oint \frac{dZ}{Z^{K-1}} (1-Z)^{-1} [Z - f(s + \lambda - \lambda Z)]^{-1}}{1 - \{ 1 - f(s) \} \frac{1}{2\pi i} \oint \frac{dZ}{Z^K} (1-Z)^{-1} [Z - f(s + \lambda - \lambda Z)]^{-1}}$$

It is easy to show that the result agrees with Cohen's Theorem [1] p. 825.

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