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THE RELEVATION TRANSFORM AND A GENERALIZATION OF THE GAMMA DISTRIBUTION FUNCTION

par Martin KRAKOWSKI

Abstract. — *Survival models arising in mathematical demography, in renewal and in replacement models lead to a generalization of the Gamma distribution function. In these models the relevation product corresponds to the addition of random variables, generally dependent. The relevation product is non-commutative, non-associative, and only left-distributive.*

In the case of auto-relevation, a relevation product of two identically distributed random variables, a measure of renewal gain or life-extension is described by an expression akin to Shannon's entropy.

A common generalization of both the relevation and convolution operations is indicated in Section 8.

Section 1

Definition. $s(t)$ is a *survivability function* if $1 - s(t)$ is the cumulative distribution function of a non-negative random variable. $s(t)$ is interpreted as the probability that a newborn individual will survive (at least) till age t; or that a new item will give at least t time units of service.

The probability density that a newborn individual will die at age t is $-s'(t)$; cf. Appendix. For simplicity of exposition we will assume, whenever needed, the differentiability of the functions considered.

Theorem. Let $A(t)$ and $B(t)$ be survivability functions. Then

$$C(t) = A(t) - B(t) \int_{x=0}^{t} \frac{\mathrm{d}A(x)}{B(x)} \quad , \quad t \geqslant 0 \tag{1.1}$$

is also a s.f.

Proof. $C(0) = 1$, since $A(0) = 1$. $C(t)$ tends to zero as t tends to infinity since $A(t)$ and $B(t)$ do so and the integral in (1) obviously remains finite. Finally,

$$C'(t) = -B'(t) \int_{x=0}^{t} \frac{dA(x)}{B(x)} \leqslant 0 ; \tag{1.1 a}$$

since $B'(t) \leqslant 0$ and $dA(x) \leqslant 0$. Therefore $C(t)$ is a s.f. It follows from (1 *a*) that $C'(0) = 0$.

To abbreviate the notation we introduce the symbol $\#$, and write

$$C(t) = A(t) \# B(t) \quad \text{or even} \quad C = A \# B. \tag{1.2}$$

The asymmetry of $\#$ suggests, correctly, the non-commutativity of the operation.

We will say that $C(t)$ is the *relevation* of $A(t)$ by $B(t)$, the term being selected because of its possible interpretations, as shown below. Indeed, it was in connection with the following two examples, Sections 2 and 3, that the notion of the relevation transform was introduced by the author. We will also refer to $C(t)$ as the relevation product, or relevation, of $A(t)$ and $B(t)$.

Section 2. An item from a population with the survivability function $A(t)$ is being replaced at the time of its failure, its age being then x, by another item of the same age x but from another population with survivability function $B(t)$. Then, the s.f. of the cumulative service life, from the beginning of the service until the failure of the second item, is the relevation $C(t)$, of $A(t)$ by $B(t)$.

Proof. The probability that either one of the two items, the original one or its replacement, is still in sevice at time t is the sum of two probabilities :

a) the probability that that the first item is still alive (in service) at time t, i.e. $A(t)$; and

b) the probability that the first item failed at some time $x \leqslant t$, and that the successor item has survived from its age x until age t, which is

$$-\int_{x=0}^{t} A'(x) \frac{B(t)}{B(x)} \mathrm{d}x = -B(t) \int_{x=0}^{t} \frac{\mathrm{d}A(x)}{B(x)};$$

here we have taken into account that the probability density of a first item failing at age x is $-A'(x)$ and that the probability that a second item selected at age x will survive at least until age $t > x$ is $B(t)/B(x)$.

Therefore, the probability $C(t)$ that the combined, or relevated, life is not terminated at time t (since beginning of service) is

$$C(t) = A(t) - B(t) \int_{x=0}^{t} \frac{\mathrm{d}A(x)}{B(x)} \tag{1.1}$$

q.e.d.

The corresponding probability density of the total life spans is $-C'(t)$ as given by (1.1a).

Section 3. The auto-relevation of $A(t)$, defined as $A(t) \# A(t)$, has an interesting interpretation. Assume that in a human population with the s.f. $A(t)$ a new miracle pill restores to a dying person, whatever his physical condition, the full remaining lifetime distribution of his age; in other words the dying person becomes statistically typical of his age group. However, this wonder medicine can be taken only once by each individual because it is entirely ineffective if taken again. Then, the new survivability function (of the double, relevated life) is the auto-relevation function $A(t) \# A(t)$.

Note that

$$C(t) = A(t) \# A(t) = A(t) - A(t) \int_0^t \frac{\mathrm{d}A(x)}{A(x)} = A(t) - A(t) \int_0^t \mathrm{d} \ln A(x)$$

and therefore

$$C(t) = A(t) \# A(t) = A(t) - A(t) \ln A(t). \tag{3.1}$$

The expected increase in the life-span due to the wonder drug, is for a new-born individual, with $C(t)$ given by (3.1),

$$H = -\int_0^\infty tC'(t)\,\mathrm{d}t = -tC(t)\Big|_0^\infty + \int_0^t C(t)\,\mathrm{d}t.$$

$tC(t)$ tends to zero, as $t \to \infty$, under all realistic conditions, e.g. when nobody lives longer than a million years; or when $A(t) \sim 1/t^{1+\epsilon}$, for large t, as is easily seen.

Therefore, the expected increase in longevity is

$$H = -\int_0^\infty A(t) \ln A(t)\,\mathrm{d}t, \tag{3.2}$$

an entropy-like expression.

Assume now that the effect of the pill is to revive the dying person by transferring him to a population characterized by the s.f. $B(t)$. Thus, if the (first) death is about to occur at an age x the pill causes the remaining lifetime y to have the s.f. $B(x+y)/B(x)$. The total life-span, composed of the successive phases A and B, has the s.f. given by (1.1), of course. Indeed, we have a re-interpretation of the scenario in section 2.

Section 4. The expression for H in (3.2) is not an entropy since $A(t)$ is not a probability density function. Of course, $A(t)$ can be normalized by multiplying it by a suitable factor to make it integrable to 1. This normalization occurs naturally when relating $A(t)$ to the associated age density of a stationary (zero-growth) population whose survivability function is $A(t)$. As shown in the Appendix for such a stationary population the age density $p(t)$ is

$$p(t) = A(t)/M \tag{4.1}$$

where the constant M is the life expectancy at birth.

Substituting $A(t) = Mp(t)$ into (3.2) we get

$$H = MEnt\,\{\,p(t)\,\} - M \ln M \tag{4.2}$$

where

$$\operatorname{Ent}\{\,p(t)\,\} = -\int_0^\infty p(t) \ln p(t)\, \mathrm{d}t. \tag{4.3}$$

By a suitable choice of the time unit, namely using the expected lifetime m as the unit span, i.e. $M = 1$, we obtain the more elegant expression (4.4) in place of (4.2) :

$$H = \operatorname{Ent}\{\,p(t)\,\} = -\int_0^\infty p(t) \ln p(t)\, \mathrm{d}t \quad ; \quad M = 1,\ A(t) = p(t). \tag{4.4}$$

(Note that $A(t)$ is dimensionless but $p(t)$ is in units 1/time.)

It is rather curious that the entropy expression should arise in a context outside information theory.

Section 5. It is easy to see that the relevation product is left-distributive, that is, if $A(t)$, $B(t)$, and $C(t)$ are survivability functions and $a + b = 1$, $a \geqslant 0$, $b \geqslant 0$, then

$$(aA + bB) \# C = aA \# B + bB \# C. \tag{5.1}$$

If we define formally the relevation product by formula (1.1) without requiring A, B, and C in (5.1) to be survivability functions, and allowing a and b to be any numbers, then the left-distributiove property still holds.

Obviously, there is no right-distributivity, generally, for the relevation product.

Commutativity holds only under the conditions described in the following Theorem.

Theorem. $A(t) \# B(t) = B(t) \# A(t)$ holds if, and only if, $A(t) = [B(t)]^a$, or equivalently if $A(t) = [s(t)]^m$ and $B(t) = [s(t)]^n$. $s(t)$ is a survivability function, and a, m, and n are non-vanishing real numbers.

Proof. It can be easily verified that

$$[s(t)]^m \# [s(t)]^n = [m[s(t)]^n - n[s(t)]^m]/(m - n). \tag{5.2}$$

From the symmetry in m and n it follows that $s^m \# s^n = s^n \# s^m$.

To prove the converse assume that

$$A(t) - B(t)\int_0^t \frac{A'(x)}{B(x)}\,dx \equiv B(t) - A(t)\int_0^t \frac{B'(x)}{A(x)}\,dx. \tag{5.3}$$

Differentiating both sides of the identity (5.3) we get

$$-B'(t)\int_0^t \frac{A'(x)}{B(x)}\,dx \equiv -A'(x)\int_0^t \frac{B'(x)}{A(x)}\,dx. \tag{5.4}$$

Differentiating both sides of the identity (5.4) results in

$$-B''(t)\int_0^t \frac{A'(x)}{B(x)}\,dx - \frac{B'(t)A'(t)}{B(t)} = -A''(t)\int_0^t \frac{B'(x)}{A(x)}\,dx - \frac{A'(t)B'(t)}{A(t)} \tag{5.5}$$

Eliminating the two integrals from (5.3), (5.4), and (5.5) we obtain after some algebraic simplifications

$$A''(t)/A'(t) - A'(t)/A(t) = B''(t)/B'(t) - B'(t)/B(t)$$

i.e.

$$\frac{d}{dt}\ln\frac{A'(t)}{A(t)} = \frac{d}{dt}\ln\frac{B'(t)}{B(t)}.$$

Integrating the last identity we obtain

$$\ln\frac{A'(t)}{A(t)} = \ln\frac{B'(t)}{B(t)} + \text{constant}$$

and

$$\frac{A'(t)}{A(t)} = a\frac{B'(t)}{B(t)}\,, \quad \text{where } a \text{ is a constant.} \tag{5.6}$$

Therefore

$$\frac{d}{dt}\ln A(t) = a\frac{d}{dt}\ln B(t)$$

and

$$\ln A(t) = a\ln B(t) + \text{constant}.$$

Since $A(0) = B(0) = 1$ the above constant vanishes and we have

$$A(t) = [B(t)]^a$$

which completes the proof.

Observe, that in view of the relation (5.6) the Theorem can be restated as follows : *Two survivability functions commute if and only if the corresponding mortality functions are proportional to each other.* Cf. Appendix ($A5$).

It follows, in particular, that any two exponential survivability functions commute in their relevation product. From (5.2)

$$e^{-\alpha t} \# e^{-\beta t} = e^{-\beta t} \# e^{-\alpha t} = [\beta\, e^{-\alpha t} - \alpha\, e^{-\beta t}]/(\beta - \alpha). \tag{5.7}$$

In this case the relevation product is equivalent to the convolution of the two cumulative distribution functions $1 - e^{-\alpha t}$ and $1 - e^{-\beta t}$. We have namely

$$e^{-\alpha t} \# e^{-\beta t} = 1 - \int_0^t [1 - B(t - x)]\, d[1 - A(x)] \tag{5.8}$$

as can be easily verified. This equivalence for exponential survivability functions can also be deduced from the interpretation of the relevation product as given in section 2.

The convolution of two s.f.'s corresponds to the case where the two life phases are independent random variables; the saved individual is as good as a newborn baby in the second population and goes through two full lives, one in each population. Under the conditions of the relevation product the individual carries over his age (at the time of imminent death) into the new population. For exponential s.f.'s mortality is a constant, independent of age, and the relevation product coincides with the convolution.

The relevation product is generally not associative, even for exponential s.f.'s, as can be easily seen. However, for any s.f. $s(t)$ and any non-vanishing, pairwise different real numbers m, n, and r we have

$$(s^m \# s^n) \# s^r = s^m \frac{nr}{(m-n)(m-r)} + s^n \frac{rm}{(n-r)(n-m)} + s^r \frac{mn}{(r-m)(r-n)} \tag{5.9}$$

as can be verified. Since the right side of (5.9) is symmetrical in m, n and r it follows that the order of the powers of $s(t)$ can be changed on the left side provided that the parentheses remain in place. For example

$$(s^m \# s^n) \# s^r = (s^n \# s^r) \# s^m. \tag{5.10}$$

It is easily verifiable that (5.10) remains valid even if the m, n, and r are not pairwise different.

The analysis of relevation products is, as a rule, much more complex than the analysis of convolutions, the latter being commutative, associative, distributive, and corresponding to the addition of independent random variables.

Section 6. We modify now the scenario of Section 3 by assuming that there are n life phases in the total relevated life; the miracle medicine is fully effective $n - 1$ times for each individual and then it becomes entirely ineffective for him; $n \geqslant 1$.

During the first $n - 1$ phases of life, whenever an individual is on the verge of dying, say at the relevated (i.e. cumulated during all lived through phases) age x, the pill restores him to the physical condition at age x and transfers him to his next life phase. « Physical condition at age x » means that his sojourn distribution in the new life phase is the same as the distribution of remaining lifespans for individuals of age x in the original pre-pill population, characterized by the s.f. $A(t)$. (The s.f. for the remaining lifespans within the new life phase is thus $A(x + t)/A(x)$; $t = 0$ at the beginning of this new phase, of course.)

Definition. Denote by $A(t;n)$ the s.f. resulting from the possibility of taking the wonder drug $n - 1$ times, $n \geqslant 1$, with full effectiveness, as explained above. (The total number of life phases is thus n.) Symbolically, we define

$$(6.1) \quad A(t;1) = A(t). \text{ and}$$

$$A(t;n+1) = A(t;n) \# A(t) \quad , \quad n \geqslant 1.$$

Theorem

$$(6.2) \qquad A(t\,;n) = A(t) \sum_{k=0}^{n-1} \frac{1}{k!} [-\ln A(t)]^k \quad , \quad n \geqslant 1.$$

Proof. We shall prove (6.2) inductively, that is by showing that if it holds for $n = m$ then it is valid for $n = m + 1$.

$$(6.3) \quad A(t\,; m+1) = A(t\,; m) \# A(t) \qquad [(\text{by definition } 6.1)]$$

$$= \left[A(t) \sum_{k=0}^{m-1} \frac{1}{k!} [-\ln A(t)]^k \right] \# A(t) \qquad [(6.2) \text{ valid for } n = m]$$

$$= \left[A(t) \left\{ \sum_{0}^{m-2} \frac{1}{k!} [-\ln A(t)]^k + \frac{1}{(m-1)!} [-\ln A(t)]^{m-1} \right\} \right] \# A(t)$$

$$= \left[A(t) \sum_{0}^{m-2} \frac{1}{k!} [-\ln A(t)]^k \right] \# A(t)$$

$$+ \left[A(t) \frac{1}{(m-1)!} [-\ln A(t)]^{m-1} \right] \# A(t) \qquad [(\text{cf. } 5.1)].$$

Since (6.2) is valid for $n = m - 1$ we have

$$(6.4) \quad \left[A(t) \sum_{0}^{m-2} \frac{1}{k!} [-\ln A(t)]^k \right] \# A(t) = A(t\,; m-1) \# A(t) = A(t\,; m).$$

Furthermore, applying formally (1.1) we get

$$\{A(t)[-\ln A(t)]^{m-1}\} \# A(t)$$
$$= A(t)[-\ln A(t)]^{m-1} - A(t) \int_0^t \frac{\mathrm{d}\{A(x)[-\ln A(x)]^{m-1}\}}{A(x)}$$
$$= A(t)[-\ln A(t)]^{m-1}$$
$$+ A(t) \int_0^t [-\ln A(x)]^{m-1} \mathrm{d}[-\ln A(x)] - A(t) \int_0^t \mathrm{d}[-\ln A(x)]^{m-1}$$
$$= A(t)[-\ln A(t)]^{m-1} + A(t) \cdot \frac{1}{m} [-\ln A(t)]^m - A(t)[-\ln A(t)]^{m-1}$$
$$= \frac{1}{m} A(t)[-\ln A(t)]^m.$$

Substituting (6.4) and (6.5) into (6.3) we get

$$A(t\,;m+1) = A(t\,;m) + \frac{1}{m!} A(t)[-\ln A(t)]^m. \tag{6.6}$$

From (6.6) and (6.2) with $n = m$ we get

$$A(t\,;m+1) = A(t) \sum_0^m \frac{1}{k!} [-\ln A(t)]^k \quad , \qquad \text{q.e.d.} \tag{6.7}$$

Section 7. Let $A(t) = \mathrm{e}^{-t/a}$. Then $-\ln A(t) = t/a$ and (6.2) becomes

$$A(t\,;n) = \mathrm{e}^{-t/a} \cdot \sum_0^{n-1} \frac{1}{k!} \left(\frac{t}{a}\right)^k \tag{7.1}$$

The corresponding probability density is

$$-\frac{\mathrm{d}}{\mathrm{d}t} A(t\,;n) = -\frac{1}{a} \mathrm{e}^{-t/a} \sum_0^{n-1} \frac{1}{k!} \left(\frac{t}{a}\right)^k + \frac{1}{a} \mathrm{e}^{-t/a} \sum_0^{n-2} \frac{1}{k!} \left(\frac{t}{a}\right)^k \tag{7.2}$$
$$= \frac{t^{n-1}\,\mathrm{e}^{-t/a}}{\Gamma(n) a^n}.$$

This can be recognized as the Gamma density distribution with parameters n (positive integer here) and positive a; cf. for example S.S. Wilks, section 7.5.

Thus, the expression (6.2) is a generalization of the Gamma distribution (in the s.f. form) which becomes the Gamma distribution when the kernel

$$A(t) = \mathrm{e}^{-t/a}.$$

The result (7.2) is not surprising because the life phases are under these conditions identically and independently distributed (that is the remaining lifetimes at age x are independent of x), as known from the properties of negative-exponential distributions.

Section 8. The s.f. $A\,(t\,;\,n)$ can be generalized by allowing the parameter n to be a real number, not necessarily an integer. The probability density corresponding to (6.2) is

$$(8.1)\qquad -\frac{\mathrm{d}}{\mathrm{d}t}A(t\,;\,n) = -A'(t)\,\frac{1}{\Gamma(n)}\,[-\ln A(t)]^{n-1}$$

as can be easily verified.

The function (8.1) remains a probability density function if we allow n to be a positive real number. This function (8.1) is positive and its integral is

$$\frac{1}{\Gamma(n)}\int_{t=0}^{\infty}[-\ln A(t)]^{n-1}\,\mathrm{d}A(t) = \frac{-1}{\Gamma(n)}\int_{1}^{0}\left[\ln\frac{1}{x}\right]^{n-1}\mathrm{d}x$$
$$= \frac{1}{\Gamma(n)}\int_{0}^{1}\left[\ln\frac{1}{x}\right]^{n-1}\mathrm{d}x = 1$$

(cf. Bjerrens de Haan, Tables of Definite Integrals).

A generalization of (1.1) is obtained by modyfing the scenario of section 2. Assume that the probability of the pill's effectiveness is $g(x)$, x being the age at which death is imminent. « Effectiveness » here means the pill transfers the patient into a population category with a s.f. $B(t)$ and that he retains his age x within the new population. If the drug is ineffective the patient dies promptly.

The probability that a newborn individual will still be around after a time t, in either his first or second life phase, is the sum of :

a) the probability that he will still be alive in his first phase, i.e. $A(t)$; and

b) the probability that he will have to take the pill at age $x < t$, that the pill will be successful, and that he will still be alive in his second phase at his relevated (that is cumulated in both phases) at time t; this is

$$\int_{0}^{t} - A'(x)g(x)\,\frac{B(t)}{B(x)}\,\mathrm{d}x = -\,B(t)\int_{0}^{t}\frac{g(x)\,\mathrm{d}A(x)}{B(x)}$$

a) and *b*) add up to $C\,(t,\,g)$ defined by (8.2)

$$(8.2)\qquad C(t,\,g) = A(t) - B(t)\int_{0}^{t}\frac{g(x)\,\mathrm{d}A(x)}{B(x)}$$

This may be called the relevation of $A(t)$ by $B(t)$ with the kernel $g(t)$. Note that $g(x)$ is not a probability density, and its integral need not be equal to one. In particular, $g(x)$ may be a constant, i.e. the probability of the pill's effectiveness does not depend on age (but, of course, the future life expectancy is age dependent; a child saved by penicilin from a deadly infection has a higher remaining life expectancy than an old person saved by the same drug from the same illness). For each x we must always have $0 \leqslant g(x) \leqslant 1$, obviously.

Let $g(x) = a$, a constant. Then (8.2) becomes

$$C(t, a) = A(t) - aB(t) \int_0^t \frac{\mathrm{d}(Ax)}{B(x)} \quad , \quad 0 \leqslant a \leqslant 1. \tag{8.3}$$

This is a convex combination of the s.f. $A(t)$ and the s.f. $A(t) \neq B(t)$.

Let now $g(x) = aB(x)$, so that the pill becomes monotonically less effective with the patient's age (the fact that $B(x)$ plays two roles in this example should not be a source of confusion). Then (8.2) becomes

$$C(t, aB) = A(t) - B(t) \int_0^t a\mathrm{d}A(x) = A(t) - aB(t)[A(t) - 1]. \tag{8.4}$$

When $a = 1$, (8.4) becomes (8.5) :

$$C(t, B) = A(t) + B(t) - A(t)B(t). \tag{8.5}$$

Curiously, the last expression is also the probability that, given a pair of newborn babies one with a s.f. $A(t)$ and the other with a s.f. $B(t)$, at least one of them will still be alive after the time t.

Because of the freedom in selecting $g(x)$ the expression (8.2) is a rich source of probability distributions.

Notice that the transformation (8.2) is linear with respect to $g(x)$.

Further generalizations of (1.1) are possible. E.g. one can assume that the distribution of the remaining lifetimes after the pill is swallowed depends on the age of the patient. We shall not pursue this topic, although it may be of interest in some biometric and renewal models, beyond showing that it is a generalization of the convolution of the functions $A(t)$ and $B(t)$.

Let the individual, saved at age x, become a member of age x, of a population with the s.f. $B(t \mid x)$. Each pill is now fully successful once (and never again) for each individual. The relevated life is then :

$$C(t) = A(t) - \int_0^t A'(x) \frac{B(t|x)}{B(x|x)} \, \mathrm{d}x. \tag{8.6}$$

Let $B(t \mid x) = B(t - x)$ and thus $B(x \mid x) = B(0) = 1$. Then (8.6) becomes

$$C(t) = A(t) - \int_0^t A'(x)B(t - x) \, \mathrm{d}x \tag{8.7}$$

and in the cumulative distribution form, 1 — s.f., more common in convolution applications

$$1 - C(t) = \int_0^t [1 - B(t - x)] \cdot \mathrm{d}[1 - A(x)] \tag{8.9}$$

as can be easily verified.

The interpretation of the assumption $B(t \mid x) = B(t - x)$ makes it clear why it leads to a convolution, namely by compensating for the age x by a transfer to an appropriately more favorable population.

The expression (8.6) generalizes both the relevation and the convolution operations at the same time.

Section 9. It is clear from (1.1) that

$$C(t) \geqslant A(t) \tag{9.1}$$

since $dA(x) \leqslant 0$.

It is also easy to show that

$$C(t) \geqslant B(t) \tag{9.2}$$

We have namely, $B(x)$ being non-increasing,

$$C(t) = A(t) - B(t) \int_0^t \frac{dA(x)}{B(x)} \geqslant A)t) - B(t) \int_0^t dA(x) \tag{9.3}$$

$$= A(t) - B(t)[A(t) - 1] = A(t) + B(t) - A(t)\, B(t).$$

Cf. the paragraph after (8.5).

Since $B(t) \leqslant 1$ we get (9.2) from the sharper inequality (9.3).

Both (9.1) and (9.2) are intuitively clear from the physical interpretation of (1.1) as a renewal or life extension.

If $A(t)$ and $B(t)$ are positive for all finite t, then (9.1) and (9.2) become strict inequalities.

It may be of interest to solve (1.1) for $A(t)$ or for $B(t)$.

Formal manipulations lead to (9.4) and (9.5) :

$$A(t) = C(t) - \frac{B(t) \cdot C'(t)}{B'(t)} \tag{9.4}$$

$$B(t) = e^{\int_0^t \frac{dC(x)}{C(x) - A(x)}} \tag{9.5}$$

Proof. From (1.1) and (1.1 *a*) one eliminates $\int_0^t \frac{dA(x)}{B(x)}$, thus getting (9.4). From (9.4), solving for $B(t)$, one gets (9.5).

(9.4) and (9.5) are necessary conditions. Given a s.f. $C(t)$ and a s.f. $B(t)$ the formula (9.4) may yield a function which is not a s.f. This may happen if, e.g. (9.2) is violated, or if $C'(t) \neq 0$, since (1.1 *a*) implies that if the first n (right-handed) derivatives of $B(t)$ vanish at $t = 0$ then so do the first $n + 1$ (right-handed) derivatives of $C(t)$. In (9.5) it is necessary, that the integral in the exponent diverge as $t \to \infty$.

Solving (3.1) for $A(t)$ in terms of $C(t) = A(t) \# A(t)$ is more difficult. Thus, it is not known yet under what conditions the recursion

$$A_{n+1}(t) = C(t) + A_n(t) \cdot \ln A_n(t) \quad , \quad n \geqslant 1, \tag{9.6}$$

where $A_1(t)$ is selected to be $C(t)$, e^{-at}, or another s.f., converges to the solution of 3.1. The successive A_n need not be survivability functions even if $C(t)$ and $A_1(t)$ are such functions.

Section 10

The (age-specific) mortality $m(t)$ corresponding to the s.f. $A(t)$ is $-A'(t)/A(t)$; Cf. in Appendix.

If the mortality $m(t)$ is non-decreasing with age then, assuming differentiability, $\frac{d}{dt} m(t) = [(A')^2 - AA'']/(A^2) \geqslant 0$. Therefore, $m(t)$ is non-decreasing when

$$(A')^2 \geqslant A \cdot A''. \tag{10.1}$$

Theorem. Let $m(t)$ be non-descreasing, i.e. (10.1) holds. Then the mortality corresponding to $A(t;n)$ in (6.2) is also non-decreasing, or equivalently

$$[A'(t;n)]^2 \geqslant A(t;n) \cdot A''(t;n). \tag{10.2}$$

Proof. It follows from (6.2) that

$$A'(t\,;n) = A' \cdot [-\ln A]^{n-1}/(n-1)\,! \tag{10.3}$$

$$A''(t\,;n) = [-\ln A]^{n-1}/(n-1)\,! - [-\ln A]^{n-2}[A']^2\,[A \cdot (n-2)\,!] \tag{10.4}$$

and, after some algebraic manipulation,

$$[A'(t\,;n)]^2 - A(t\,;n)A''(t\,;n) \tag{10.5}$$

$$= \frac{[-\ln A]^{n-2}}{(n2-!)} \left\{ \frac{[-\ln A]^n}{(n-1)(n-1)\,!}[(A')^2 - A \cdot A''] + [A']^2 + \sum_1^{n-1} \frac{[-\ln A]^k}{(k-1)\,!}\left[\frac{[A']^2}{k} - \frac{A \cdot A}{n-1}\right] \right\}$$

It follows from (10.1) and $k \leqslant n-1$ that $[A']^2/k \leqslant AA''/(n-1)$. Since $-\ln A \geqslant 0$ the right side of (10.5) is $\geqslant 0$, qed.

APPENDIX

Survivability, Mortality, and Age Distribution in Stationary Populations

The following proof outlines are for completeness of presentation. The relations are classical and can be found, e.g. in Keyfitz (cf. Bibliography) although with different notation.

The c.d.f. corresponding to the s.f. $s(t)$ is $1 - s(t)$ and hence

(A1) $f(t) = - s'(t) =$ probability density of the corresponding lifespans.

The expected lifespan M of a new entry is

(A2) $$M = - \int_0^\infty ts'(t)\mathrm{d}t.$$

If $ts(t) \to 0$ as $t \to \infty$, as is the case in all pratical applications, then

(A3) $$M = \int_0^\infty s(t)\mathrm{d}t.$$

The mortality function corresponding to $s(t)$ is defined by (A4) :

(A4) $m(t)\mathrm{d}t =$ probability that an individual of age t will die (exit) within $\mathrm{d}t$.

The function $m(t)$ is also called the hazard function or the power of mortality in actuarial contexts.

The probablity that a new entry will survive till age $t + \mathrm{d}t$ is the probablity that he will survive till age t multiplied by the conditional probablity that he will not die then within $\mathrm{d}t$. Hence,

$$s(t + \mathrm{d}t) = s(t)\,[1 - m(t)\mathrm{d}t]$$

and

(A5) $$m(t) = - s'(t)/s(t).$$

A stationary population is one whose constant rate of entries (births) equals its rate of exits (deaths) and whose age distribution does not change with time. At any instance the fraction $p(x)\mathrm{d}x$ of the population aged between x and $x + \mathrm{d}x$ are survivors of the fraction aged between 0 and $\mathrm{d}x$ as of x time units earlier. The age density being independent of calendar time we have

(A6) $$p(t) = p(0)\, s(t).$$

Integrating both sides of (A6) from 0 to ∞ we get $p(0) = M$; cf. (A3).

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