REVUE FRANÇAISE D'AUTOMATIQUE, INFORMATIQUE, RECHERCHE OPÉRATIONNELLE. RECHERCHE OPÉRATIONNELLE

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Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle, tome 7, nº V1 (1973), p. 97-106

<http://www.numdam.org/item?id=RO_1973__7_1_97_0>

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(7e année, V-1, 1973, p. 97-106)

AN INTERMITTANT $MI^{(X)}/G^{(Y)}/1$ SYSTEM WITH MULTIPHASED CAPACITY OF THE SERVICE CHANNEL

by R. S. GAUR (*)

Résumé. — *The paper studies the behaviour of an $MI^{(X)}G^{(Y)}/1$ system in which the capacity of the server varies, in phases, with the length of queue. The server is made intermittently available. Probability generating functions of the queue length when the server is free and busy are obtained.*

INTRODUCTION

Making investigations in bulk queues on the lines of Bailey (1954), Jaiswal (1960), Conolly (1960) and Foster (1961), Sharda (1968) finds the transient state queue length probabilities of an intermittent $MI^{(X)}/G^{(Y)}/1$ system assuming the capacity of the server, a random variable in the model, to be fixed. According to the assumption, the size of the batch is determined at the beginning of each service and is either equal to the total number of units waiting or to the capacity of the service channel, determined afresh before each service, whichever is less. The probability that the service channel can serve j units is u_j where $\sum_{j=1}^{M} u_j = 1$. The idea that the capacity may vary, in phases, with the queue length provides grounds for further investigations made in this paper. Sharda's results have been shown to follow as a particular case. As is evident from its nature the model will be found to apply to many practical situations.

The Model Description

The system is described as follows :

Customers arive in a Poisson process with parameter λt in groups of size C_n with the distribution

$$Pr.(C_n = i) = c_i, \qquad (i = 1, 2, ...)$$

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where $\sum_{i=1}^{\infty} c_i = 1$. The batches are preordered for service.

2. The queue discipline is first come first served.

3. The capacity of the service channel is a random variable and has $k + 1$ phases, each phase operating over a definite length-range of the queue. A phase is designated as the lth phase ($l = 0, 1, \ldots k$), the 0th phase being the lower most. When the capacity is in the 1th phase the maximum size of the service batch is M_l. The capacity remains in the lth phase so long as the queue length is less than $M_{l+1}(l = 0, 1, \ldots k - 1)$ and steps to the $l + 1$th phase as soon as the queue length becomes M_{l+1}. Of course $M_1 < M_{l+1}$. The capacity operates in the highest phase, that is, the kth phase for a queue $\geqslant M_k$. The size of the batch is determined at the beginning of each service and is either equal to the total number of units waiting or to the capacity of the service channel determined afresh before each service, whichever is less. The probability that the service channel can serve j units in the lth phase of its capacity is $u_j^{(l)}$ where $\sum_{j=1}^{Ml} u_j^{(l)} = 1$.

4. Service time of each batch is general with the probability density $D(x)$. $\eta(x)\Delta$ is the first order conditional probability that the corresponding service will be completed in time x and $x + \Delta$ if the same has not been completed till time x. $\eta(x)$ is related to $D(x)$ by the relation

$$D(x) = \eta(x) \exp. \left(- \int_0^x \eta(x)\, dx\right).$$

4. The server is intermittently available, the availability distribution being negatively exponentiel with mean v^{-1}.

Formulation of Equations

Define :

$P_{n,F}^{(t)} \equiv$ probability that there are n units in the queue at time t and the server is «free» in relation to the queue, i.e., there is neither a unit in service nor is any to be taken at that intant.

$P_{n,B}^{(x,t)} \equiv$ probability that there are n units waiting in the queue at time t and the server is « Busy » in relation to the queue and that the time passed since the service started lies between x and $x + \Delta$. « Busy » in relation to the queue means that a batch is being served or else the same is to be taken for service just then.

For the initial condition, we decide to reckon time from the instant when there is no unit in the queue and the server is free in relation to the queue. Thus

$$P_{0,F}^{(0)} = 1. \tag{1}$$

From the probability considerations of the process we have the following difference-defferential equitions governing the system :

$$\frac{\partial P_{0,F}^{(t)}}{\partial t} + \lambda P_{0,F}^{(t)} = \int_0^\infty P_{0,B}^{(x,t)} \eta(x)\, \mathrm{d}x. \tag{2}$$

$$\frac{\partial P_{n,F}^{(t)}}{\partial t} + (\lambda + v) P_{n,F}^{(t)} = \lambda \sum_{i=1}^{n} c_i P_{n-i,F}^{(t)} + \int_0^\infty P_{n,B}^{(x,t)} \eta(x)\, \mathrm{d}x, \quad 1 \leqslant n < \infty. \tag{3}$$

$$\frac{\partial P_{0,B}^{(x,t)}}{\partial x} + \frac{\partial P_{0,B}^{(x,t)}}{\partial t} + (\lambda + \eta(x)) P_{0,B}^{(x,t)} = 0. \tag{4}$$

$$\frac{\partial P_{n,B}^{(x,t)}}{\partial x} + \frac{\partial P_{n,B}^{(x,t)}}{\partial t} + (\lambda + \eta(x)) P_{n,B}^{(x,t)} = \lambda \sum_{i=1}^{n} c_i P_{n-i,B}^{(x,t)}, \; 1 \leqslant n < \infty. \tag{5}$$

The relevent boundary condition is obtained from the fact that at the moment of taking a batch for service, there should be n units waiting in the queue and the server should be free in relation to the queue. Thus we have

$$P_{0,B}^{(0,t)} = v \sum_{r=1}^{M_0} P_{r,F}^{(t)} \sum_{j=r}^{M_0} u_j^{(0)} + v \sum_{l=1}^{k} P_{M_l,F}^{(t)} u_{M_l}^{(l)}, \tag{6}$$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_0} P_{n+j,F}^{(t)} u_j^{(0)} + v \sum_{l=1}^{k} \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(t)} u_j^{(l)}, \quad 1 \leqslant n \leqslant M_1 - M_0 - 1;$$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_i} P_{n+j,F}^{(t)} u_j^{(i)} + v \sum_{l=i+1}^{k} \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(t)} u_j^{(l)}, \quad M_i - 1 \leqslant n \leqslant M_{i+1} - M_i - 1, \; 1 \leqslant i \leqslant k-1; \tag{7}$$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_l-n-1} P_{n+j,F}^{(t)} u_j^{(i-1)} + v \sum_{l=i}^{k} \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(t)} u_j^{(l)}, \quad M_i - M_{i-1} \leqslant n \leqslant M_i - 2, \; 1 \leqslant i \leqslant k;$$

$$P_{n,B}^{(0,t)} = v \sum_{j=1}^{M_k} P_{n+j,F}^{(t)} u_j^{(k)}, \quad M_k - 1 \leqslant n < \infty.$$

We define the Laplace transforms as follows :

$$P_{n,F}^{(s)} = \int_0^\infty \mathrm{e}^{-st} P_{n,F}^{(t)}\, \mathrm{d}t, \; P_{n,B}^{(x,s)} = \int_0^\infty \mathrm{e}^{-st} P_{n,B}^{(x,t)}\, \mathrm{d}t. \tag{8}$$

Taking the Laplace transforms of (2)-(7) and using (1) we have :

$$(s + \lambda + v)P_{n,F}^{(s)} = \lambda \sum_{i=1}^{n} c_i P_{n-i,F}^{(s)} + \int_0^\infty P_{n,B}^{(x,s)} \eta(x)\,\mathrm{d}x. \tag{9}$$

$$(s + \lambda + v)P_{0,F}^{(s)} = 1 + vP_{0,F}^{(s)} + \int_0^\infty P_{0,B}^{(x,s)} \eta(x)\,\mathrm{d}x. \tag{10}$$

$$\frac{\partial P_{0,B}^{(x,s)}}{\partial x} + (s + \lambda + \eta(x))P_{0,B}^{(x,s)} = 0. \tag{11}$$

$$\frac{\partial P_{n,B}^{(x,s)}}{\partial x} + (s + \lambda + \eta(x))P_{n,B}^{(x,s)} = \lambda \sum_{i=1}^{n} c_i P_{n-i,B}^{(x,s)}. \tag{12}$$

$$P_{0,B}^{(0,s)} = v \sum_{r=1}^{M_0} P_{r,F}^{(s)} \sum_{j=r}^{M_0} u_j^{(0)} + v \sum_{l=1}^{k} P_{M_l,F}^{(s)} u_{M_l}^{(l)}. \tag{13}$$

$$\begin{aligned}
P_{n,B}^{(0,s)} &= v \sum_{j=1}^{M_0} P_{n+j,F}^{(s)} u_j^{(0)} + v \sum_{l=1}^{k} \sum_{J=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)}, \\
&\qquad 1 \leqslant n \leqslant M_1 - M_0 - 1; \\
P_{n,B}^{(0,s)} &= v \sum_{j=1}^{M_i} P_{n+j,F}^{(s)} u_j^{(i)} + v \sum_{l=i+1}^{k} \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)}, \\
&\qquad M_i - 1 \leqslant n \leqslant M_{i+1} - M_i - 1, \\
&\qquad 1 \leqslant i \leqslant k - 1; \\
P_{n,B}^{(0,s)} &= v \sum_{j=1}^{M_i-n-1} P_{n+j,F}^{(s)} u_j^{(i-1)} + v \sum_{l=i}^{k} \sum_{j=M_l-n}^{M_l} P_{n+j,F}^{(s)} u_j^{(l)}, \\
&\qquad M_i - M_{i-1} \leqslant n \leqslant M_i - 2, \\
&\qquad 1 \leqslant i \leqslant k; \\
P_{n,B}^{(0,s)} &= v \sum_{j=1}^{M_k} P_{n+j,F}^{(s)} u_j^{(k)}, \\
&\qquad M_k - 1 \leqslant n < \infty.
\end{aligned} \tag{14}$$

Define the following generating functions :

$$P_F^{(a,s)} = \sum_{n=0}^{\infty} P_{n,F}^{(s)} a^n;\quad P_B^{(x,a,s)} = \sum_{n=0}^{\infty} P_{n,B}^{(x,s)} a^n,\quad C(a) = \sum_{i=1}^{\infty} c_i a^i \tag{15}$$

Multiplying (9) by a^n, summing over all n and using (10) and (15) we have :

$$(s + \lambda + v - \lambda C(a))P_F^{(a,s)} = 1 + vP_{0,F}^{(s)} + \int_0^\infty P_B^{(x,a,s)} \eta(x)\,\mathrm{d}x. \tag{16}$$

Multiplying (12) by a^n, summing over all n and using (11) and (15) we have :

$$\frac{\partial}{\partial x} P_B^{(x,a,s)} + (s + \lambda + \eta(x) - \lambda C(a))P_B^{(x,a,s)} = 0. \tag{17}$$

Multiplying (14) by a^n, summing over all n and using (13) and (15) we have

$$(18)\qquad P_B^{(0,s,a)} = v \Bigg\{ \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} P_{0,F}^{(s)} \Bigg\}$$

Integrating (17) w.r.t.x. between 0 and x and using (18) we get

$$(19)\qquad P_B^{(x,a,s)} = v \Bigg\{ v \sum_{J=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{J=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_1} \frac{u_j^{(1)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) - \sum_{j=1}^{M_k} P_{0,F}^{(s)} \frac{u_j^{(k)}}{a^j} \Bigg\} e^{-(s+\lambda-\lambda C(a))x - \int_0^x \eta(x)\,dx}$$

Putting the value of $P_B^{(x,a,s)}$ in (16) and integrating we get

$$1 + vP_{0,F}^{(s)} \left\{ 1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} D(s + \lambda - \lambda C(a)) \right\} + v \Bigg\{ P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right) + \sum_{r=2}^{M_0} P_{r,F}^{(0)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \Bigg\}$$

$$P_F^{(a,s)} = \frac{D(s+\lambda-\lambda-\lambda C(a))}{s+\lambda+v-\lambda C(a)-v\sum_{j=1}^{M_k}\frac{u_j^{(k)}}{a^j}D(s+\lambda-\lambda C(a))} \tag{20}$$

where

$$D(s+\lambda-\lambda C(a)) = \int_0^\infty e^{-(s+\lambda-\lambda C(a))x} D(x)\,dx.$$

Also

$$\begin{aligned} P_B^{(a,s)} = \int_0^\infty P_B^{(x,a,s)}\,dx = v\Bigg\{ &\sum_{j=1}^{M_k} P_F^{(s,a)}\frac{u_j^{(k)}}{a^j} \\ &+ P_{1,F}^{(s)}\left(1 - a^r\sum_{J=1}^{M_k}\frac{u_j^{(k)}}{a^j}\right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)}\left(\sum_{j=2}^{M_0} u_j^{(0)}\right. \\ &\left. + a^r\sum_{j=1}^{r-1}\frac{u_j^{(0)}}{a^j} - a^r\sum_{j=1}^{M_k}\frac{u_j^{(k)}}{a^j}\right) \\ &+ \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)}a^r\left(\sum_{j=1}^{M_0}\frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k}\frac{u_j^{(k)}}{a^j}\right) \\ &+ \sum_{l=1}^{k-1}\sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)}a^r\left(\sum_{j=1}^{M_l}\frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k}\frac{u_j^{(k)}}{a^j}\right) \\ &- \sum_{j=1}^{M_k} P_{0,F}^{(s)}\frac{u_j^{(k)}}{a^j}\Bigg\}\left\{\frac{1-D(s+\lambda-\lambda C(a))}{s+\lambda-\lambda C(a)}\right\}. \end{aligned} \tag{21}$$

To find $P_{r,F}^{(s)}$ $(r = 0, 1, 2, \ldots.. M_k - 1)$ in (20) we apply Rouche's theorem to its denominator. Let

$$f(a) = a^{Mk},\ g(a) = \frac{vD(s+\lambda-\lambda C(a))\sum_{j=1}^{M_k} u_j^{(k)} a^{Mk-j}}{s+\lambda+v-\lambda C(a)}.$$

Now $f(a)$ and $g(a)$ are analytic inside and on the contour $|a| = 1$. Also on the contour $|a| = 1$, $|f(a)| = 1$ and

$$\begin{aligned} |g(a)| &= v\left|\frac{D(s+\lambda-\lambda C(a))\sum_{j=1}^{M_k} u_j^{(k)}a^{Mk-j}}{s+\lambda+v-\lambda C(a)}\right|_{|a|=1} \\ &= v\left|\frac{\int_0^x D(x)e^{-(s+\lambda-\lambda C(a))x}dx\sum_{J=1}^{M_k} u_j^{(k)}a^{Mk-j}}{s+\lambda+v-\lambda C(a)}\right|_{|a|=1} \\ &\leqslant \frac{v}{\xi+v}\int_0^\infty \left|D(x)e^{-(s+\lambda-\lambda C(a))x}\right|\,|dx|, \end{aligned}$$

since $|s + \lambda + v - \lambda C(a)|_{|a| = 1} \geqslant \xi + \lambda + v - \lambda|C(a)|_{|a| = 1} \geqslant \xi + v$. Here $s = \xi + i\eta$.

Because $|e^{sx}| > 1$, if $\xi > 0$, we have

$$|g(a)|_{|a|=1} \leqslant \frac{v}{\xi + v} \int_0^\infty \frac{D(x)}{e^{(s+\lambda-\lambda C(a))x}} |dx| < \frac{v}{\xi + v} < 1 = |f(a)|_{|a|=1}.$$

Therefore on the contour $|a| = 1$, $|f(a)| > |g(a)|$. Hence all the conditions of Rouche's theorem are satisfied. Therefore $f(a) - g(a)$ and $f(a)$ have the same number of zeros inside the unit circle $|a| = 1$. But $f(a)$ has M_k zeros inside the unit circle so that $f(a) - g(a)$ also has M_k zeros inside this circle.

Since $P_F^{(a,s)}$ is singular inside and on the contour $|a| = 1$, the numerator of (20) must vanish for these M_k zeros of the denominator. Thus the M_k unknowns $P_{n,F}^{(s)}(r = 0, 1, 2, \ldots M_k - 1)$ can be uniquiely determined. Hence $P_F^{(a,s)}$ can be completely determined and $P_{n,F}^{(s)}$ can be picked up as the coefficient of a^n in the expansion of $P_F^{(a,s)}$. Substituting the values of $P_F^{(a,s)}$ and $P_{r,F}^{(s)}(r = 0, 1, 2, \ldots M_k - 1)$ in (21) $P_B^{(a,s)}$ can be determined and $P_{n,B}^{(s)}$ can be picked up as the coefficient of a^n in the expension of $P_B^{(a,s)}$. Hence the queue length distributions when the server is « free » and « busy » are completely determined.

Particular Cases

1. When the service time distribution is exponentiel with mean u^{-1}

$$D(s + \lambda - \lambda C(a)) = \frac{u}{s + \lambda + u - \lambda C(a)}.$$

Hence $P_F^{(a,s)}$ and $P_B^{(a,s)}$ are given by

$$1 + vP_{0,F}^{(s)} \left\{ 1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \frac{u}{s + \lambda + u - \lambda C(a)} \right\}$$

$$+ v \left\{ P_{1,F}^{(s)} \left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right.$$

$$+ \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right)$$

$$+ \sum_{M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right)$$

$$\left. + \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right\}$$

$$(22) \qquad P_F^{(a,s)} = \frac{\dfrac{u}{s+\lambda+u-\lambda C(a)}}{s+\lambda+v-\lambda C(a) - uv \sum_{j=1}^{M_k} \dfrac{u_j^{(k)}}{a^j} \dfrac{1}{s+\lambda+u-\lambda C(a)}}$$

and

$$(23) \qquad P_B^{(a,s)} = \frac{v}{s+\lambda+u-\lambda C(a)} \left\{ \sum_{j=1}^{M_k} P_F^{(s,a)} \frac{u_j^{(k)}}{a^j} + P_{1,F}^{(s)} \left(1 - a \sum^{M_k} \frac{u_j^{(k)}}{a^j} \right) \right.$$

$$+ \sum_{r=2}^{M_0} P_{r,F}^{(s)} \left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j} - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right)$$

$$+ \sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right)$$

$$+ \sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right)$$

$$\left. - P_{0,F}^{(s)} \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} \right\}.$$

For applying Rouche's theorem to the denominator of (22) take

$$f(a) = a^{M_k}, \; g(a) = \frac{uv \sum_{j=1}^{M_k} u_j^{(k)} a^{M_k - j}}{(s+\lambda+v-\lambda C(a))(s+\lambda+u-\lambda C(a))}.$$

$f(a)$ and $g(a)$ are analytic inside and on the countour $|a| = 1$. Also because

$$\left| \frac{u}{s+\lambda+u-\lambda C(a)} \right| < 1 \quad \text{and} \quad \left| \frac{v}{s+\lambda+v-\lambda C(a)} \right| < 1,$$

$$|g(a)|_{|a|=1} = \left| \frac{uv \sum_{j=1}^{M_k} a^{M_k - j}}{(s+\lambda+v-\lambda C(a))(s+\lambda+u-\lambda C(a))} \right|_{|a|=1}$$

$$\leqslant \frac{uv \sum_{j=1}^{M_k} |a^{M_k-j}|_{|a|=1}}{|s+\lambda+v-\lambda C(a)|_{|a|=1} \, |s+\lambda+u-\lambda C(a)|_{|a|=1}}$$

$$< 1 = |f(a)|_{|a|=1}$$

Therefore $|f(a)| > |g(a)|$ on $|a| = 1$.

Thus all the conditions of Rouche's theorem are satisfied and so $f(a)$ and $f(a) - g(a)$ have the same number of zeros inside the countour $|a| = 1$. But since $f(a)$ has M_k zeros inside the unit circle $|a| = 1$, $f(a) - g(a)$ has only M_k

zeros inside this circle. As the numerator of (22) must vanish for these M_k zeros, the M_k unknowns $P_{r,F}^{(s)}$ $(r = 0, 1, 2 \ldots\ldots M_k - 1)$ can be uniquely determined and hence $P_F^{(a,s)}$ is completely known.

Putting the values of $P_F^{(a,s)}$ and $P_{r,F}^{(s)}$ $(r = 0, 1, 2 \ldots M_k - 1)$ in (23) $P_B^{(a,s)}$ can also be completely determined. $P_{n,F}^{(s)}$ and $P_{n,B}^{(s)}$ can be piecked up as coefficients of a^n in the expensions of $P_F^{(a,s)}$ and $P_B^{(a,s)}$ respectively.

(2) When the capacity of the service channel has only one phase we get the generating functions by taking $k = 0$ and consequently

$$u_j^{(l)} = u_j^{(0)} = u_j\, M_1 = M_0 = M$$

$$(l = 1, 2, \ldots k; j = 1, 2, \ldots M_0), \text{ in (20) and (21).}$$

Now since

$$1 - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j} D(s + \lambda - \lambda C(a)) = 1 - \sum_{j=1}^{M} \frac{u_j}{a^j} D(s + \lambda - \lambda C(a)),$$

$$P_{1,F}^{(s)}\left(1 - a \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right) + \sum_{r=2}^{M_0} P_{r,F}^{(s)}\left(\sum_{j=r}^{M_0} u_j^{(0)} + a^r \sum_{j=1}^{r-1} \frac{u_j^{(0)}}{a^j}\right.$$

$$\left. - a^r \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right) = P_{1,F}^{(s)}\left(\sum_{j=1}^{M} u_j - a \sum_{j=1}^{M} \frac{u_j}{a^j}\right)$$

$$+ \sum_{r=2}^{M} P_{r,F}^{(s)}\left(\sum_{j=r}^{M} u_j + a^r \sum_{j=1}^{r-1} \frac{u_j}{a^j} - a^r \sum_{j=1}^{M} \frac{u_j}{a^j}\right)$$

$$= P_{1,F}^{(s)}\left(\sum_{j=1}^{M} u_j - a^r \sum_{j=1}^{M} \frac{u_j}{a^j}\right)$$

$$+ \sum_{r=2}^{M} P_{r,F}^{(s)}\left(\sum_{j=r}^{M} u_j - a^r \sum_{j=r}^{M} \frac{u_j}{a^j}\right)$$

$$= \sum_{r=1}^{M} P_{r,F}^{(s)}\left(\sum_{j=r}^{M} u_j - a^r \sum_{j=r}^{M} \frac{u_j}{a^j}\right)$$

$$= \sum_{r=1}^{M} P_{r,F}^{(s)} \sum_{j=r}^{M} u_j\left(1 - \frac{a^r}{a^j}\right) = \sum_{r=1}^{M-1} P_{r,F}^{(s)} \sum_{j=r+1}^{M} u_j\left(1 - \frac{a^r}{a^j}\right),$$

and the terms

$$\sum_{r=M_0+1}^{M_1-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_0} \frac{u_j^{(0)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right),$$

$$\sum_{l=1}^{k-1} \sum_{r=M_l}^{M_{l+1}-1} P_{r,F}^{(s)} a^r \left(\sum_{j=1}^{M_l} \frac{u_j^{(l)}}{a^j} - \sum_{j=1}^{M_k} \frac{u_j^{(k)}}{a^j}\right)$$

cease to exist we have

$$1 + vP^{(s)}_{0,F} \left\{ 1 - \sum_{j=1}^{M} \frac{u_j}{a^j} D(s + \lambda - \lambda C(a)) \right\} +$$

$$P_F^{(a,s)} = \frac{v \sum_{r=1}^{M-1} P^{(s)}_{r,F} \sum_{j=r+1}^{M} u_j \left(1 - \frac{a^r}{a^j}\right) D(s + \lambda - \lambda C(a))}{s + \lambda + v - \lambda C(a) - v \sum_{j=1}^{M} \frac{u_j}{a^j} D(s + \lambda - \lambda C(a))} \tag{24}$$

and

$$P_B^{(a,s)} = v \left\{ \sum_{j=1}^{M} P_F^{(s,a)} \frac{u_j}{a^j} + \sum_{r=1}^{M-1} \sum_{j=r+1}^{M} u_j \left(1 - \frac{a^r}{a^j}\right) - P^{(s)}_{0,F} \sum_{j=1}^{M} \frac{u_j}{a^j} \right\} \left\{ \frac{1 - D(s + \lambda - \lambda C(a)}{s + \lambda - \lambda C(a)} \right\}. \tag{25}$$

As shown for the denominator of (20), the denominator (24) has only M zeros inside the unit circle $|a| = 1$ and since the numerator of (24) must vanish for these M zeros, the M unknowns $P^{(s)}_{r,F}$ $(r = 0, 1, 2 \ldots M - 1)$ can be uniquely determined and hence $P_F^{(a,s)}$ is completely known.

Putting the values of $P_F^{(a,s)}$ and $P^{(s)}_{r,F}$ $(r = 0, 1, \ldots M - 1)$ in (25) $P_B^{(a,s)}$ can also be completely determined. The $P^{(s)}_{n,F}$ and $P^{(s)}_{n,B}$ can be picked up as coefficients of a^n in the expensions of $P_F^{(a,s)}$ and $P_B^{(a,s)}$ respectively.

The results (24) and (25) agree with those obtained by Sharda (1968).

ACKNOWLEDGEMENT

My deep sense of gratitude is for Dr. N. N. Aggarwal, Assistt. Professor, Regional Engineering College, Kurukshetra University, who introduced me to the field of research in mathematical knowledge.

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