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Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle, tome 6, n° V2 (1972), p. 95-97

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0-1 OPTIMIZATION AND NON-LINEAR PROGRAMMING

by I. G. ROSENBERG ⁽¹⁾

Abstract. — The optimization problem for pseudo-Boolean functions is : Find the points of minima of f on the vertices of the n -dimensional unit cube where f is a real polynomial linear in each variable. This discrete problem is equivalent to the continuous problem : Find the points of minima of f on the n -dimensional unit cube.

1. INTRODUCTION

It is well known [1] that many problems in operations research, switching theory, combinatorics, graph theory etc. can be reduced to the following problem P : Let $f(X)$ be a real polynomial with n variables which is linear in each variable. Find the minimum of f on the set $\{0, 1\}^n$. There are ways (at least at the theoretical level) to reduce any 0-1 program to this problem [3]. In this note we present the following (apparently so far unrecorded) simple fact : In P we can replace $\{0, 1\}^n$ by $[0, 1]^n$ in other words, P can be treated as a continuous non-linear problem : Minimize $f(X)$ subject to very simple constraints $0 \leq x_i \leq 1$ ($i = 1, \dots, n$). It is hoped that this problem will be easier to solve than P .

Using an idea of Picard and Ratliff we show that for such polynomials of an even degree it suffices to investigate only those without linear terms provided $\{0, 1\}$ is replaced by $\{-1, 1\}$. It has been shown in [5] that P can be reduced to a similar problem with a quadratic polynomial provided that a sufficient number of slack variables are introduced. In view of this the following problem is of prime interest : Find the minimum of $g(X)$ on $[-1, 1]^n$, where g is a quadratic polynomial without linear terms or squares.

A special case of Proposition 1 was found also by P. L. Hammer (oral communication).

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2. MAIN RESULT

We will need the following result :

Lemma : Let $p(x_1, \dots, x_n)$ be a polynomial linear in each variable. Let $a_i < b_i$ ($i = 1, \dots, n$), let $U = [a_1, b_1] \times \dots \times [a_n, b_n]$, and let

$$X^* = (x_1^*, \dots, x_n^*) \in U.$$

If $m = f(X^*)$ is the minimum of f on U , then f is constant and equal to m on

$$W_{X^*} = \{ (w_1, \dots, w_n) \in U \mid x_i^* \in \{a_i, b_i\} \Rightarrow w_i = x_i^* \}.$$

Proof : Let $I = \{ 1 \leq i \leq n \mid x_i^* \in \{a_i, b_i\} \}$ and let $J = \{1, \dots, n\} \setminus I = \{j_1, \dots, j_k\}$. Let $g(y_1, \dots, y_k)$ be the function obtained from P by setting $x_i = x_i^*$ ($\forall i \in I$) and $x_{j_i} = y_i$ ($i = 1, \dots, k$). In other words, we have fixed all variables for which $x_i^* \in \{a_i, b_i\}$ and kept all variables for which $a_i < x_i^* < b_i$. Since m is the minimum of p on U , m is also the minimum of p on $W_{X^*} \subseteq U$ and therefore $m = g(x_{j_1}^*, \dots, x_{j_k}^*)$ is the minimum of g on

$$V = [a_{j_1}, b_{j_1}] \times \dots \times [a_{j_k}, b_{j_k}].$$

The function g is clearly linear in y_1 ; hence $g(y_1, x_{j_2}^*, \dots, x_{j_k}^*) = ay_1 + b$. Since $m = ax_{j_1}^* + b$ is the minimum of g , it follows that $a = 0$ and $g(y_1, x_{j_2}^*, \dots, x_{j_k}^*)$ is constant and equal m . Continuing in the same way we get that

$$g(y_1, y_2, x_{j_3}^*, \dots, x_{j_k}^*)$$

is constant and equal m and finally we obtain that g is constant and equal m on V . But this proves the lemma.

Let $S \subseteq R^n$ and let $f: R^n \rightarrow R$ (R reals). We set $\Omega_S^f = \{ X \in S \mid f(X)$ is minimum of f on $S \}$. Now we have :

Proposition 1. Let f be a polynomial linear in each of its n variables and let $S = [a_1, b_1] \times \dots \times [a_n, b_n]$ and $T = \{a_1, b_1\} \times \dots \times \{a_n, b_n\}$. Then Ω_S^f is the set of all faces C of S satisfying

$$C \cap T \subseteq \Omega_T^f,$$

and Ω_T^f is the set of the integer points of Ω_S^f .

Proof : If $X^* \in \Omega_S^f$, then, by the lemma, f is constant and equal $m = f(X^*)$ on W_{X^*} ; in particular f takes the value m on $W_{X^*} \cap T$. Since $T \subseteq S$, m is also the minimum of f on T and this, in fact, proves the statement.

Corollary 1. If f is linear in each of its variables then $\Omega_{\{0,1\}^n}^f$ is the set of all integer points of $\Omega_{[0,1]^n}^f$.

3. REMOVAL OF LINEAR TERMS

In the problem P the simplest nontrivial case is the case of a quadratic polynomial. This is of special interest because the general case can be transformed to the quadratic one by adding enough slack variables [5]. Also there are some results concerning the quadratic case [2]. If we wish to use the theory of quadratic forms we have to eliminate the linear terms. The ordinary approach ($x_i = y_i + \alpha_i$) requires solution of a linear system and therefore if it can be used at all, presents practical obstacles. Picard and Ratliff [4] have indicated a simple method which can be slightly generalized as follows :

Let

$$x_i = \frac{1}{2}[1 - \xi_0 \xi_i]. \quad (i = 1, \dots, n) \quad (1)$$

It is easy to check that for each i (1) defines a mapping of $\{-1, 1\}^2$ onto $\{0, 1\}$. Moreover if $\xi_0 \in \{-1, 1\}$ and $1 \leq i_1 < \dots < i_k \leq n$ then

$$x_{i_1} \dots x_{i_k} = \frac{1}{2^k} [r - \xi_0 s] \quad (2)$$

where r and s are polynomials in $\xi_{i_1}, \dots, \xi_{i_k}$ containing only terms of even and odd degrees, respectively. Since the degrees of r and s do not exceed k we have :

Proposition 2. *Let f be a polynomial linear in each variable. If f has an even degree, then the substitution (1) together with $\xi_0^2 = 1$ yields a polynomial g of the same degree and without linear terms such that the points of*

$$\Omega_{\{0,1\}^n}^f \text{ and } \Omega_{\{-1,1\}^{n+1}}^g$$

are related by (1).

Now we can apply Proposition 1 :

Corollary 2. *The set $\Omega_{\{-1,1\}^n}^g$ is the set of all integer points from $\Omega_{\{-1,1\}^n}^g$.*

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