Revue française d'informatique et de recherche opérationnelle. Série verte

John E. Walsh
Grace J. Kelleher

Greatly increased practical usefulness of two-person game theory by adoption of median criterion

Revue française d'informatique et de recherche opérationnelle. Série verte, tome 4, nº V3 (1970), p. 81-88
<http://www.numdam.org/item?id=RO_1970__4_3_81_0>

© AFCET, 1970, tous droits réservés.

L'accès aux archives de la revue « Revue française d'informatique et de recherche opérationnelle. Série verte » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

R.I.R.O.
(4e année, V-3, 1970, p. 81-88)

GREATLY INCREASED PRACTICAL USEFULNESS OF TWO-PERSON GAME THEORY BY ADOPTION OF MEDIAN CRITERION (1)

par John E. WALSH (2) and Grace J. KELLEHER

Abstract. — *Considered is discrete two-person game theory where the players choose their strategies independently. Two approaches using a median criterion have been developed for obtaining optimum solution. In the first approach, the payoffs are ranked according to desirability separately for each matrix (with agreement between the players on these rankings). For the second approach, the possible outcomes of the game (pairs of payoffs, one to each player) are ordered according to desirability separately by each player.. Only situations of players behaving competitively are considered for the first approach The class of games with optimum solutions for the first approach is huge compared to (and includes) the class with minimax solutions for expected-value game theory, but is a very small subclass of all games. A strong application advantage of the first approach is that ranking of the payoffs for each player, plus accurate evaluation of at most two payoffs for each player, is sufficient for application. Rather general situations can be considered for the second approach, which is usable for virtually all games. Ordinarily, however, virtually all payoffs need to be accurately evaluated for the second approach, and determination of the ranking for outcomes can require huge effort. An exception occurs for a case where the two approaches are combined. For both approaches, the payoffs can be of a very general nature (some payoffs may not even be numbers). In this paper, the two approaches are outlined and their application properties are discussed.*

INTRODUCTION AND DISCUSSION

The case considered is that of two players with finite numbers of strategies. Separately and independently, each player selects one of his strategies. Each possible combination of strategies determines an outcome for the game, where each outcome consists of a specified pair of payoffs (one to each player). The payoffs to a player for the various

(1) Southern Methodist University.
(2) Research partially supported by Mobil Research and Development Corporation. Also associated with ONR Contract N00014-68-A-0515 and NASA Grant NGR 44-007-028.

strategy combinations can be conveniently expressed as a matrix, where the rows represent his strategies and the columns the strategies of the other player. The two payoff matrices are known to both players.

A player uses a mixed strategy when he assigns probabilities (sum to unity) to his possible strategies and randomly selects one of them according to these probabilities. Game theory has probabilistic aspects when at least one player uses a randomly chosen strategy. Then the payoff to each player is a random variable, with a distribution determined by the probabilities used by the players. The distributions for these two random payoffs constitute the maximum information that is obtainable about the outcome for the game.

A fundamental problem of game theory is to make an optimum choice for the probabilities of the mixed strategies (with unit probabilities possible). Such a choice has many complications when all the properties of probability distributions are taken into consideration. However, this determination is greatly simplified if all that is considered is some reasonable « representative value » for a distribution. The well known expected-value method uses the distribution mean (expected payoff to the player) to represent a distribution. Another reasonable way is to represent a distribution by its median. This is the basis for median game theory.

It is desirable to have optimum solutions that are of a « forcing » nature. That is, an optimum choice of strategies controls the game outcome according to some meaningful criterion (such as expected payoff). The minimax method of solution for expected-value game theory has this property (for example, see ref. 1). Also, the two approaches that have been developed for median game theory are of this nature. However, both approaches for median game theory have huge advantages over the minimax method with respect to practical application.

The purpose of this paper is first to outline the two approaches to median game theory. Then, comparisons are made between these approaches and minimax game theory with respect to generality of application and effort needed for application (at the end of each outline). Finally, the two approaches are compared with each other.

The first approach, which is based on rankings within payoff matrices, has advantages over the second approach in the effort needed for application but strong disadvantages in generality of application. The second approach, which is based on ranking of the outcomes, has general applicability. The minimax method and the first approach are for situations where the players act competitively toward each other. The second approach is usable for almost any kind of situation (including that of competitive players). A special case occurs (for competitive players) that is a combination of the two approaches and has advantages of both.

A strong advantage of the two median approaches is that the payoffs can be of a general nature. In fact, some payoffs might not even be numbers (for example, could represent categories). The allowable payoffs must satisfy the arithmetical operations and ordinarily are expressed in the same unit for game theory with an expected-value basis.

The next section is concerned with the first approach for median game theory. This is followed by a section devoted to the second approach. The final section contains a comparison between the two approaches.

FIRST APPROACH

A ranking of the payoffs within each of the two matrices provides the basis for the first approach. In development of the current results for the first approach, the payoffs within a matrix were considered to be numbers for which a natural ranking (according to increasing desirability occurs. Then the rankings within the matrices are necessarily in agreement for the two players. However, the results obviously also hold for payoffs of a general nature when the rankings (according to increasing desirability) are in agreement for the players in the two matrices. For simplicity the description given here is phrased as if the payoffs have « values ». Subject to the condition that the players agree on the rankings, however, these results are applicable for any kinds of payoffs.

The first approach is intended for use only when the players act competitively. The concepts of a player behaving protectively, or vindictively, are useful for situations with competitive players. That is, a protective player tries to maximize his payoff without consideration of the payoff to the other player. A vindictive player attempts to minimize the payoff to the other player without consideration of the payoff to himself. A strategy that allows a player to be protective and vindictive simultaneously is optimum for him when the players behave competitively.

Let the players be designated as I and II. The following properties always hold : A largest payoff P_I (P_{II}) occurs in the matrix for player I (II) such that, acting protectively, he can assure himself at least this payoff with probability at least 1/2. Also, a smallest payoff P_I' (P_{II}') occurs in the matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 1/2, that player I (II) receives at most this payoff. The relations $P_I' \leqslant P_I$ and $P_{II}' \leqslant P_{II}$ hold, with equality possible. Methods for evaluation of P_I, P_{II}, P_I', P_{II}' are given in refs. 2 and 3, where the viewpoint of ref. 3 is ordinarily preferable. These references also contain methods for determination of protective median optimum strategies and of vindictive medium optimum strategies.

Payoff matrices occur such that a player can be protective and vindictive simultaneously (according to the median critetion). When this can happen for a given player, the game is said to be one player median competitive (OPMC) for him. A game is median competitive if and only if it is OPMC for both players. A subclass of the median competitive games is identified in ref. 2. The complete class is identified in ref. 3.

Identification of whether a game is OPMC for a player is not difficult. Consider the possible outcomes. Those outcomes such that the payoff to player I (II) is at least P_I (P_{II}) and also the payoff to player II (I) is at most P'_{II} (P'_I) constitute set I (II). The game is OPMC for player I (II) if and only if player I (II) can assure, with probability at least 1/2, that an outcome in set I (II) occurs. A procedure for determination of whether a game is OPMC for a player, and for determining a median optimum strategy for him, is given in ref. 3.

Now, let us compare the results for the first approach with those for the minimax method. In expected-value game theory, the players can be protective and vindictive simultaneously, which corresponds to existence of minimax solutions, when the payoff matrices satisfy a zero-sum condition (sum of payoffs is zero for every strategy combination) or one of some mild modifications of this condition. These « zero-sum » games are a very small subclass of the median competitive games. Also, the OPMC concept, for just one player, does not seem to have an analogue in expected-value game theory. Thus, the first approach has very strong advantages over expected-value game theory with respect to generality of application.

The first approach also has very strong advantages over expected-value game theory with respect to application effort. For realistic games, a very large number of strategy combinations can occur. As an example, suppose that each player has 100 strategies. Then, there are 10,000 strategy combinations. An accurate payoff value must be determined for virtually every strategy combination in expected-value game theory. This can result in a huge amount of effort. For the first approach, it is sufficient to know the order of the payoffs within each matrix and to have accurate values for P_I, P_{II}, P'_I, P'_{II}. Moreover, knowledge of the order within the matrix for player I (II) identifies the matrix locations for P_I (P_{II}) and P'_I (P'_{II}). Also, knowledge of ordering among a set of the largest payoffs of a matrix, and among a set of the smallest payoffs, is not needed for use of the first approach. Hence, the first approach has very strong advantages with respect to application effort.

A moderately thorough discussion of application advantages of the first approach over the expected-value method, for discrete two-person games, is given in ref. 4. These advantages are augmented by the more general types of payoffs that can be considered, as described at the beginning of this section.

SECOND APPROACH

Consider all the possible outcomes for a game. Suppose that, separately for each player, these outcomes can be ranked according to increasing desirability to that player (with equal desirability possible). This should virtually always be possible on a paired comparison basis. Each player could use almost any possible way of ranking the outcomes. If (p_I, p_{II})

is a general outcome, however, virtually any ordering method should be such that, for player I (II), desirability is a nondecreasing function of p_I (p_{II}) for fixed p_{II} (p_I), when payoffs can be ordered according to increasing desirability (separate orderings can occur for each player).

Once an ordering according to increasing desirability has been determined for player I (II), he can identify a smallest set of outcomes S_I (S_{II}) such that the other outcomes are less desirable and such that he can assure an outcome of S_I (S_{II}) with probability at least $1/2$. A procedure for determining S_I and S_{II} is outlined in the Appendix. A procedure for determining optimum median strategies is also outlined there. These results are « the best obtainable », according to the median criterion used. The idea of ranking outcomes in a general fashion is based on a method used in ref. 5.

Now consider a special case of the second approach that has some aspects of the first approach, including competitive behavior of the players. In addition, the payoffs are ranked for each matrix and the players agree on the rankings. The methods for rankings of the outcomes are such that all outcomes of set I (II) have maximum desirability to player I (II). Also, the relations among desirability that are stated in terms of p_I and p_{II} at the beginning of this section, are satisfied. The material using this type of ranking is taken from ref. 6.

Finally, let us compare results for the second approach with those for expected-value game theory in the general case. First, the players do not necessarily behave in a competitive manner for the second approach. Second, the second approach is applicable for virtually all discrete two-person games. Thus, the second approach is hugely preferable to expected-value game theory with respect to generality of application.

Next, consider the effort needed for applying a method. The second approach and the expected-value method are about the same with respect to necessity for accurately evaluating payoffs. That is virtually all payoffs need to be accurately evaluated in both cases. The ranking of outcomes can require a huge amount of effort if a paired comparison method is used, due to a huge number of possible pairs, which is $N(N - 1)/2$ if N is the number of outcomes. However, ranking is easy when a suitable function of p_I and p_{II} is available for this purpose.

The special case of the second approach has strong advantages over expected-value game theory with respect to effort needed for application. Within set I, and set II, it is sufficient to know relative order among payoffs and accurate values for at most two payoffs (which are identified by the orderings). Often, set I (II) contains the predominant number of the outcomes in S_I (S_{II}), so that a relatively few additional outcomes are needed to obtain S_I (S_{II}). However, this special case has less advantages with regard to generality of application. That is, the behavior of the players must be competitive and the players must be in agreement with respect to the orderings within payoff matrices.

COMPARISON OF APPROACHES

Now, the two approaches are compared with regard to generality of application and effort for application. The properties considered have already been described in the comparisons of the approaches with the expected-value method and, for brevity, will not be stated in much detail.

The greatest advantage of the first approach over the second approach is with respect to effort for application. The necessity to accurately evaluate payoffs is almost entirely replaced by the requirement that they be ordered (separately in each payoff matrix). At most two payoffs need to be accurately evaluated for each matrix and their matrix positions are identified by the orderings. As discussed in ref. 4, great amounts of time and effort can be needed to accurately evaluate very large numbers of payoffs, and huge numbers of payoffs can easily occur in practice. Also, there can conceptual difficulties in even approximate determination of some payoffs, and the first approach can have strong advantages when this is the case (ref. 4). Ordinarily, virtually all of the payoffs need to be accurately evaluated for the second approach although exceptions occur (one, the special case oft he preceding section, is discussed later).

The ordering of payoffs within matrices is virtually always very much easier that the ordering of outcomes. This is especially the case if paired comparisons, which is a general method, needs to be used for ordering the outcomes. Paired comparisons seldom needs to be used for ordering payoffs, even when some of the payoffs are not numbers. Also, if there is difficulty in ordering payoffs, the difficulty in ordering outcomes is usually increased correspondingly. One case, however, where ordering outcomes is not difficult is that where, for a player, relative desirability is expressible as an explicit function of (p_I, p_{II}), where (p_I, p_{II}) is a general outcome. Unfortunately, the devlopment of a suitable function of (p_I, p_{II}) can itself be very difficult.

The two approaches have roughly the same properties with regard to allowable kinds of payoffs. However, for very general types of payoffs, ordering of outcomes can be much more difficult than ordering of payoffs.

The second approach has an exceeding strong advantage over the first approach with respect to generality of application. First, the situation need not be that where the players act as if they were competitors. In fact, one player could prefer that the other player receives larger payoffs. Second, the second approach can be used by a player whenever he is able to rank the outcomes according to their desirability to him. It would seem that this ordering is possible for virtually all situations, so that the second approach is (virtually) always applicable. On the other hand, the players act competitively for the first approch. Also, the players are required to agree on the rankings for the payoffs in the two matrices. Most important, however, is that an optimum solution occurs for a player only if the game is OPMC for him. The set of games that are

OPMC for at least one player is a very small subclass of the class of all discrete two-person games.

The special case of the second approach has properties of both approaches, which results in advantages and disadvantages. The disadvantages, compared to the ordinary second approach, is that the players are required to behave as competitors and that they must agree on the orderings of the payoffs in the two matrices. That is, the generality of application is reduced somewhat. However, there are strong advantages with respect to the necessity to evaluate payoffs and with respect to the effort needed to order outcomes (as discussed in the preceding section).

Finally, it should be noted that only subsets of payoffs and subsets of outcomes actually need to be determined. Only the set of payoffs at least as desirable to player I (II) as P_I (P_{II}) and the set of payoffs at most as desirable to player I (II) as P'_I (P'_{II}) need to be determined for the first approach. Only the outcomes of sets S_I and S_{II} need to be determined for the second approach. Also, situations occur where cooperation is definitely preferable to optimum solutions for median game theory. Cases of this nature are considered in ref. 5.

APPENDIX

Consider determination of S_I and S_{II}. For player I (II), first mark the position(s) in his matrix for the payoff(s) in the outcome(s) with the highest level of desirability. Then also mark the position(s) for the payoff(s) in the outcome(s) with the next to highest level of desirability. Continue this marking, according to decreasing desirability, until the first time some one of the marked positions can be assured by player I (II) with probability at least 1/2. The resulting outcomes with marked payoffs constitute S_I (S_{II}). It is to be noted that outcomes not in S_I (S_{II}) are less desirable to player I (II) than the outcomes of S_I (S_{II}).

As for determining P_I and P_{II} (ref. 3), the procedure is to replace each marked payoff by unity and each unmarked payoff by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis. The set S_I (S_{II}) is the smallest set obtainable using the marking procedure such that the value of this game to player I (II), using his converted matrix (to ones and zeroes), is at least 1/2.

Finally, consider determination of median optimum strategies for players I and II when the second approach is used. For player I (II), let all payoffs in his matrix that correspond to outcomes which belong to S_I (S_{II}) be replaced by unity and all others replaced by zero. The resulting matrix of ones and zeroes is considered to be for a zero-sum game with an expected-value basis. An optimum strategy for player I (II) in this game is median optimum for him.

REFERENCES

[1] Owen, Guillermo, *Game Theory*, W. B. Saunders Co., 1968.

[2] Walsh, E. John, « Discrete two-person game theory with median payoff criterion », *Opsearch*, vol. 6 (1969), pp. 83-97. Also see « Errata », *Opsearch*, vol. 6 (1969), p. 216.

[3] Walsh, E. John, « Median two-person game theory for median competitive games », *Journal of the Operations Research Soicety of Japan*, vol. 12, décembre 1969, pp. 11-20.

[4] Walsh, E. John and Kelleher, J. Grace, « Difficulties in practical application of game theory and a partial solution ». Submitted to *Journal of the Canadian Operational Research Society.*

[5] Walsh, E. John, « Identification of situations where cooperation is preferable to use of median game theory ». *Opsearch*, vol. 7, june 1970, pp. 89-95.

[6] Walsh, E. John, « Generally applicable solutions for two-person median game theory ». *Journal of the Operations Research Society of Japan*, vol. 13 (1970), pp. 1-5.