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Secrétariat :
Nathalie Christiaën
Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie, 75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99 / Fax : (33) 01 40 46 90 96
Mél : `revues@smf.ens.fr` / URL : `http//smf.emath.fr/`

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SELF-PORTRAITS WITH ÉVARISTE GALOIS (AND THE SHADOW OF CAMILLE JORDAN)

Frédéric Brechenmacher

Abstract. — This paper investigates the collections of 19th century texts in which Evariste Galois's works were referred to in connection to those of Camille Jordan. Before the 1890s, when object-oriented disciplines developed, most of the papers referring to Galois have underlying them three main *networks of texts*. These groups of texts were revolving around the works of individuals: Kronecker, Klein, and Dickson. Even though they were mainly active for short periods of no more than a decade, the three networks were based in turn on specific

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F. Brechenmacher, Univ. Lille Nord de France, 59000 Lille (France), U. Artois, Laboratoire de mathématiques de Lens (EA 2462), rue Jean Souvraz S.P. 18, 62300 Lens (France).

Courrier électronique : `frederic.brechenmacher@euler.univ-artois.fr`

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references to the works of Galois that occurred in the course of the 19th century. By questioning how mathematicians were portraying themselves and their mathematics through their references to Galois, this paper therefore sheds new light on some collective interpretations of Galois's works. It especially highlights the important role played in the long term legacy of Galois by some practices of *reduction* modeled on the *analytic representation* of the decomposition of linear substitutions into two forms of actions of cycles.

Complementary to the local study of these networks, the article proposes a more global analysis. Galois's works were often related to the problem of the "classification and transformation" of the "irrationals." Contrary to what has become, in the 20th century, a commonplace of the historiography of algebra, and distinct from the teaching of courses in *Algèbre supérieure*, Galois's works were fitted into classifications of mathematical knowledge neither under the heading of the theory of equations nor as part of the theory of substitutions. For most of the 19th century, the problem of the irrationals involved elliptic (or abelian) functions (and therefore complex analysis). The impossibility of solving *general* algebraic equations of degree greater than four by radicals highlighted the necessity of characterizing the *special* nature of the irrational quantities and functions defined by both algebraic and differential equations.

RÉSUMÉ (Auto-portraits avec Évariste Galois (et l'ombre de Camille Jordan))

Cet article questionne les dimensions collectives des relations entre les travaux de Galois et Jordan au XIX^e^ siècle. Avant les années 1890 et le développement de disciplines centrées sur des objets, les références à Galois se répartissaient majoritairement au sein de trois *réseaux de textes* centrés sur des travaux d'individus : Klein, Kronecker et Dickson. Bien que ces réseaux n'aient été chacun essentiellement actif que sur le temps court d'une décennie, tous s'appuyaient sur des références spécifiques à Galois qui impliquaient le temps long du XIX^e^ siècle. En envisageant de telles références comme des autoportraits de mathématiciens et de leurs mathématiques, cet article porte un nouvel éclairage sur les travaux de Galois et leurs circulations. Il montre notamment l'importance du rôle joué sur le temps long par des pratiques de réductions prenant modèle sur la *représentation analytique* de la décomposition des substitutions linéaires en deux formes d'actions des cycles.

En complément de l'étude locale de ces trois réseaux, cet article propose également une analyse à un niveau plus global. Contrairement à ce qui était devenu au XX^e^ siècle un lieu commun de l'historiographie de l'algèbre, et à l'exception du domaine de l'enseignement de l'Algèbre supérieure, les travaux de Galois ont pendant longtemps été envisagés collectivement dans des cadres différents de ceux de la théorie des équations ou de la théorie des substitutions. A l'échelle d'un demi-siècle en Europe, ces travaux ont été effectivement largement commentés dans le cadre du problème de la « classification et la transformation » des « irrationnelles ». Pendant une large partie du XIX^e^ siècle, ce problème impliquait notamment les fonctions elliptiques et abéliennes—et par conséquent l'analyse complexe. L'impossibilité de résoudre par radicaux des équations algébriques *générales* de degré supérieur ou égal à cinq démontrait en effet la nécessité de caractériser la nature *spéciale* des grandeurs ou fonctions irrationnelles définies par des équations aussi bien algébriques que différentielles.

INTRODUCTION[1]

Opening Camille Jordan's 1870 *Traité des substitutions et des équations algébriques*—which will simply be called the *Traité* in the sequel—one reads in the preface that Galois was first to have grounded the theory of equations on a "definitive base by showing that to each equation corresponds a group of substitutions in which its essential characteristics are reflected" [Jordan 1870, p. v]. But the *Traité* aimed at going beyond Galois: "... the solution of equations by radicals ... now appears just as the first link in a long chain of questions concerning the transformations of irrational numbers and their classification." To achieve this, Jordan presented an "essential" "method of reduction" of a group into chains of (normal) subgroups [Jordan 1870, p. 392],[2] which transformed Galois's ideas into a fully fledged theory, a "*corps de doctrine.*"

The present paper, in a way, follows a similar method; it aims at investigating how the decomposition of a corpus of mathematical papers into networks of texts can unveil certain collective dimensions in the historical process of mathematics. More precisely, this paper aims at investigating the *collective* dimensions of the *relations* between the works of Évariste Galois and Camille Jordan.

Jordan's *Traité*, the starting point of this paper, has often been taken to be a midpoint in the historiography of Galois Theory. In a retrospective perspective, Jordan's Galois would mark a turning point in the unfolding of the abstract group concept [Wussing 1984], of Artin's Galois theory [Kiernan 1971], or more generally of the rise of a structural image of algebra [Corry 1996]. Recently, in the opposite historical direction, the *Traité*'s

[1] This paper will appeal to quotation marks quite often for the purpose of mentioning expressions or terminologies as they were used by the actors (and solely for this purpose).
In this paper, the term "substitution group" designates a permutation group on a finite number of letters, and "linear" substitutions/groups designate both linear and affine substitutions/groups. Moreover, p is a prime number, n is an integer, ξ is a primitive nth root of unity, and g is a primitive root mod p.

[2] An irreducible equation $f(x) = 0$ is solvable by radicals if and only if its Galois group can be "reduced" by a series $G = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_m = I$ in which every H_k is a normal subgroup of G and all H_k/H_{k+1} are abelian. This theorem is closely related to the Jordan-Hölder theorem, but the chain of subgroups here is not necessarily a composition series, with simple abelian quotients (and therefore cyclic of prime order). This criterion of solvability was first stated in [Jordan 1864], while the theorem on the invariance of the orders of the successive quotients in a composition series was stated in [Jordan 1869a] and proved in [Jordan 1870, p. 42–48].

Galois has been studied with a focus on Galois's "Mémoire sur les conditions de résolubilité des équations par radicaux" (the *Mémoire,* for short) by Caroline Ehrhardt [Ehrhardt 2007].

Our emphasis, however, will not be on the Galois theory of *general* equations, but rather, following [Neumann 1997], on three applications of that theory in Galois's writings: to equations of prime degree, to primitive equations of prime power degree, and to modular equations. To be sure, these *special* equations intervened in Galois's works not so much as *applications* of, but rather as *models* for the general theory. They are all associated with linear substitutions [Goldstein & Schappacher 2007a, p. 34], i.e., to what Jordan was to transform into a *general* object of investigation.[3] Rather than concentrating on the dichotomy between abstract and concrete, this paper will thus peruse the intertwining of the general and the special. Moreover, we shall see that one of the main specificities of Jordan's Galois was not the development of the group concept, but the *circulation* of a practice of *decomposition* (Galois), or *reduction* (Jordan), in which the *analytic representation of substitutions* played the key role.

Let us briefly recapitulate the stages of Jordan's treatment of Galois prior to the *Traité.* He commented for the first time on Galois in the few pages he added to his thesis with a view to competing for the *Grand prix* of the Paris Academy in 1860.[4] Between 1864 and 1870, he published a series of notes and memoirs on issues which he related to Galois: solvable groups, solvable equations, and irrational numbers. Jordan then published three commentaries on "Galois's fundamental theorem," i.e., on the relationship between the "adjunction of roots to an equation" and the "reduction" of a group.[5] But while the *Préface* of the *Traité* presented the whole book as a commentary on the works of Galois [Jordan 1870, p. VIII], Jordan did not refer to Galois any longer after 1870, not even when the latter entered the mathematical pantheon around the turn of

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3 In modern parlance, if one considers the symmetric group S on p^n letters as the group of permutations of the field $GF(p^n)$ a subgroup of S is solvable if and only if it is affine. Affine groups have been designated for a long time as "linear groups" with a distinction between general linear (i.e., affine) and homogeneous linear (i.e., linear).

4 The *Grand prix* of 1860 concerned the problem of the number of values of a function, i.e., one of the main lines of development of the theory of substitutions [Ehrhardt 2007, p. 397].

5 First a note in the *Comptes rendus* [Jordan 1865], next in "Lettre à M. Liouville sur la résolution algébrique des équations" [Jordan 1867b], and finally in the "Commentaire sur Galois" published in *Crelle's* Journal [Jordan 1869a]. Recall that when Liouville had edited Galois's works in his journal in 1846, he had claimed he would comment further on the *Mémoire* in a forthcoming paper [Ehrhardt 2007, p. 189–210].

the century.[6] Already by that time, however, Jordan was often presented as the mathematician who had emancipated the notion of group from Galois's *Mémoire* by developing it into an autonomous theory of (substitution) groups.

Focusing now on Jordan's *Traité*, a quick survey of its four books (*livres*) shows the limitations of the common view that its focus is on the theory of equations:

The "Théorie de Galois" alluded to in the very short *Livre I* is all about higher congruences $f \equiv 0 \pmod{P}$, for an irreducible polynomial of degree n with integer coefficients. It thus deals with what would nowadays be called finite fields, or Galois fields, in the tradition of the number-theoretical imaginaries which Galois had introduced in his 1830 "Note sur la théorie des nombres."[7] But Galois's imaginaries bear only a very indirect relation to the general principles of his *Mémoire*, and therefore also to the correspondence between fields and groups which is today perceived as the very essence of Galois theory.[8] Indeed, for Galois, number-theoretic imaginaries were above all useful in enabling a practice for dealing with the substitutions involved in the investigation of primitive equations of prime power degree [Galois 1832, p. 405–407], [Galois 1830b, p. 410]. More precisely, one of the first general principles of the *Mémoire* had been to consider as *rational* "every rational function of a certain number of determined quantities which are supposed to be known a priori. ... we shall [then] say that we *adjoin* them to the equation to be solved" [Galois 1846, p. 418].[9] The *Mémoire*'s first proposition stated that: "Let a given equation have the m roots $a, b, c, \ldots$ There will always be a group of permutations of the letters $a, b, c, \ldots$ [...] such that every function of the roots, invariant under the substitutions of the group, is rationally known" [Galois 1846, p. 421]. The known rational functions can be retrospectively understood

6 The constitution of disciplinary pantheons has been recently discussed in [Weber 2012] in connection to the issue of the *grandeur savante* in France at the turn of the 20th century.

7 Independently of the legacy of Galois, finite fields had been developed in the legacy of Gauss by Schönemann, Dedekind, and Kronecker. See [Frei 2007].

8 In 1846, Liouville had insisted on the distinction between Galois's imaginaries and the solvability of equations when he pointed out that the representation afforded by primitive roots did not imply any result on the solvability of higher congruences by radicals (Liouville *in* [Galois 1846, p. 401]).

9 This recourse to "known rational functions" was not original with Galois in 1830. Lagrange had developed the notion of "similar" functions as early as 1770 (cf. [van der Waerden 1985, p. 81]). Two functions f and g of the roots of a given equation are called *similar*, if all substitutions leaving f invariant also leave g invariant. It then follows that g is a rational function of f and of the coefficients of the initial equation.

as forming a *field*. But the substitutions were acting on indeterminate letters or on arrangements of letters, not directly on the *field*.[10] In the case of an equation of prime power degree, the p^n roots could be indexed by number-theoretic imaginaries. These in turn could be substantiated via cyclotomy, thus providing an analytic representation for the substitutions involved [Galois 1830b, p. 405].

Only a few references to Galois can be found in Jordan's *Livre II* on substitutions, and none at all in its opening chapter "On substitutions in general," which may today be described as group theory. The main allusion to Galois occurs in the section on the "Analytic representation of substitutions" (chap. II, § I), precisely in connection with the Traité's first use of number-theoretic imaginaries for the indexing mentioned above [Jordan 1870, p. 91]. This resumption is crucial as it leads to the "origin of the linear group" (chap. II, § II), i.e., to central objects of *Livre II*.[11] Indeed, underlying the indexing of p^n letters was *one* type of substitution (a cycle) appearing in *two analytic forms*: $(i \quad i+1)$ and $(i \quad gi)$. The "linear form" $(i \quad ai+b)$ originated from the composition of these two forms.[12] As we shall see, the analytic representation of n-ary substitutions in the general linear group would be one of the main specificities of Jordan's Galois in the long run.

The opening chapter of *Livre III* presents the *Mémoire*'s approach to general equations. However, the association between groups and equations was inscribed in the broader framework of a "General theory of irrationalities." While the "Algebraic applications" (chap. II) to Galois's theory of equations represented only a small part of *Livre III*, the emphasis was on "Geometric applications" (chap, III) and on "Applications to the theory of

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10 Even though substitutions were considered in relation to the invariance of functions of the roots, they did not form the automorphism group of a field, but the permutation group of the roots.

11 The groups considered are $GL_m(p^n)$ along with its subgroups $SL_m(p^n)$, $PSL_m(p^n)$, $Sp_{2n}(p)$, $PSp_{2n}(p)$, $O_n(p^n)$, etc. Cf. [Dickson 1901] as well as [Dieudonné 1962].

12 See [Galois 1846, p. 430–432], [Jordan 1870, p. 91]. Finite fields were considered both as additive and multiplicative abelian groups but without direct relation to the rational functions of the roots.
Because this paper will especially consider finite fields $GF(p^n)$, which are separable extensions (and even Galois extensions) of F_p, I will not deal with the retrospective linear algebraic standpoint of Artin's Galois theory: most of the texts under consideration were not resorting to the notions of vector space, normality, separability, field extension, or even to a clear separation between groups and finite fields. Moreover, Galois's theory of general equations was related to the formally different values of functions on n variables (or roots of *general* equations), it can therefore be applied to *special* equations with no multiple roots. For this reason, the equations considered in this paper will be supposed to have distinct roots.

transcendental functions" (chap. IV). For the geometric applications, the groups were usually not introduced by the adjunction of roots to equations as in Galois's method, but for instance by permutations between the lines on a surface, the invariance of an algebraic form, etc. And when it came to transcendental functions, Jordan considered the adjunction of certain "irrationals" corresponding to non-solvable equations (e.g., the modular equations of order $n > 3$) or to values of transcendental functions (e.g., trigonometric, elliptic, and abelian functions).

Livre IV is devoted to "Solution by radicals." It is based on a theorem which Jordan claimed for himself, thus going beyond Galois. This theorem provided the basis for the classification of maximal solvable subgroups of transitive groups through what Jean Dieudonné designated as "enormous machinery" consisting of successive reductions of *general* groups to a chain of *special* ones [Dieudonné 1962, p. xxxix-xlii]. All the rest of the treatise, almost one third of the book, is devoted to this problem. The general linear group played a key role, as the largest group of the chain whose substitutions could be analytically represented.[13]

Let us now come back to the usual history of the relation between Jordan and Galois. The main problem is that its main categories are indeterminate. "Groups," "equations," "algebra" have had changing meanings in various times and spaces—not to mention the terms Galois groups, Galois theory, Galois's ideas, etc. [Ehrhardt 2007, p. 1–45]. Let us consider two examples: First, Felix Klein and his followers in the 1880s used the phrase "*the* Galois groups" to designate the three special groups $\mathrm{PSL}_2(p)$ with $p = 5, 7, 11$. Second, Eliakim Hastings Moore stated in 1893 that every finite field is the abstract form of a Galois field $\mathrm{GF}(p^n)$ in the tradition of the Galois theory from *Livre I.*, i.e., with no direct relation to the result that every finite field can be represented as a Galois extension of F_p.[14]

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13 As shall be seen in greater detail later, the first step had been to reduce solvable transitive groups to primitive groups. A minimal normal subgroup A of a solvable primitive group G is abelian of type $(1, 1, \ldots, 1)$, i.e., isomorphic to a direct product of cyclic groups, i.e., to $\mathrm{GF}(p^n)^*$. Now G is acting on A as a linear group. In modern parlance, Jordan had introduced the linear group precisely as the maximal group in which $\mathrm{GF}(p^n)^*$ can be a normal subgroup, i.e., as the group of automorphisms of $\mathrm{GF}(p^n)$. This involved considering linear substitutions as generated by the two analytic forms of cycles.

14 As shall be seen in greater detail later, Moore's Galois fields were *defined* as classes of equivalences of an irreducible polynomial P of degree n on $F_p = Z/pZ$, i.e., $F_p[X]/(P)$. This definition was not directly related to Galois's works and had been inspired by [Serret 1866]. As any finite field of p^n elements can be represented as the splitting field of $P(X) = X^{p^n} - X$ on F_p, every finite field can be represented as a Galois field. But such a splitting field was not considered as a Galois extension of F_p: there

Moreover, the Jordan-Galois relation has often been presented as an exclusive relation between the one (Jordan) and the ideas of the other. But this relation has never been equitable. Usually, in fact, it has been connected to the celebration of the Great Mathematician Galois. Further, implicit collective epistemic and moral values accompanied the fulsome expression of Galois's grandeur. As a matter of fact, after Galois's ascension to the mathematical pantheon at the turn of the 20th century, some authoritative figures in French mathematics appealed to the Jordan-Galois relation to successively advance two claims: one for the universality of the French style of thinking in Analysis as opposed to the specialization of German arithmetic and algebra (1900–1930); and a second claim, symmetrically, that celebrated German conceptual algebra as opposed to the older French computational approaches (1930–1970). Let compare the introduction of the 1897 reprinting of Galois's works to that in Robert Bourgne and Jean-Pierre Azra's 1962 critical extended edition. The authors, Émile Picard and Jean Dieudonné, both celebrated Galois for his introduction of the notion of group. But while the former insisted on the analysis of groups of operations, the latter celebrated ideas that lie at the roots of modern algebra.

The 1897 reprinting was published two years after Sophus Lie had been invited to lecture on "Galois's influence on mathematics" at the celebrations of the centenary of the *École normale supérieure* [Ehrhardt 2007, p. 628–649]. Picard's 1897 account followed the role Lie had attributed to Jordan, namely as the one who had "clarified, developed, and applied" substitution groups in regard to the solvability of equations [Lie 1895, p. 4].[15] Jordan was thus presented as the immediate follower, the one who had generalized Galois's distinction between simple and compound groups to the notion of composition series (Picard *in* [Galois 1897, p. viii]). Picard's claims were to circulate at an international level, and played a key role in the consideration of Galois as the main founder of group theory.[16] Previous works on the history of the theory of equations had highlighted other aspects such as effective methods of solution [Aubry 1894] or Charles Hermite, Leopold Kronecker and Francesco Brioschi's approaches to the general quintic [Pierpont 1895]. But, after 1897, histories of the theory of

was no concern for the interplay between groups and fields which is characteristic of Galois theory.

15 [Ehrhardt 2007, p. 1–45] has historicised the category of the "intelligibility" of Galois's writings.

16 As will be seen later, Klein had already played an important role in the presentation of Galois as one of the founders of group theory.

equations would usually adopt a three-act structure: before Galois, Galois, and how Jordan had made Galois theory "become public" [Pierpont 1897, p.340].

But in addition to his clarification of Galois theory, the role of the researcher who closed the algebraic issue of the solvability of equations was also, somewhat incidentally, assigned to Jordan. Lie and Picard indeed both claimed that, unlike the previous works of Joseph-Louis Lagrange and Augustin-Louis Cauchy, Galois groups had exceeded the boundaries of algebra in introducing ideas whose "far reaching impact appears to us more and more every day" [Fehr 1897, p.756]. The seeds Galois had sown in the special case of equations were to blossom into a general notion of analysis. This claim should nevertheless not only be considered as having aimed at promoting Picard's or Lie's contributions to continuous group theory and differential equations. Picard, in particular, clearly took on the role of an official public authority on mathematics. Recall that, in France, the mathematical sciences were mainly divided between analysis, geometry and applications. At the turn of the century, several authorities such as Jules Tannery, Picard, Henri Poincaré, Jacques Hadamard contrasted the "richness" of the power of unification of analysis with the "poverty" of considering algebra and/or arithmetic as autonomous disciplines.

These official lines of discourse usually pointed to recent developments in Germany in the legacies of Kronecker or Richard Dedekind.[17] Promptly following Picard [Galois 1897, p. x], a review of Heinrich Weber's 1895 *Lehrbuch der Algebra* highlighted how Galois had introduced the "fundamental ideas" of Algebra as it was practiced in Germany; had he lived longer, all "French Science" would have had a different orientation [D'Esclaybes 1898, p. 416]. The celebration of the centenary of the *École normale* had aggrandized Galois's reputation, to the level of one who merited entry into the pantheon of Science, as we have mentioned above. Like other *grands savants*, Galois became involved in nationalistic anti-German discourse.[18] In the early 1920s, his image would be one of

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17 See [Brechenmacher 201?b]. Picard's *Traité d'analyse* proposed an exposition of Galois's theory very faithful to the original *Mémoire*. But although Picard appealed to the main notions introduced by Kronecker, algebraic Galois theory was not treated as an autonomous topic but as a first step toward differential Galois theory.

18 On the recurrent media depiction of Galois as a hero of French science, see [Picard 1900, p. 63], [Picard 1902, p. 124–125], [Picard 1914, p. 98–99], [Picard 1916, p. 12], [Picard 1922, p. 281–283]. Picard's views were reproduced in various journals and monographs. The future President of the French Republic Paul Deschanel appealed to Picard's Galois in his 1916 "Les Allemands et la science" which would be reprinted in *La France victorieuse. Paroles de guerre* [Deschanel 1919, p. 122].

the icons of post-war discussions on the universality of the French style of thinking. In an issue of the journal *France et monde* devoted to the topic of "the great ideas of mankind and the French style of thinking," Hadamard would claim: "Thanks to Galois's method, which might be the deepest thing a human being ever conceived in mathematics, the general problem of algebra—to which one can reduce almost everything that was studied during antiquity—is (theoretically) solved as much as it can be [...]" [Hadamard 1923, p. 339].[19]

Depending on their editorial orientations, the media reacted more or less positively to official expressions of the *grandeur savante*. But even though Galois may have been discussed with irony [Chevreuse 1912, p.1] and even mocked on some occasions [Beaunier 1908, p.3], he does not seem to have ever been seriously discredited. Recall that Galois had become a public figure early on in the 1830s. By the turn of the century, his image was already multifaceted and had an extended historical dimension. It transcended a number of categories, including the one of hero of national science. For instance, in 1923, a paper of the communist daily *L'Humanité* reported on a group of pupils of the *École Normale Supérieure* who had commemorated the 1871 *Commune de Paris*. The students were presented as followers of the revolutionary Galois who the paper opposed to "official science," especially "the French [official science]" [L'humanité 1923, p. 4].

At the turn of the 1930s-1940s, the founders of the Bourbaki group symmetrically reversed the previous categories of the official history of Galois as an icon of mathematical Frenchness. The latter's works came to be celebrated for having paved the way to algebraic number theory. The public to whom Jordan had mediated Galois's ideas changed, in this account: it now involved the genealogy Dedekind, David Hilbert, Emmy Noether, and Emil Artin. But the relation Galois-Jordan was not affected by the inversion of the roles attributed to analysis, algebra, France, and Germany. When he wrote the introduction to the 1962 edition of Galois's works, Dieudonné

19 See also [France et Monde 1922], [Adhemard 1922]. Moreover, Picard would present Galois as a hero of the "universality of French mathematics" at the occasion of the celebration of the fifty years of the *Société mathématique de France* [Picard 1924a, p. 31] (see also [Picard 1924b; 1925]). In 1923 the name of Galois had also appeared in the journal "*La pensée française, libre organe de propagation nationale et d'expansion française*," which was published in Strasbourg. The front page of the journal represented the French "Marianne" with storks wings (the symbol of Alsace), sitting on the globe of the world, with the notice "*La pensée française reigne sur le monde comme l'expression même de la liberté féconde et généreuse.*" Galois had been referred to at the occasion of a discussion on the masterpieces of French literature [Dunand 1923, p. 18].

was also involved in the edition of Jordan's works. Gaston Julia's *Préface* would once again present Jordan as the immediate follower of Galois, the one who was in direct contact with the latter's ideas [Jordan 1961–1964, p. vi].

In a word, the Jordan-Galois relation in official discourse on mathematics was ultimately quite stable for the *longue durée* of the 20th century. It was one of the main elements of epistemic continuity in the three-act story of the "predecessors," the "origins," and the "influence" of "Galois's ideas" [van der Waerden 1985, p. 76–133]. But as we have seen, despite the bilateral appearance of the relation Jordan-Galois, the actual meanings attached to this relation depended on third parties and resorted to implicit collective dimensions.

It is the aim of this paper to investigate some of these collective dimensions by considering various "portraits" of mathematicians with Galois. The metaphor of the self-portrait highlights the relational nature of mathematics. It suggests that the actors who referred to Galois through Jordan were also portraying certain individual and collective dimensions of their own mathematics.

The period under consideration is 1830–1914, with a focus on 1860–1900. On the one hand, Galois published his first papers in 1830 and Jordan defended his thesis in 1860. On the other hand, the mathematical apotheosis of Galois partially changes the nature of the corpus to be investigated for post-1900 investigations. The constitution of a corpus is nevertheless problematic even before the turn of the century. The opposing collective dimensions put to the fore by Picard and the Bourbaki group illustrate that neither national nor theoretical identities should be taken for granted when dealing with the Jordan-Galois relation. Selecting some texts because of their group-theoretical nature would imply resorting to the canonical role devoted to Jordan; while considering together some authors because they were French or German would prevent questioning the actual roles played by national dimensions.

The constitution of a corpus for the period of time preceding the *Traité*, to which the first section of this paper is devoted, is less problematic than the-post 1870 investigations developed in the second and third sections. Indeed, considering Jordan as a reader of Galois makes it possible to investigate the direct textual references of the one to the other,[20] while the

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20 On the historical notion of readers of a text, see [Goldstein 1995].

inverse problem of intertextual relations is much more difficult. This problem has been tackled by systematic investigations of the references to Galois and Jordan in the reviews of the *Jahrbuch über die Fortschritte der Mathematik* between 1870 and 1914. A first global corpus has been constituted by these texts, and then completed by their implicit and explicit web of references. Investigations of intertextual connections have then aimed at decomposing the global corpus into networks of texts.[21] Most of the papers can be organized into three main networks. As we shall see, a specific reference to Jordan-Galois lies beneath the identity of each group. Moreover, in each group the intertextual references converge to an individual whose name I shall use for designating the whole collective: Klein, Kronecker, and Leonard Dickson. Finally, each network was mostly active during a single ten-year period, while making references on a scale of half a century.

The second section of this paper investigates Dickson's network, which is mainly constituted of texts that have been published by French and American authors between 1893 and 1907. The main shared references are Moore's introduction of Galois fields in 1893, and Dickson's 1901 *Linear groups with an exposition of the Galois field theory*.[22] We shall see that the coherence of the collection of texts is based on specific roles devoted to both linear groups and Galois fields in the legacy of Jordan's *Traité*, but with no interest in Galois Theory. Another main characteristic of the Dickson network is that certain papers of the 1860s of Hermite, Joseph-Alfred Serret, Émile Mathieu, and Jordan are shared references for the whole network. These references were not exclusive of others, such as those to more recent works of Georg Frobenius, Alfred Loewy, or Klein, whose influence in the U.S.A has been well documented [Parshall & Rowe 1994, p. 147–455]. But none of these played as important a role for the collective identity of the network as the works of the 1860s. Dickson's network thus revolved around a two-fold periodization. We shall see that the two times and spaces involved were articulated by the *Traité*: Dickson's network can actually be understood as the space of circulation of a specific relation Jordan had established to the works of Galois in the 1860s.

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21 These networks are detailed in [Brechenmacher 2012a]. On methodological issues related to the use of networks, see [Goldstein 1999, p. 204–212], [Goldstein & Schappacher 2007b, p. 72–75], and [Brechenmacher 2007a;b; 2010].

22 As will be seen in greater detail later, Dickson's monograph had followed the thesis the latter had completed under the supervision of Moore [Parshall 1991].

The two other networks will be considered in the third section of this paper. They had developed in the interval of the above-mentioned two-fold periodization. They revolved around Kronecker's 1882 *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, and Klein's 1884 *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Here, the references to Galois were usually not directly connected to Jordan's *Traité*. On the one hand, Kronecker and his followers were mainly referring to Galois's *general* approach to equations (but not in the perspective of Jordan's *Livre III*).[23] On the other hand, Klein and his followers followed Hermite in focusing on the three special Galois groups of modular equations. In both networks, numerous references were nevertheless made to Jordan's works. It must therefore be pointed out that the problem of the collective dimensions of the relation Jordan-Galois is not equivalent to the one of the reception of the *Traité*.

Groups of texts such as the ones mentioned above should nevertheless not be considered as constellations in an empty sky. First, each author potentially belonged to several networks, which pointed to various topics, times and spaces. Second, in laying the emphasis on *textual* interrelations, this paper does not aim at discussing the main collective dimensions in which the *actors* were involved, such as the emergence of an American research community or the institutionalization of finite group theory. Third, some collective references to Jordan-Galois will not be in the scope of the present investigation. Either they did not lay much emphasis on Jordan (as for the case of differential Galois theory), or they lay outside the period considered here, as in the case of Dedekind's Galois. Even though Dedekind had lectured on Galois's works in Göttingen in the mid-1850s, his perspective remained disconnected from Jordan's Galois before the turn of the 20th century.[24]

But this investigation of the circulations of Jordan's Galois in networks of texts nevertheless aims to shed light on the identities that were collectively attributed to Galois's works in the 19th century. Galois's works have

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23 In short, Kronecker had developed a constructive presentation of finite field extensions of certain ground fields. See [Petri & Schappacher 2007], [Goldstein & Schappacher 2007b, p. 81–88].

24 Even though Dedekind would inspire [Bachmann 1881] and [Weber 1893b; 1895–1896], he would not publish his ideas on Galois theory until 1894. Before the mid-1890s, Kronecker's theory of algebraic quantities as a theory of forms (i.e., functions of n variables) was much more influential than Dedekind's fields and ideals. On Dedekind's Galois, see [Kiernan 1971, p. 129–133], [Edwards et al. 1982], [Scharlau 1981; 1982], [Corry 1996, p. 75–80, 110–112, 129–130], [Ehrhardt 2007, p. 470–504], [Goldstein & Schappacher 2007b, p. 78–81].

often been commented on with reference to a long-term history of algebra.[25] But algebra has taken on changing and complex identities that cannot be reduced to a changing focus from equations to algebraic structures. The question especially arises as to the relation of Jordan's Galois to the field of research of arithmetic algebraic analysis that developed between the 1820s and the late 1850s [Goldstein & Schappacher 2007a;b], or to more local theoretical and disciplinary organizations such as the several lines of development of group theory or the theory of equations as it was taught in various time periods and social spaces [Ehrhardt 2007].

This paper might not open the "book with seven seals" as Klein called the *Traité* when recalling his time with Lie in Paris in 1870 [Klein 1921–1923, p. 51]. But at least two facets of the book have to be distinguished. On the one hand, the *Traité*'s Galois has often been considered as a "masterpiece of mathematical architecture. The beauty of the edifice erected by Jordan is admirable" [van der Waerden 1985, p. 117]. On the other hand the *Traité* as a whole has also been described as a "a note-worthy event. [It] collects and unifies the results of his predecessors and contains an immense amount of new matter" [Pierpont 1904, p. 143]. The *Traité* was neither a textbook (like [Serret 1866] or [Netto 1882]) nor a compilation of papers (like [Hermite 1859]). It presented both an original structure and a synthesis in continuity with previous works. Both facets were articulated by the claim that:

> Le but de cet Ouvrage est de développer les méthodes de Galois et de les constituer en corps de doctrine, en montrant avec quelle facilité elles permettent de résoudre tous les principaux problèmes de la théorie des équations [Jordan 1870, p. VII].

As we shall see, we shall have to distinguish between two different images of Galois in the *Traité*. The first one issues from Jordan's specific approach to Galois, and ran through the book as a chain of generalizations from cyclotomy to the analytic representation of linear substitutions and eventually to the "essential method of reduction" of a group. The *second* Galois in the *Traité* was to be found in Livre *III*. There, Galois Theory attached Jordan's Galois to some parts of the book that were either in continuity with some previous works that had not previously been directly related to Galois (such as the general developments on substitution groups in Livre *II* and

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25 See, among others, [Aubry 1894], [Pierpont 1897], [Kiernan 1971], [Dieudonné 1978], [van der Waerden 1985], [Toti Rigatelli 1996]. Galois was already presented as the conclusion of a genealogy of works on the "theory of equations" in the context of the delimitation of algebra in [Serret 1849, p. 1–4].

most applications of Livre *III*) or were specific to Jordan but disconnected from Galois (such as the classification of solvable groups in Livre *IV*).

The structure given to this paper aims at echoing the differences between these two images of Galois. This paper will therefore deal with the time periods 1830–1870 and 1890–1900 related to the first Galois in the *Traité*, before considering the period 1870–1890 and the second image of Galois in Jordan's book. The first section of this paper aims at identifying the specific character of Jordan's Galois as compared to the image in other collective references to Galois's works. The second section of this paper will investigate the discontinuous circulation of the first Galois of the *Traité* by a micro-historical analysis of the small-scale development of Dickson network after 1890. Finally, the third section of this paper will question the second image of Galois with regard to its circulation and non-circulation in the Klein and Kronecker networks between 1870 and 1890.

1. THE ANALYTIC REPRESENTATION OF SUBSTITUTIONS (1830–1870)

1.1. *Analytic representations and applications in Galois's works*

This section aims at discussing the retrospective point of view of Jordan's early works on Galois's works and their circulations between 1830 and 1870. I will therefore limit my attention to the papers of the 1846 edition of Galois's works to which Jordan appealed. Recall that because the *Mémoire* was lost twice, its last version is more recent than most of Galois's writings. These are, then:

– April 1830. "Analyse d'un Mémoire sur la résolution algébrique des équations" published in the *Bulletin de Ferrusac* (*Analyse*, for short).
– ... 1830. The fragment of the *second Mémoire* "Des équations primitives qui sont solubles par radicaux" as it was edited by Joseph Liouville.
– July 1830. "Note sur la théorie des nombres," published in the *Bulletin des Ferrusac.* (*Note*, for short).
– 1831. Last version of the *Mémoire.*
– May 1832. Letter to Auguste Chevalier, published in the *Revue encyclopédique* (*Letter*, for short).

It is well known that the introduction of Galois's *Mémoire* had laid the emphasis on a distinction between the "general principles" of a theory and its three "applications" [Galois 1846, p. 417]. The first application was also the concluding theorem of the *Mémoire*, i.e., the *criterion* that: "in order that an equation of prime degree be solvable by radicals, it is necessary and sufficient that, if two of its roots are known, the others can be expressed rationally" [Galois 1846, p. 432]. Details on the two other applications had been given in the *Analyse*, the *second Mémoire*, the *Note*, and the *Letter*. What I shall designate as the second application was the characterization of solvable primitive equations of degree p^n. The third application was relative to modular equations.

1.1.1. *Primitive equations of prime power degrees*

The *Analyse* started with the introduction of the distinction between primitive and imprimitive equations: a "non-primitive equation of degree mn is an equation that can be decomposed into m factors of degree n, by appealing to a single equation of degree m" [Galois 1830a, p. 395]. These equations were also designated as Carl Friedrich Gauss's equations. In the *second Mémoire* Galois indeed appealed to "M. Gauss's method of decomposition" for reducing the problem of finding solvable irreducible equations of composite degree to the one of finding solvable primitive equations of degree p^n [Galois 1846, p. 434].

Unlike Alexandre Théophile Vandermonde (1774) and Lagrange's (1771) approaches to the special cases $x^5 - 1 = 0$ and $x^{11} - 1 = 0$, Gauss had introduced a general method of successive factorizations for proving the solvability by radicals of (irreducible) cyclotomic equations of degree $p-1$. The factorizations resorted to organizations of the roots in a specific order by appealing to the two indexings provided by a pth primitive root of unity ξ and by a primitive root g mod p. For any factorization $p - 1 = ef$, let $h = g^e$, and consider the equation of degree e whose roots correspond to the following e "periods" of sums of f terms:

$$\eta_i = \xi^i + \xi^{ih} + \cdots + \xi^{ih^{f-1}} \qquad (1 \leq i \leq e).$$

Such decompositions of the roots into periods allows factorizing the initial (imprimitive) cyclotomic equation into e factors of degree f.

In 1808, Lagrange had given a new proof of the solvability of cyclotomic equations. The successive auxiliary equations attached to Gauss's periods were replaced by the direct consideration of an auxiliary function of the coefficients and of roots of unity, i.e., the Lagrangian resolvent $\xi + \alpha\xi^g + \alpha^2\xi^{g^2} + \cdots + \alpha^{p-1}\xi^{g^{p-2}}$ (with α a primitive pth root of unity).[26] In the context of his review of [Lagrange 1808], Louis Poinsot commented on the

26 The Lagrangian resolvent line of development of Galois theory has been well documented. See [Kiernan 1971, p. 103–110], [van der Waerden 1985, p. 76–88], [Neumann 2007, p. 112], [Ehrhardt 2007, p. 78–103].

two approaches of Gauss and Lagrange. At this occasion, he had designated Gauss's periods as "groups" in a sense Galois would also use later on [Boucard 2011, p. 59–62]. Groups in this sense involved *both* partitions of "permutations of letters" (i.e., arrangements of the roots or indexing lists) *and* decompositions of "systems of substitutions" (the operations from one permutation to another).[27] Even though there might not have been any direct connection between Poinsot and Galois, Jordan deduced conclusions identical to those of Galois's early works from Poinsot's approach.

From the retrospective point of view of Jordan's 1860 thesis, the Gauss-Poinsot method consisted in dividing the letters into groups, each of the same cardinal, while systems of substitutions were simultaneously partitioned into a "combination of displacements between the groups and of permutations of the letters within each of the groups" [Jordan 1860, p. 5]. From a modern perspective, the "groups of permutations" correspond to a decomposition of the field into blocks of imprimitivity under the action of an imprimitive substitution group,[28] which is itself decomposed into a primitive quotient group.

Gauss's decomposition resorted to a single kind of substitution (i.e., cycles). But two forms of actions had to be distinguished depending on whether the cycles were acting within the groups or between the groups. Poinsot had discussed these two forms of actions from a geometric perspective. The roots generated by a primitive root of unity could be represented "as if they were in a circle" [Boucard 2011, p. 68]. They could then be made to move forward by translations, i.e., by the operation $(i \quad i+1)$ on their indices. But they could also be made to move by rotations of the full circle i.e., $(i \quad gi)$. In 1815, Cauchy introduced cycles by appealing to a similar circular representation [Cauchy 1815a, p. 75–81] even though he did not consider the analytic representations induced by the two forms of actions of cycles $(i \quad i+1)$ and $(i \quad gi)$. On the contrary, the analytic

27 The ambivalence of the terminology "group" as regard to the distinction between the "permutations of the roots" and the "substitutions" has often been considered as a limitation of Galois's approach (e.g. [Dahan-Dalmedico 1980, p. 282], [Radloff 2002]). But it should be pointed out that this ambivalence was the very nature of "groups" as they originated from the decomposition of imprimitive groups by the consideration of blocks of imprimitivity of letters.

28 Let G be a transitive group operating on a set Ω. A subset Γ of Ω is called a *block of imprimitivity* if $\Gamma \neq \varnothing$ and for every $g \in G$, either $\Gamma^g = \Gamma$ or $\Gamma^g \cap \Gamma = \varnothing$. If Γ is such a block and $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are the distinct sets Γ^g for $g \in G$, then $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ is a partition of Ω. G is said to be imprimitive if there is a non-trivial proper block. G is primitive if it is not imprimitive. See [Neumann 2006] for a discussion on primitivity in Galois's works. On the roles played by primitivity in Jordan's classification of solvable transitive groups, see [Brechenmacher 2006, p. 195–202].

representation of substitutions played an important role in Galois's second memoir.

The aim of the second memoir was to characterize solvable primitive equations by the general characteristic that their degree had to be a power of a prime. Galois had considered a primitive equation of degree N that turned into Q imprimitive equations by the adjunction of a radical of prime degree λ. The group of the equation was then partitioned into λ conjugated imprimitive groups. Let H be one of these imprimitive groups; its letters were decomposed on the model of Gauss's method into a table of p columns whose rows correspond to systems of imprimitivity:

$$a_0, a_1, a_2, \ldots, a_{p-1},$$
$$b_0, b_1, b_2, \ldots, b_{p-1},$$
$$c_0, c_1, c_2, \ldots, c_{p-1}.$$

Galois then argued that $N = p^n$, p prime. More importantly, the above allowed the introduction of n series of p indices for the indexing of the letters, and thereby to give an analytic representation to substitutions on p^n letters:[29]

> La forme générale des lettres sera
>
> $$a_{k_1, k_2, k_3, \ldots, k_\mu}$$
>
> $k_1, k_2, k_3, \ldots, k_\mu$ étant des indices qui peuvent prendre chacun les P valeurs $0, 1, 2, 3, \ldots P-1$. [...] dans le groupe H, toutes les substitutions seront de la forme
>
> $$[a_{k_1, k_2, k_3, \ldots, k_\mu} \quad a_{\varphi(k_1), \psi(k_2), \chi(k_3), \ldots, \sigma(k_\mu)}]$$ [Galois 1846, p. 436].

Galois then investigated further the case of primitive equations of degree p^2. A cycle, or a "circular substitution" as he said following Cauchy, would have the following form:

$$(a_{k_1, k_2}, a_{k_1+\alpha_1, k_2+\alpha_2}).$$

But then, Galois argued, because the substitutions of the group have to transform cycles into cycles, they must have a "linear form" [Galois 1846, p. 439]:

$$(a_{k_1, k_2}, a_{m_1 k_1 + n_1, m_2 k_2 + n_2}).$$

29 Modern formulations of this result resort to the notion of vector space, i.e., a field of p^n letters can be considered as a n dimensional vector space over F_p.

As will be seen in greater detail later, this argument would play a key role in the proof of the *criterion* and would circulate throughout the 19th century. It would be later understood as the statement that the maximal group in which a direct product of cyclic groups is a normal subgroup is the general linear group or that the group of automorphisms of an abelian group of type $(1, 1, \dots, 1)$ is a linear group.

Galois then successively computed the number of linear substitutions on p^2 letters and looked for solvable "divisors" (i.e., subgroups) of the group by investigating substitutions of the following form:

$$\left(b_{\frac{k_1}{k_2}}, b_{\frac{m_1k_1+n_1k_2}{m_2k_1+n_2k_2}}\right).$$

In the *Analyse*, Galois had already made it clear that the groups formed by the above substitutions were related to the modular equations of elliptic functions.

1.1.2. *Number-theoretic imaginaries*

The analytic representation of substitutions of primitive groups of p^n letters was improved by the introduction of number-theoretic imaginaries.

Let $f(x) \equiv 0 \pmod{p}$ be an irreducible higher congruence of degree n. As was common at the time, Galois legitimized the introduction of imaginary roots j by appealing to the analogy carried on by the process (of factorization) used for the case of ordinary equations.[30] He expressed the rational functions of the roots as "general expressions" $aj^{n-1}+bj^{n-2}+\dots+1$ (with $a, b, \dots$ mod p) and first proved that these p^n "algebraic quantities" could be considered as the roots of $x^{p^n} \equiv x \pmod{p}$. Reciprocally, he argued that the roots of the latter equation "all depend on one congruence of degree n" [Galois 1830b, p. 399–402].

But the aim of the *Note* was actually to show that any system of p^n indices could be reindexed "in analogy with" the indexing of p letters $l_1, l_2, \dots, l_p$, by the roots of Gauss's congruence $x^p \equiv x \pmod{p}$,[31] i.e., by the iterated powers of a primitive root j of $x^{p^n} \equiv x \pmod{p}$.[32] Galois later presented his note as a lemma for the investigation of primitive substitutions

30 On these issues, see [Durand-Richard 1996; 2008]. Jordan would still resort to such an analogy in 1870.

31 On Gauss's proof of the existence of primitive roots of cyclotomic equations, see [Neumann 2007].

32 In modern parlance, a Galois field GF(p^n) is both an additive group, which can be represented as an n-dimensional vector space on F_p, and a multiplicative cyclic group of p^{n-1} elements. Before Galois, higher congruences had been considered in the "missing section eight" of Gauss's *Disquitiones Arithmeticae* [Frei 2007] as well as by Poinsot [Boucard 2011].

[Galois 1832, p. 410]. Indeed, the conclusion of the *Note* was devoted to the characterization of solvable primitive equations [Galois 1830b, p. 405]. The roots x_i of a primitive equation of degree p^n could now be indexed by the solutions of the congruence $i^{p^n} \equiv i \pmod p$. Galois then claimed that if any function of the roots invariable by the substitutions of the form $(i \quad (ai+b)^{p^r})$ has a rational value, then the equation is solvable, and reciprocally. The proof was presented as a direct consequence of the decomposition of linear substitutions into a product of the two forms of cycles, i.e., in the form $a'(i+b')^{p^r}$: "those who are accustomed to the theory of equations will have no trouble seeing this" [Galois 1846, p. 406].

1.1.3. *The* criterion *and the last version of the* Mémoire

The proof Galois alluded to in the *Note* was given in the *Mémoire* for the case $n = 1$ corresponding to the criterion of solvability of irreducible equations of prime degree, which Galois presented as an application of his general principles. This application followed *proposition V*, which presented the problem of the solvability by radicals as resorting to the interplay between successive adjunctions of roots and successive decompositions of a group.[33] For the case of prime degree equations, the key argument was that the smallest non-trivial group in the successive reductions had to be generated by a cycle. Here, Galois explicitly referred to Cauchy even though he did not appeal to the latter's representation of substitutions as products of cycles [Dahan-Dalmedico 1980, p. 286–295] but to analytic representations.

Galois looked for the penultimate group in the successive reductions of the given equation. He showed that if its substitutions are represented by $(x_i, x_{f(i)})$, then $f(i+a) = f(i) + A$ (i.e., the group is the maximal group in which the cyclic group $(i \quad i+a)$ is a normal subgroup). Thus $f(i+2a) = f(i) + 2A, \ldots, f(i+ma) = f(i) + mA$. If $a = 1$, $i = 0$, then $f(i) = ai + b$, from which Galois eventually deduced that:

> Ainsi, pour qu'une équation irréductible de degré premier soit soluble par radicaux, il *faut* et il *suffit* que toute fonction invariable par les substitutions
>
> $$x_k, x_{ak+b}$$
>
> soit rationnellement connue [Galois 1846, p. 431].

[33] For some systematic comments on the general principles of Galois's *Mémoire*, see [Radloff 2002] and [Ehrhardt 2007, p. 55–77]

As will be seen later in greater detail, the introduction of the general linear group in Jordan's *Traité* would "originate" from the exact same argument.

1.1.4. *Modular equations*

The third application was relative to the modular equations of the transformations of elliptic functions. Let $u \longrightarrow \varphi(u,k)$ be the doubly periodic complex functions introduced as the inverse functions of integrals such as

$$u(\varphi, k^2) = \int_o^{\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

with periods $K = u(1,k)$ and $K' = u(1/k,k)$ (with k the modulus). The *modular equation* of order p (p prime) links all those moduli k' to a given modulus k for which it is possible to write the transformation

$$\frac{dy}{\sqrt{(1-y^2)(1-k'y^2)}} = \frac{dx}{p\sqrt{(1-x^2)(1-k^2x^2)}}$$

where $y = U(x)V(x)^{-1}$, with appropriate relatively prime polynomials U, V. Let L, L' and K, K' be the respective periods attached to the moduli k' and k, then the above expression thus gives a *transformation* of order p of $\frac{K}{K'}$ considered as a (modular) function of k: $\frac{L}{L'} = p\frac{K}{K'}$. This transformation yields a polynomial relation between k^2 and λ^2 of degree $p+1$: the modular equation of p.[34] As Catherine Goldstein discusses in further detail in this volume, Galois stated in 1832 that the degree of those equations can be reduced to p if and only if $p = 5, 7, 11$.[35] This statement was related to the investigation of the substitutions of the form $\frac{ai+b}{ci+d}$ $(ad - bc \neq 0)$.

1.1.5. *Three applications and some general principles*

In sum, Galois's three applications were associated to three analytic "forms" of linear substitutions. First, the criterion for solvable equations of prime degree was associated to $(i \quad ai + b)$. Second, the investigation of

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34 The modular function is invariant by the group of unimodular linear fractional substitutions $SL_2(\mathbb{Z})$ and the modular equation can thus also be introduced as the transformation equation of elliptic integrals by such substitutions. See Houzel [1978; 2007].

35 In 1830 he had first stated that p had to be equal to 5. In modern parlance, the problem consists in finding an irreducible equation of the smallest degree whose roots generate the splitting field of F_p. $PSL_2(p)$ can then be realized as a transitive subgroup of $Sym(m)$ for $m < p+1$. But the index of a maximal subgroup of the group $SL_2(p)$ associated with a modular equation is $p+1$ except for $p = 5, 7, 11$ when it is p. This subgroup is not a normal subgroup (recall that $PSL_2(p)$ is simple for $p > 3$), i.e., it gives rise to a non-proper decomposition in Galois's vocabulary.

solvable equations of composite degree involved considering linear substitutions on p^n number-theoretic imaginaries. Third, modular equations of the transformations of elliptic functions were associated with binary fractional linear substitutions.

But we have seen also that the three applications were intrinsically interlaced with one another in the evolution of Galois's investigations. They were not limited to *applications* but played also the role of *special model* cases for the *general* principles of the *Mémoire*. Each application modeled a special form of decomposition of a group.

First, as Galois would make it clear in the *Letter*, the "simplest decompositions are the ones of M. Gauss" by which the investigation of solvable transitive equations of composite degree was reduced to the one of solvable primitive equations of prime power degree. But, wondered Galois, "what are the decompositions that can be practiced on an equation that Gauss's method would not simplify?" [Galois 1832, p. 409]. As has been seen above, the decomposition of primitive equations of p^n or p degrees was modeled on the decomposition of linear substitutions into the two forms of representation of cycles. Recall that there was no clear concept of factor group yet. In the reduction of $(ai + b)$ into two cyclic substitutions, the two analytic forms $(i \quad i + 1)$ and $(i \quad gi)$ provided a model for the operations involved in composition series. It was on this model that Galois stated that the substitutions of primitive solvable equations of degree p^n had to have the linear form $x_{k,l,m,\ldots} \mid x_{ak+bl+cm+\cdots+h, a'k+b'l+c'm+\cdots+h', a''k+\cdots}$ [Galois 1832, p. 410].

Moreover, the reduction of the degree of the modular equation gave an example of improper decomposition, i.e., of non-normal subgroups of a group. In the *Letter*, Galois had distinguished this type of decomposition from the "proper decomposition":

> [...] on voit une grande différence entre adjoindre à une équation une des racines d'une équation auxiliaire ou les adjoindre toutes. Dans les deux cas, le groupe de l'équation se partage par l'adjonction en groupes tels que l'on passe de l'un à l'autre par une même substitution, mais la condition que ces groupes aient les mêmes substitutions n'a lieu que dans le second cas. Cela s'appelle *la décomposition propre.* [...] En d'autres termes, quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que
>
> $$G = H + HS + HS' + \cdots$$
>
> Et aussi, il peut se décomposer en groupes qui ont tous les mêmes substitutions, en sorte que
>
> $$G = H + TH + T'H + \cdots$$

Ces deux genres de décompositions ne coïncident pas ordinairement. Quand ils coïncident, la décomposition est dite *propre.* [Galois 1832, p. 408–409].

According to Galois, the difference between improper and proper decompositions[36] was the difference between adjoining one root or all the roots of an auxiliary equation to an equation. Galois's auxiliary equations could involve non-solvable equations on the model of the reduced modular equations. In 1832 Galois indeed claimed he had not focused all his attention on solvability by radicals but had also investigated "all possible transformation on an equation, whether it is solvable by radicals or not" [Galois 1832, p. 408].

The "proper decomposition" had been modeled on the traditional use of auxiliary equations $x^p = a$ in issues of solvability by radicals (and therefore of the binomial equation $x^n - 1 = 0$). This situation may be illustrated by propositions II and III of the *Mémoire.* The first described the proper decomposition of a group relative to the adjunction of a root to an equation. The second stated that if "one adjoins to an equation *all* the roots of an auxiliary equation, the groups of Theorem II would have the additional property of possessing the same substitutions" [Galois 1846, p. 423–425]. But this proposition had been previously stated differently. Its original formulation was that if one considers all the pth roots of unity to have been adjoined to an equation, then the same decomposition of the original group would originate from the adjunction of any of the roots of $x^p = a$. In that case, the adjunction of a root would imply the adjunction of all roots, i.e., the situation to which the *proposition III* had been generalized afterward.

The general quartic, and quintic had also played the role of model cases for Galois's investigations [Galois 1846, p. 428, 433]. But it must be pointed out that all the "applications" were pointing to the legacy of Gauss, while general equations were related to the legacy of Lagrange. The two legacies of Gauss and Lagrange did not play the same role in the *Mémoire.* In short, and on the one hand, three forms of decompositions had been modeled on Gauss's equations. On the other hand, Lagrange's legacy was related to the consideration of the number of values of rational functions of the roots under the action of substitutions, a problem that would become one of the main lines of development of the theory of substitutions.[37]

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36 This difference corresponds to distinguishing between non-normal and normal subgroups respectively.

37 This problem is tantamount to finding the possible orders for subgroups of the symmetric group. Given a function $\varphi(x_1, x_2, \ldots, x_n)$ of n "letters," a "value" of

1.2. *The shadow of Poinsot on Jordan's early works*

One of the two theses Jordan had defended in 1860 was devoted to the problem of the number of values of functions. Its main result was the introduction of a type of "conjugate systems" (i.e., the equivalent of a group in [Cauchy 1844]) of n-ary linear substitutions (i.e., $\mathrm{GL}_n(p)$) by a "method of reduction" of a "permutation group." But Jordan had not studied Galois yet. As a matter of fact, he acknowledged in a footnote that he had "discovered recently in the works of Galois the statement of the theorem" that he had concluded his thesis with, i.e., the order of $\mathrm{GL}_n(p)$.[38] Jordan's notion of "permutation group" amounted to the simultaneous consideration of blocks of imprimitivity and substitution groups. Unlike Galois, though, Jordan appealed to a precise distinction between permutation groups and conjugate systems of substitutions.[39]

Jordan had explicitly attributed the notion of group to Poinsot. When the number of values of a function was less than $n!$, he had considered that a "symmetry occurred within the function" as an application of "what Poinsot has distinguished from the rest of mathematics as the theory of order" [Jordan 1860, p. 3]. According to Jordan, other examples of applications of this theory were Cauchy's determinants, Abel's works on the general quintic, as well as Galois's works on "the conditions of algebraic solvability, the whole theory of equations considered in its full generality, and the classification of algebraic irrationals" [Jordan 1860, p. 3].

When he first referred to Galois, Jordan thus aimed at stressing the generality of the theory of order as opposed to "most geometers who have considered this question [of the many valued functions] in the aim of applying it to the theory of equations." Similar claims about the generality of a broad framework related to both symmetry and groups could be found

φ was a function obtained by permuting the variables, i.e., for any $\sigma \in \mathrm{Sym}(n)$, $\varphi^{\sigma}(x_1, x_2, \ldots, x_n) = \varphi(x_{1\sigma}, x_{2\sigma}, \ldots, x_{n\sigma})$ was a value of φ. If φ takes only one value, then it is symmetric and can therefore be expressed as a rational function of the elementary symmetric functions. If $x_1, \ldots, x_n$ are the roots of an equation with coefficients on a given "rational domain," this means that φ can be expressed as a rational function on the rational domain. In general, φ can take up to $n!$ distinct values and for intermediary cases between 1 and $n!$, normal subgroups of the symmetric group can potentially be associated to φ by considering the set of substitutions leaving φ invariant. If φ takes, for instance, ρ distinct values $\varphi_1, \varphi_2, \ldots, \varphi_\rho$, these values can be considered as the roots of an equation of degree ρ whose coefficients are the symmetric functions of the initial variables.

38 The *Traité* did not attribute this theorem to Galois. But Netto, who systematically tracked down the results that could be considered to have been stated before Jordan, attributed this theorem to Galois [Netto 1882, p. 155].

39 On Poinsot's notion of group, see [Boucard 2011].

in the contemporary writings of Théodore Despeyrous [1861, p. 417], another follower of Poinsot. Unlike Jordan, Despeyrous nevertheless never attributed any role to Galois as regards permutation groups [Ehrhardt 2007, p. 400–403].

Jordan's thesis was organized on a two-step reduction of the general problem of the number of values of functions. First, a general transitive system of substitutions was reduced to a system T of substitutions associated with groups of p^n letters (i.e., a primitive quotient group). Second, the substitutions of T were decomposed into substitutions "of the first and of the second species." This reduction was explicitly presented as modeled on Poinsot's reformulation of Gauss's decomposition. Each species of substitution corresponded to one of the two forms of representation of cycles. Their products generated linear substitutions; T was therefore what Jordan would designate later as a linear group.

Even though he acknowledged his method was not efficient for applications, Jordan claimed that in the aim of "studying the problem of the symmetry in itself, the method is not only more direct, it is also more natural and is actually the only way that leads to the true principles" [Jordan 1860, p. 4]. Jordan also highlighted the analogy between his method and the reduction of a helicoidal motion into motions of translation and rotation. This implicitly referred to Poinsot. Moreover, he eventually appealed to the legacies of Gauss and Abel to claim that what could be designated as the unscrewing of the method of reduction of groups was the "very essence" of the question:

> On pourrait voir une image de ce résultat dans le théorème de mécanique qui ramène le mouvement général d'un corps solide à un mouvement de translation combiné avec une rotation autour du centre de gravité. Ce principe du classement des lettres en divers groupes est le même dont Gauss et Abel ont déjà montré la fécondité dans la théorie des équations: il me semble être dans l'essence même de la question, et sert de fondement à toute mon analyse. [Jordan 1860, p.5].

The question of how Jordan accessed Poinsot's theory of order is open. But the echoes between Galois's decomposition and Jordan's early works might have been the consequence of a perspective on Gauss and Lagrange that Poinsot, Galois, and Jordan had shared. It was indeed in the framework of the reduction of imprimitive groups to primitive groups on the model of Gauss's decomposition that the general linear group had originated from the two forms of representations of cycles in the works of both Galois and Jordan.

Jordan first commented on Galois in the seven-page supplement he added to the memoir sent to the *Académie* for the *Grand Prix* of 1860 on the problem of the number of values of functions. He immediately focused on Galois's distinction between imprimitive and primitive equations.[40] More importantly, he began to consider later steps of reduction that would thus resort to other forms of decomposition (i.e., into normal subgroups).

In 1864, Jordan gave the exact opposite meanings to the terms "groups" and "systems" that they had had in the 1860 thesis. Groups were now substitution groups, while systems became partitions of letters. This change of attitude would go with the adoption of a symbolic representation for groups. But as will be seen later, Jordan did not attribute the move he (Jordan) had made from substitutions to groups to Galois. Moreover, the adoption of a symbolic representation was not intended to hide the parallel important use of the analytic representation of linear substitutions.

We have seen that Galois had made use of symbolic representations when he distinguished between the proper and improper decompositions of a group [Galois 1832, p. 408–409]. This situation has often been considered as an important step toward the consideration of groups as single abstract objects. But a completely opposite interpretation could be developed. It must indeed be pointed out that Galois's use of the symbolic notation was limited to the context of the description of the forms of decompositions of groups. As we have seen, Galois's (and later Jordan's) decompositions had been *modeled* on the analytic representations of two *special* forms of arithmetic actions of cycles, like an addition or a product. Galois's symbolic operations on the groups can therefore be understood in analogy with actual arithmetic operations. Resorting to such an analogy was quite normal in the 1820s-1830s. Gauss had already appealed to it to represent symbolically quadratic forms in 1801. Several works had developed the analogies between iterated arithmetic powers and the symbolic law of exponents of differential operators ([Koppelman 1971], [Durand-Richard 1996]). Moreover, Fourier and Cauchy had developed symbolic approaches to linear differential equations in the 1820s in analogy with algebraic equations [Dahan-Dalmedico 1992, p. 197].

In Galois's decomposition or Jordan's reduction, special analytic representations were not opposed to general symbolic presentations; each was both a special model case for, and an application of, the other.

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40 Jordan had aimed at providing a new proof that solvable primitive equations have a degree of a power of a prime but the proof failed because groups had been partitioned on the model of blocks of imprimitivity [Neumann 2006, p. 413].

1.3. *An overview on the long term circulation of Galois's applications*

Apart from Enrico Betti's and Jordan's systematic comments on Galois's works,[41] the three applications were not usually presented together in the framework of a comprehensive theory. This section aims at providing a brief overview of the different lines of development in which the applications were involved.

1.3.1. *The* criterion

In the *Avertissement* to the 1846 edition of Galois's works, Liouville had claimed that Galois had laid the grounds for a "general" theory of the solvability of equations by radicals. It is well known that he did not comment further on the content of such a general theory. But Liouville had nevertheless celebrated "Galois's method" through its "particular" use for the proof of the *criterion.* The presentation of the *criterion* as a particular application of a general theory of equations dominated public discourse on Galois's works until the mid-1890s (at which point it disappeared). Liouville's presentation of Galois was in fact reproduced word for word in publications targeting larger audiences than specialized mathematical journals, e.g. the 1848 biography of Galois in the *Magasin encyclopédique* or the many notices that would be published in several encyclopedic dictionaries.

But the citation of Liouville citing Galois could also be found in Serret's *Cours d'algèbre supérieure.* Despite the fact that the first edition of 1849 had made almost no use of Galois's works, its introduction presented Galois's *criterion* as the endpoint of a *longue durée* history of the "theory of equations" involving Cardano, Lagrange, Ruffini, and Abel among others [Serret 1849, p. 1–4]. In 1854, Serret's second edition included two additional notes relative to the criterion. The first consisted of a translation of [Kronecker 1853] involving a discussion on Galois's theorem with regard to Abel's approach. The second was a new proof of the *criterion* by Hermite.

Serret would include a presentation of Galois's general theory of equations in the third edition of his *Cours* in 1866. The criterion would then be presented as the conclusion of the theory. Apart from Jordan's *Traité* and Klein's *Icosahedron,* the solvable prime degree "Galois equations" or "metacyclic equations" would conclude most presentations of Galois theory until the turn of the century, e.g. [Netto 1882, p. 278], [Bolza 1891], [Borel &

41 On Betti's work, see [Ehrhardt 2007, p. 271–285], [Mammone 1989], [Wussing 1984, p. 123–128], [Kiernan 1971, p. 103–110].

Drach 1895, p. 334], [Vogt 1895, p. 188], [Weber 1895–1896, p. 597, 648], [Picard 1896, p. 481], [Pierpont 1900].

1.3.2. *Modular equations and* the *three Galois groups*

Unlike textbooks, papers published in specialized journals rarely referred to the *criterion.* When Hermite first referred to Galois publicly in 1851, he already expressed his interest in the cases in which the degrees of the modular equations could be reduced, a problem Betti would investigate in 1853 (cf. C. Goldstein's paper in this volume). In 1858–1859, Hermite would appeal to Galois's works again at the occasion of the series of papers he would devote to the modular equation of degree 5 and the general quintic.

From this point on, Galois's third application was usually referred to in connection to the works of "Galois-Betti-Hermite." This became one of the main types of reference to Galois in the second half of the century, both in periodical publications and in treatises such as [Jordan 1870], [Briot & Bouquet 1859], [Klein 1884], and [Klein & Fricke 1890] (as well as in other textbooks although implicitly as will be seen later). At the turn of the 1870s–1880s, the expression Galois groups was used in the Klein network for designating the groups associated to the three modular equations. Later on, at the Chicago congress of 1893, Joseph Perott still designated the group of order 660 of the modular equation of order 11 as the Galois group, while Moore aimed at generalizing the Galois groups by introducing abstract Galois fields.

1.3.3. Les imaginaires de Galois

We have seen that even though number-theoretic imaginaries were tightly linked to substitutions, Galois presented his *Note* as an autonomous topic in number theory. In the 1854 edition of Serret's *Algèbre,* Galois's imaginaries were presented as the conclusion of a series of three lectures devoted to the theory of congruences. Unlike the notion of primitive root of binomial congruences, they were not connected to cyclotomic (and abelian) equations or to the additional notes of Hermite and Kronecker.[42] Apart from the short note of [Allegret 1856a],[43] Galois imaginaries were not used again in connection with equations until the works of Jordan in the mid-1860s. When he started to develop his approach on higher

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[42] For a detailed account of the references to Galois's "Mémoire" in the various editions of Serret's *Algèbre,* see [Ehrhardt 2007, p. 358–392].

[43] To my knowledge, Alexandre Allégret's works have never been mentioned in connection to Galois before.

congruences in 1857, Dedekind alluded to both the presentations of Schönemann in the legacy of Gauss and Abel, and those of Serret in the legacy of Galois.

At the turn of the 1850s-1860s, Galois's imaginaries were used as a way to extend analytic forms of substitutions from p to p^n variables. Most texts actually dealt with binary linear fractional substitutions ([Serret 1859; 1866], [Mathieu 1860; 1861a;b]). Jordan would nevertheless investigate general linear substitutions on n variables, and Mathieu and Hermite would consider non-linear cases as will been seen in greater detail later.

In the third edition of the *Cours* in 1866, Serret went further, inscribing number-theoretic imaginaries in a comprehensive theory of congruences. The presentation included a development on integer polynomials modulo a "modular function" [Serret 1865; 1866].[44] Serret's approach on Galois imaginaries was endorsed later by treatises such as [Jordan 1870], [Borel & Drach 1895], and [Vogt 1895]. All these presentations nevertheless also systematically included Galois's original approach. Jordan in particular appealed to a traditional way of legitimizing the use of imaginaries by resorting to the analogies carried on by "instruments of computation." In 1867 he insisted that:

> La considération des racines imaginaires des congruences irréductibles s'introduit d'elle même dans mon analyse, qui n'aurait certainement pas abouti si j'avais hésité à l'adopter. Je serais heureux d'avoir contribué par ces exemples à montrer la puissance de ce nouvel instrument d'analyse, que d'éminents géomètres paraissent regarder encore avec une certaine défiance [Jordan 1867a, p. 269].

But Kronecker's 1882 *Grundzüge* contested the legitimacy of traditional presentations of irrationals such as algebraic imaginaries. Later on, when algebraic number theory came to incorporate the legacy of Dedekind, number-theoretic imaginaries were presented as a special case of *endlicher Körper*: the *Congruenzkörper* [Weber 1893b, p. 534]. As will be seen in the next section, a traditional perspective on Galois imaginaries nevertheless circulated from Serret and Jordan to [Gierster 1881], [Klein & Fricke 1890], [Moore 1893], and [Burnside 1894].

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44 This presentation was therefore close to the ones of Schönemann and Dedekind, which were nevertheless not mentioned by Serret.

1.3.4. *Jordan and the application to primitive equations of degree p^n*

The second application had had few echoes until the mid-1860s. Allégret had nevertheless published two notes in 1856 with the aim of generalizing Galois's *criterion* to equations of composite degree. Referring to the works of Kronecker, Betti, and Pierre-Laurent Wantzel, he had considered the "group of linear substitutions defined by Galois" in connection to congruences and cyclotomic equations. The interest Allégret had for groups was nevertheless not shared by most of the authors who dealt with substitutions at the time. As has been pointed out before, when they referred to Galois in the context of substitutions, most authors pointed to number-theoretic imaginaries and not to groups.[45]

In the first note related to Galois that Jordan addressed to the *Comptes rendus* in 1864, he reactivated the issue of the determination of solvable equations of prime power degree. His aim was to lay the emphasis on his "method" whose "essence" was to "reduce" a group into a "chain" of subgroups:

> Voici mon théorème fondamental:
> Pour qu'un groupe L appartienne à une équation résoluble par radicaux, il faut et il suffit qu'il soit le dernier terme d'une série de groupes partiels $I, F, G, H, \ldots$ jouissant des propriétés suivantes: 1° chacun de ces groupes contient le précédent; 2° ses substitutions sont échangeables entre elles, aux substitutions près du précédent; 3° toutes les substitutions de L lui sont permutables.
> L'essence de ma méthode consiste à déterminer successivement les groupes partiels $F, G, H, \ldots$ [Jordan 1864, p. 965].

Jordan argued that the problem had to be reduced further than Galois's two-step decomposition of solvable transitive equations to primitive equations with linear substitutions. One thus had to devote specific attention to linear substitutions:

> Si l'on distingue les racines les unes des autres par n indices indépendants $x, x', x,'' \ldots$, variables chacun de 0 à $p-1$, le groupe [of primitive equations of degree p^n] dérive de la combinaison de deux sortes de substitutions:
> 1° Celles qui remplacent la racine générale $a_{x,x',x,''\ldots}$ par la suivante
>
> $$a_{x+\alpha \bmod p, x'+\alpha' \bmod p, x''+\alpha'' \bmod p, \ldots}$$

45 It may be added that most authors who were working on the theory of equations at the time did not mention Galois either. For instance, Mathieu referred to Galois (imaginaries) for the first time in 1859 after he had already published a memoir on algebraic equations (1856) and three notes on multiply transitive substitutions acting on many valued functions (1858) [Ehrhardt 2007, p. 406].

$\alpha, \alpha', \alpha''$ étant des entiers variables d'une substitutions à l'autre.

2° Un certain nombre de substitutions remplaçant $a_{x,x',x,''\ldots}$ par

$$a_{ax+bx'+cx,''\ldots \bmod p,\ a'x+b'x'+c'x,''\ldots \bmod p,\ a''x+b''x'+c''x,''\ldots \bmod p,\ldots}$$

[...]. J'étudie à part ces dernières substitutions que je désigne par la notation suivante:

$$\left| \begin{array}{ll} x & ax + bx' + cx'' + \cdots \\ x' & a'x + b'x' + c'x'' + \cdots \\ x'' & a''x + b''x' + c''x'' + \cdots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right|$$

[Jordan 1864, p. 964].

In the following years, Jordan repeatedly pointed out the incorrectness of Galois's claim [1830b, p. 406] that the condition of linearity was sufficient for characterizing solvable primitive groups:

> Galois avait annoncé que les équations primitives et solubles par radicaux rentreraient dans un type unique, sauf pour le neuvième et le vingt-cinquième degré, qui présenteraient certains types exceptionnels. On voit par les énoncés qui précèdent qu'il faut prendre presque exactement le contre-pied de cette assertion [Jordan 1868a, p. 113].

Later on, most of *Livre II* dealt with the problem of characterization and classification of subgroups of $\mathrm{GL}_n(p)$, while most of *Livre IV* investigated the roles played by general linear groups in the chain reduction of solvable transitive groups. But the n-variable generality of Jordan's approach on linear groups was not taken on by most later presentations of the theory of substitutions. [Netto 1882], [Klein 1884], [Klein & Fricke 1890], [Bolza 1891], [Borel & Drach 1895], [Weber 1895–1896], [Picard 1896], [Pierpont 1900] focused rather on the binary linear and fractional linear substitutions associated with solvable equations of prime degree and with modular equations.

1.3.5. *Galois's theory of general equations*

Let us now consider the interactions between Galois's general principles and the three applications. After Betti's series of memoirs in the early 1850s, Galois's theory of general equations was presented for the most part in textbooks and treatises. To begin with, in the 1866 edition of Serret's *Algèbre,* the second volume was divided into three sections. The first was devoted to properties of integers and was dominated by the theory of congruences. The second presented the theory of substitutions. The third was devoted to the algebraic solution of equations. Its last chapter presented a commentary on Galois's *Mémoire.*

The theory of substitutions had three main interactions with reference to Galois. First, the notion of Galois resolvent connected substitutions to equations as will be seen in greater detail below.[46] Second, the problem of the analytic representation of substitutions connected number-theoretic imaginaries to substitutions as will be discussed in the next paragraph. Third, binary linear fractional substitutions were the only special type of substitutions to be considered. This focus implicitly pointed to modular equations. But the *Cours* considered neither Galois's application to primitive equations nor general linear groups. Until the mid 1890s, and apart from Jordan's *Traité*, most later treatises would follow both the separation between substitutions and equations and the above list of points of contact (e.g. [Netto 1882], [Bolza 1891], [Borel & Drach 1895], [Vogt 1895]).

This situation can be understood in the light of one of the main roots of the theory of substitutions, i.e., the problem of the number of values of functions. Interest in the latter problem had originated in the 18th century, when the solvability by radicals of an nth degree equation had been connected to the number of values a resolvent function of n variables could take. But even though such issues had originally been closely related to equations, they would give rise to autonomous developments on *substitutions*, *permutations* and *arrangements*, such as in [Cauchy 1815a;b; 1844]. Joseph Bertrand, (1845) and Serret (1849), in particular, stated limits for the number of values of functions. At the turn of the 1850s-1860s, substitutions themselves would become the main focus of the series of papers published by authors such as Mathieu, Jordan, or Thomas Kirkman [Ehrhardt 2007, p. 291–393].

The main point of contact with Galois's *Mémoire* was also the unique aspect of the latter Serret had already presented in the first edition of his *Algèbre* in 1849. There, it was designated as "Galois's equation in V." This designation would be used again by Hermite in 1851 and 1858–1859 and later by Jordan. It would eventually be replaced by the designation of "Galois resolvent" Betti had used in 1852 on the model of the already traditional Lagrangian resolvent. In most presentations of the second half of the 19th century, the notion of Galois resolvent was used for associating equations to systems of substitutions or groups. On the contrary, Jordan's

46 A Galois resolvent is a function of n variables that takes $n!$ values under the action of Sym (n). In the sense of Lagrange's 1770 notion of similar function, a Galois resolvent is similar to any rational function of the roots of an equation of degree n and of its coefficients (recall that the roots are supposed to be distinct).

Livre III would hide the resolvent in the proof of Galois's fundamental theorem:[47]

> THÉORÈME FONDAMENTAL. — Théorème I. Soit $F(x) = 0$ une équation dont les racines $x_1, \dots, x_m$ sont toutes inégales, et à laquelle on peut supposer qu'on ait adjoint certaines quantités auxiliaires $y, z, \dots$ Il existera toujours entre les racines $x_1, \dots, x_m$ un groupe de substitutions tel que toute fonction des racines, dont les substitutions de ce groupe n'altèrent pas la valeur numérique, soit rationnellement exprimable, et réciproquement. [Jordan 1870, p. 257].

Let F be an irreducible equation of degree n with simple roots $x_1, \dots, x_n$. A Galois resolvent is a function Ψ that takes exactly $n!$ distinct values under the action of all the substitutions leaving the roots (x_i) globally invariant [Galois 1846, p. 419]. All the roots can then be expressed as rational functions of Ψ [Galois 1846, p. 420].[48] Consider $u_1, \dots, u_n$ "general" values so that $u_1x_1 + \cdots + u_nx_n$ takes $n!$ distinct values $\Psi_1, \dots, \Psi_{n!}$ by the action of $\mathrm{Sym}(n)$. Then $\Psi(v) = (v - \Psi_1)(v - \Psi_n) \cdots (v - \Psi_{n!})$ is a Galois resolvent of the equation F. Now, consider a maximal irreducible factor $\varphi(v)$ of $\Psi(v)$ and the group G leaving φ globally invariant; then "adjoin" a rational function of the roots $\varphi_1(x_1, \dots, x_n)$ so that the resolvent Ψ would break up into smaller degrees. The substitutions of G leaving φ_1 globally invariant form a normal subgroup of G corresponding to the reduction of the equation by the adjunction of φ_1 [Galois 1846, p. 424].

Despite Jordan's emphasis on the notion of group of an equation, the notion of Galois resolvent continued to be widely used in the following decades. As will be seen in greater detail later, Kronecker's reworking of the notion of resolvent on a rational domain laid the ground for most presentations of Galois groups of equations in the 1890s, such as the ones of [Bolza 1891], [Borel & Drach 1895], [Picard 1896], [Vogt 1895], [Hölder 1899], [Pierpont 1900].

Kronecker's legacy was nevertheless challenged in the mid-1890s. In 1893, Oskar Bolza openly discussed the respective merits of Jordan's and Kronecker's definitions of Galois groups, while [Weber 1893b] and [Hilbert 1894] appealed to the legacy of Dedekind. In Weber's 1895 *Lehrbuch der Algebra,* Galois theory resorted to the consideration of how

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47 Compare to [Galois 1846, p. 421, trans. Van der Waerden, 1985, p. 107]:
PROPOSITION 1. — There is a group of permutations of the letters $a, b, c, \dots$ such that every function of the roots, invariable under the substitutions of the group is rationally known.
Conversely, every function of the roots rationally known is invariable under the group.

48 The theorem of the primitive element gives another formulation to this. See [Edwards 1984, p. 44–45], [van der Waerden 1985, p. 106].

"Körper" could be obtained one from another by the adjunction of a quantity, while the the group which left the initital Körper invariant was divided into normal subgroups. But the constructive character of the Galois resolvent would nevertheless still play an important role [Weber 1895–1896, p. 467].

1.4. *The problem of the analytic representation of substitutions*

The analytic representation of substitutions had circulated with both Galois's applications and general principles. Even in Weber's *Lehrbuch* the two analytic forms of cycles still played a key role in the conclusion of the presentation of Galois theory (i.e., the criterion of solvability for prime degree metacyclic equations) [Weber 1895–1896, p. 637].

In 1852, Betti had begun his commentary on Galois with representing substitutions $(x_i \quad x_{\varphi(i)})$ by a bijective function φ (with the indices i either integer mod p or Galois imaginaries). Finding all the possible expressions of such functions was later identified as the problem of the analytic representation of substitutions. Betti had nevertheless not given any specific expression to φ until he had discussed Galois's notion of decomposition of a group. Following Galois in 1830 and preceding Jordan in 1860, Betti had then raised the issue of determining the "maximal multiplier of a group," i.e., the last step of a decomposition [Betti 1852, p. 45]. For the case of a prime number of letters, this group was generated by $(i \quad i+1)$ or $(i \quad gi)$ while the composition of both forms generated the substitutions $(i \quad ai+b)$. Following Galois, Betti also extended his investigations to groups of prime power order and therefore to n-ary linear substitutions.

In comparison to Cauchy's 1844–1846 papers on substitutions,[49] the focus on the decomposition of analytic representations was specific to works which referred to Galois, such as Betti's or [Hermite 1851]. The latter paper succeeded Puiseux's 1850 "Recherches sur les fonctions algébriques." In both the framework of Cauchy's complex analysis and of Hermite's 1844 investigations of the division equation of abelian functions, Puiseux

49 See [Dahan-Dalmedico 1980, p. 296–310]. Even though in 1844 Cauchy also presented the two forms of analytic representation of cycles, which he had called "arithmetic and geometric substitutions" [Cauchy 1844, p. 178], he had favored other modes of representations of substitutions such as products of cycles, the two lines notation, the symbolic notation, and some tabular representations. Moreover, unlike Betti's decompositions, he composed the "conjugated system" of substitutions by the two forms of cycles (i.e., the linear group) but for the sole purpose of computing its order and with no interest in the analytic form of the resulting substitutions.

had considered algebraic functions $f(z, w) = 0$ on the complex plane.[50] He had shown that in the neighborhood of any point z_0 which is not a branch point, the roots $w_1, w_2, \ldots, w_n$ can be expanded as convergent power series in $z - z_0$. If one makes z move on a closed circuit avoiding the branch points, the roots are permuted by a system of substitutions, which Puiseux had investigated by appealing to Cauchy's representation of cycles [Puiseux 1850, p. 384]. In 1851, Hermite responded to Puiseux by representing analytically the substitutions involved for stating a criterion of the solvability for equations with parameters, analogous to Galois's criterion. Hermite indeed stated that for equations of a prime degree, "the necessary and sufficient condition for the solvability by radicals is that all the functions of the roots invariant for the substitutions of the following special form $\binom{u_k}{u_{ak+b}}$ [...] are rationally known" [Hermite 1851, p. 461]. The proof was based on the decomposition of the above form into products of $\binom{u_k}{u_{k+1}}$ and $\binom{u_k}{u_{\rho k}}$, with ρ a primitive root mod p.

When Hermite referred to Galois again in 1858–1859, he appealed to the analytic representation of linear fractional substitutions. The reduction of the degree of modular equations, as he said, "depends on a deeper investigation of the substitutions $\frac{ai+b}{ci+d}$" [Hermite 1859, p. 58]. Following Hermite, Serret and Mathieu considered linear fractional substitutions with number-theoretic imaginaries as variables in 1859.

In 1863, Hermite eventually stated a necessary and sufficient condition for an analytic function to represent a substitution on p^n letters. Moreover, he systematically stated all the reduced forms of analytic expressions of substitutions on 3, 5, and 7 letters. This approach laid the groundwork for most later presentations of the problem until the turn of the century (e.g. [Serret 1866, p. 383], [Jordan 1870, p.88], [Netto 1882, p. 140], [Borel & Drach 1895, p. 306], [Dickson 1901, p. 59]).[51]

Given a substitution S operating on $p-1$ letters $l_1, l_2, \ldots, l_{p-1}$, the problem is to find an analytic function φ of which a finite number of values correspond to the values of S, i.e., $S(l_z) = l_{\varphi(z)}$. Such a function can be found by the use of Lagrange's interpolation formula. Given two functions φ and Ψ of degree $p - 1$ associated to two substitutions S and T on $p - 1$ letters, the substitution ST is then associated with the function $\varphi\Psi$. Now, in order to keep the degree of $\varphi\Psi$ equal to $p - 1$, it is necessary to consider both the indices of the $p - 1$ letters and the coefficients of φ and Ψ modulo p.

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50 The polynomial $f(z, w)$ in w is irreducible in the field of rational functions of z.

51 In the introduction of his 1882 thesis, Edmond Maillet attributed to Hermite the introduction of the analytical notation $(x_i \quad x_{\varphi(i)})$ itself [Maillet 1892, p. 2].

In case of substitutions acting on p^n letters, one has to consider Galois's number-theoretic imaginaries.

Most treatises presented the problem of the analytic representation of substitutions just before they introduced linear substitutions (i.e., the form generated by the two forms of cycles). Moreover, this presentation usually played the role of an intermediary between substitutions and equations. Most authors therefore limited themselves to the considerations of the substitutions $(i \quad ai+b)$ and $\frac{\alpha i+\beta}{\gamma i+\delta}$ related to solvable equations of prime degree and to modular equations.

But Jordan's *Traité* assigned a different role to the analytic representation of substitutions. This problem gave rise to the investigations on the general linear group, which was presented as originating from direct products of cyclic groups as in Galois's and Betti's approaches [Jordan 1870, p. 88–92]. On this occasion, Jordan insisted that, in modern parlance, $\mathrm{GF}(p^n)$ could either be represented as $F_p(j)$ with j a root of $x^{p^{n-1}} \equiv 0$ or as a direct product of copies of F_p (i.e., as a vector space over F_p). In the first case, $\mathrm{GF}(p^n)$ is immediately associated to a multiplicative cyclic group generated by the substitution (i, ji). In the second case, it is associated to an additive cyclic group generated by $(i^{(k)}, i^{(k)}+1)$, i.e., $|i, i', \ldots, i+\alpha, i'+\alpha', \ldots|$. The linear group $\mathrm{GL}_n(p)$ then originates from the problem of finding the maximal group in which the above group is a normal subgroup. Its substitutions were shown to have the form $|i, i', \ldots, ai+bi'+\cdots, a'i+b'i'+\cdots|$.

The higher level of generality of Jordan's groups of n-ary substitutions was nevertheless problematic. A "general" development was indeed supposed to be valid for *all* the objects under consideration, such as in Cauchy's theory of substitutions. But Jordan's n-ary linear substitutions did not provide any *general* solution to the problem of the number of values of functions for which they had been introduced. In a sense, they had missed their *object.* As was alluded to above, Hermite approached the problem very differently in 1863 when he both stated a truly *general* result and investigated systematically some *special* cases.[52] As Jules Drach would point it out in 1895, unlike the binary fractional linear substitutions associated to Galois's first and third applications, the application to equations of composite degree pointed to substitutions that cannot be represented analytically in general:

52 For instance, for $p=5$, the reduced forms of analytic substitutions were x, x^2, x^3+ax.

> L'étude des equations de degré composé est beaucoup plus difficile, car la representation analytique des substitutions, qui nous a été si utile, devient moins simple; on est en effet obligé dans le cas le plus simple où le degré de l'équation est une puissance p^{ν} d'un nombre premier, soit de prendre pour indices les p^{ν} valeurs incongrues d'une imaginaire de Galois [...] soit d'affecter chaque racine de ν indices, dont chacun prend p valeurs incongrues (mod p). [...] Aussi n'entrerons-nous pas dans cette etude, nous contentant de renvoyer aux savantes recherches de M. Jordan. [Borel & Drach 1895, p. 311].

In the 1870s, Jordan's general linear groups were explicitly criticized by Kronecker for their false generality and formal nature [Brechenmacher 2007a]. Indeed, Kronecker accused Jordan of having mixed up tools relative to the orientation he had given to his investigations (i.e., n-ary linear substitutions) with the inherent significations of "objects of investigation." In the introduction of his 1882 treatise on substitutions, Eugen Netto, a former student of Kronecker, insisted that:

> It is unquestionable that the sphere of application of an Algorithm is extended by eliminating from its fundamental principles and its general structure all matters and suppositions not absolutely essential to its nature, and that through the general character of the objects with which it deals, the possibility of its employment in the most varied directions is secured. [...] If, on the other hand, it is a question of the application of an auxiliary method to a definitely prescribed and limited problem, the elaboration of the method will have to take into account only this one purpose. The exclusion of all superfluous elements and the increased usefulness of the method is a sufficient compensation for the lacking, but not defective, generality. A greater efficiency is attached in a smaller sphere of action. [Netto 1882, transl. Cole in Netto, 1892, p. iii].

In contrast with cyclic, abelian, metacyclic, and modular groups, linear groups were not considered by Netto as a special type of group: they were limited to the object of investigation of the problem of the analytic representation of substitutions [Netto 1882, p. 128–139].

Before analyzing further the specificity of Jordan's general linear groups, it is therefore customary to investigate first the status taken on by substitutions in the 1850s-1860s.

1.5. *The general, the special, and the status of substitutions*

It must first be pointed out that while Jordan's focus on n-ary linear groups was specific, it was on the contrary normal in the mid-19th century to deal with n-ary linear substitutions as *processes* of changes of variables. But the generality of the approach had to be adapted to its object.

Although "substitution" was a generic expression used for designating various procedures of transformations of variables, special types of substitutions were nevertheless related to special *objects* of investigations. Such objects could be quite complex and were usually not limited to a single notion. Let us consider the example of the "equation of the secular inequalities in planetary theory" [Brechenmacher 2007b]. Most of the many texts that were published on this topic were not concerned at all with celestial mechanics. Their object of investigation was on the contrary identified by the reference to the special equation. First, this reference delimitated a corpus of previous works mostly published in periodical journals, i.e., a network of texts. Second it pointed to problems shared throughout the network, such as proving that the roots are real or dealing with multiple roots. Third, the reference to the special equation was the mode of circulation of a practice that involved n-ary linear orthogonal substitutions with real coefficients.

In the 18th century, the special equation had originated from investigations on the small oscillations of a string loaded with n weights or of n planets on their orbits. The mathematisation of these problems involved symmetric linear systems of n differential equations with constant coefficient. Specific processes had been developed for "transforming" such systems into an integrable form (i.e., a diagonal matrix). These processes were eventually identified with the special equation they involved (the characteristic equation of the system) and had then circulated with the latter equation between various domains of the mathematical sciences, e.g. differential equations, celestial mechanics, analytic geometry, quadratic forms, the theory of invariants etc.

As an object of investigation, the "equation of the secular etc." was attached to a certain form of generality in which it was legitimate to deal with n variables. For instance, Cauchy had appealed to it in 1830 for generalizing to n-ary quadratic forms his geometric method for finding the principal axis of conics or quadrics. Later on in the 1850s, Hermite generalized Cauchy's approach to n-ary Hermitian forms. The homogenous expression Weierstrass gave in 1858 to the roots of this equation whatever their multiplicity were later celebrated by Kronecker as a model of "true generality" [Brechenmacher 201?a].

But in contrast to the general linear substitutions associated with this equation, the problem with Jordan's n-ary linear group was that its form of generality was neither truly general nor adapted to contemporary works on the number of values of functions or on the solvability of equations. The latter issue was indeed considered by Hermite and Kronecker in the

1850s as related to the *special* irrationalities defined by general equations of a given degree. The issue was thus of a different nature than the *general* properties of equations of degree n that were given by invariant theory, i.e., the investigations of n-ary algebraic forms that "reproduced themselves" by linear transformations that had been especially developed by Cayley and Sylvester.[53]

The series of papers Hermite published in 1858–1859 is discussed by C. Goldstein in this volume. Hermite considered that the impossibility of solving the general quintic by radicals raised the issue of expressing the roots by "uniform and distinct functions" of auxiliary variables. Because the roots of the modular equations were associated with substitutions having the analytic form $\frac{ai+b}{ci+d}$, they could be used to express analytically the roots of the general quartic or quintic by elliptic functions.

The aim, Hermite insisted, was to develop some knowledge about the "special nature" of the quantities defined by the general quintic in Jerrard's reduced form $x^5 - x - a = 0$, to "grasp what is proper and essential to the mode of existence of these quantities about which all we know is that they are not expressed by radicals" [Hermite 1859, p. 1]. This *special* nature was discussed in connection to the *general* framework of invariant theory.[54] The invariants, Hermite argued, provide the "elements that characterise the essential properties of the roots of algebraic equations, i.e., those that remain the same through the various transformations" [Hermite 1859, p. 19]. Hermite especially mentioned the example of Sturm's theorem. Sylvester had shown that the number of distinct roots of the "equation of the secular inequalities of planetary theory" was given by the invariant—the "inertia"—of a quadratic form for n-ary real linear substitutions [Sinaceur 1991, p. 124–132]. Hermite then introduced a distinction between two classes of equivalences of quadratic forms: while the "arithmetic equivalence" referred to the traditional number-theoretic context of a quadratic form with integer coefficients, n-ary linear substitutions with real coefficients had been associated to the "algebraic equivalence" of forms in reference to the algebraic nature of Sturm's theorem.

But in 1858–1859, Hermite showed that the investigation of general quartic and quintic equations made it compulsory to consider forms and substitutions of a *special* type. His analysis was based on the consideration of the discriminant, i.e., the algebraic form D given by the product of all

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53 See [Crilly 1986] and [Parshall 1989].

54 This problem was typically presented as a generalization of the solution of the general cubic by trigonometric functions. See [Jordan 1870, p. 366–378].

the differences of the roots of the equation. He first highlighted the role played by D in the "affinities" and "analogies" he had discovered between the modular equation of degree 3 and various other *special* quartics related to the ternary cubic forms involved in geometric problems of contact. In general, D is a two valued function under the action of substitutions on the roots (while D^2 is a symmetric function of the roots).[55] But Hermite pointed out that, in various special cases of quartics, D was a single-valued function, and was therefore a rational function of the coefficients. All these special equations could be transformed into a modular equation. As for the general quartic, the adjunction of a square root of D would split its 24th degree Galois resolvent into an equation of the 2nd degree and an equation of the 12th degree. The solutions of the quartic could thus be expressed as rational functions of $\sqrt{D}$ and of the roots of the modular equation of degree 3.

Hermite then generalized his approach to the general quintic. Unlike the groups Betti had used for reducing the degree of the modular equation of degree 5, Hermite appealed to the framework of Lagrange's "similar forms," i.e., forms invariant by substitutions [Hermite 1859, p. 62]. He proved Galois's claim that the functions invariant under the substitutions $\frac{ai+b}{ci+d}$ with $ad-bc \equiv 1$ provide rational non-symmetric functions of the roots of the reduced modular equation of degree 5. Moreover, he insisted that this property is a consequence of the fact that the Galois resolvent of the equation split into irreducible factors of degree less than $5! = 120$. Because this specificity was shared by the reduced modular equations of 5, 7, and 11, Hermite argued that these "equations with affects" constituted special "orders of irrationalities":[56]

> Les équations du septième et du onzième degré présentant cette propriété que les fonctions non symétriques de leurs racines invariables par les substitutions ainsi définies ont une valeur rationnelle, constituent un ordre spécial d'irrationnalité qui les distingue nettement des équations les plus générales de ces degrés. Ce sont, suivant l'espression de M. Kronecker, des équations

[55] Ruffini and Cauchy had shown that the number of values that a non-symmetric rational function of 5 variables attains cannot be lesser than 5 unless it is 2.

[56] The Galois group of the reduced modular equation of order 5 is $\mathrm{PSL}_2(5)$; it is isomorphic to $A(5)$, which can be understood as the reduction of $\mathrm{Sym}(5)$ by the adjunction of the discriminant of the general quintic. In short, an equation with affect is an equation whose Galois group is smaller than the symmetric group. The notion was introduced by Kronecker for equations of degree 7: while all equations of degree 5 can be solved by elliptic functions, an analytical solutions of equations of degree 7 is only possible in the case of equations with affects.

douées d'affections, et qu'il sera sans doute possible de ramener analytiquement à celles dont la théorie des fonctions analytiques a donné la première notion [Hermite 1859, p. 64].

Conclusion

Let us come to some conclusions about the two examples of substitutions given by the equation of the secular inequalities in planetary theory, and by modular equations. First, both were attached to two forms of analytic representations (i.e., n-ary linear substitutions with real coefficients and unimodular binary fractional linear substitutions (in F_p)). Second, both were associated to classes of forms (i.e., to algebraic and arithmetic equivalences respectively). Third, both left invariant some functions and forms (i.e., Sturm's functions and quadratic forms, elliptic functions and discriminants). Fourth, both were related to the nature of the roots of the attached special equations.

But unlike the *general* linear n-ary substitutions and algebraic forms involved in invariant theory, Hermite considered that the extension of the problem of the solvability by radicals to equations of degrees higher than four involved identifying special "orders of irrationalities" attached to *special* algebraic forms and special forms of substitutions. He thus concluded his investigations on the general quintic by listing all the possible forms of analytic representation of substitutions on 5 or 7 letters, i.e., the issue he later developed in his influential 1863 memoir.

On the contrary, the n-ary linear substitutions Jordan introduced in 1860 originated without a clear object of investigation. They provided neither special nor general solutions to the problem of the number of values of functions. Recall that Jordan placed the emphasis on the essential nature of the *relations* given by his method of reduction, not on specific objects. Galois's second application would nevertheless provide an object for Jordan's reduction of n-ary linear substitutions. As Jordan claimed in the introduction to the supplement to his thesis: "the definitions and the theorems given in the memoir [i.e., Jordan's thesis] are really getting into the heart of the matter, the distinctions I have introduced are truly fundamental" because they show "some essential property of the equation associated with the system of substitutions under consideration" [Jordan 1861, p. 187], i.e., that primitive solvable equations correspond to linear groups on p^n letters.

As will be seen in the two following sections of this paper, for decades Galois's legacy opposed two approaches which both aimed at reaching the "essence" of mathematics. On the one hand, some texts aimed at characterizing the *special* nature of *general* equations of a given degree. On the other hand, other texts focused on the *relations* between *classes* of solvable equations (or groups) of a *general* degree n. The two approaches were nevertheless both presented in Jordan's *Traité*. The first went with Jordan's specific practice of reduction. It structured the *Traité* in a complex chain of generalizations of special model cases. As we shall see in the next section, this approach had almost no circulation until it was developed in Dickson's network in the 1890s. The second approach was not specific to Jordan but was nevertheless presented in the synthesis of the *Traité*'s *Livre III*. As will be seen in the third section of this paper, it was either rejected or partially adopted in the 1870s-1880s in the Kronecker and Klein networks.

2. LINEAR GROUPS IN GALOIS FIELDS (1890–1900)

We now change perspective, and consider the collective dimensions of networks of texts that referred to Jordan and Galois after the publication of the *Traité*. As has been alluded to above, two images of Galois in the *Traité* have to be distinguished. We shall begin by considering the circulation of Jordan's Galois in the Dickson network. First, we shall discuss the identity of this group of texts in connection with other collective dimensions of larger scale. Second we shall question further the first Galois of the *Traité* from the perspective of some of its readers in the mid-1890s.

2.1. *A space of circulation of algebraic practices*

Recall that the designation 'Dickson's network' points to a group mainly constituted of texts that have been published between 1893–1907 by French and American authors. This group initially involved actors in Chicago (Moore, Dickson, Ida May Schottenfels, Joseph H. Wedderburn, William Bussey, Robert Börger) and in Paris (Jordan, Émile Borel and Jules Drach, Raymond Le Vavasseur, Jean-Armand de Séguier, Léon Autonne) but quickly extended to actors in Stanford (George A. Miller, William A. Manning, Hans Blichfeldt), and to other individuals such as William. L. Putnam, Edward V. Huntington, or Lewis Neikirk.[57]

The institutionalization of finite group theory is an important collective trend in which the Dickson network participated. Most texts were indeed

[57] A finer characterization of the Dickson network will be given in [Brechenmacher 2012a].

classified in the *Jahrbuch*'s subsection "substitutions and group theory." Moreover, reports were produced ([Miller 1898; 1902; 1907], [Dickson 1899]), monographs were published ([Burnside 1897], [Dickson 1901], [Séguier 1904a], [Le Vavasseur 1904]), and discussions were developed on issues related to the teaching and the history of finite groups [Ehrhardt 2007, p. 628–648].

Dickson's network originated between Otto Hölder's abstract formulation of the notion of quotient group [Hölder 1889a] and the emergence of group representation theory.[58] The determination of all groups of a given order was often proclaimed as a general goal. This question had already been presented as the "general problem" of substitutions in the third edition of Serret's *Algèbre* [Serret 1866, p. 283]. But the texts of the Dickson network usually pointed to the "abstract" formulation Cayley had given to the "general problem of groups" in the first volume of the *American Journal of Mathematics* [Cayley 1878, p. 50]. The use of the composition series of the Jordan-Hölder theorem potentially reduced the general problem to the one of the determination of all simple groups.

The latter problem was much related to the development of abstract group theory (i.e., groups defined by symbolic, and later axiomatic, operations). First, the use of the Jordan-Hölder theorem required the consideration of quotient groups that were not introduced by substitutions but by symbolic laws of operations [Nicholson 1993, p.81–85]. Second, Hölder's use of Sylow's theorems for determining simple groups of order less than 200 [Hölder 1892], or groups of orders p^3, pq^2, pqr, p^4 [Hölder 1893], was based on the identification of abstract groups up to isomorphism.[59] The identification of classes of simple groups therefore raised difficult issues related to the various concrete forms of representations of abstract groups such as substitutions groups or collineation groups [Silvestri 1979, p. 338].

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58 For a comparison between Jordan and Hölder's approaches on Jordan-Hölder theorem, see [Nicholson 1993] and [Corry 1996, p. 24–34].

59 In the mid-1860s, the use of Jordan's "method of reduction" of groups into composition series had raised representation issues [Jordan 1867b, p. 108]. The notion of isomorphism had been appropriated by Jordan from the framework of crystallography Scholz [1989] and had been presented as a general notion of the theory of substitutions [Jordan 1870, p. 56]. It would play a key role in the connections Klein would develop between various types of groups in the late 1870s and would become "abundant" in the 1890s [Frobenius 1895a, p. 168]. First, Hölder's introduction of abstract quotient groups would point to the isomorphism theorems [Frobenius 1895b]. Second, the actual composition of groups from factor-groups could not be undertaken unless all the automorphisms of the groups involved would be known [Hölder 1893, p. 313], [Hölder 1895a, p. 340].

In this context, the analytic representation of substitutions played an important role in Dickson's reading of the *Traité*. The latter's thesis was indeed entitled "The Analytic Representation of Substitutions on a Power of a Prime Number of Letters with a Discussion of the Linear Group." It provided a synthesis between the approaches of Hermite and Jordan to substitutions.

Hölder's approach to abstract groups was a shared reference in the Dickson network. In his paper of 1889, Hölder initially appealed to the abstract approach developed by [Dyck 1882] in the legacy of Cayley. A symbolic approach was nevertheless developed earlier in 1877–1878 by Frobenius, partly in the legacy of Cayley as well as [Hawkins 2008]. After Hölder eventually appealed to Frobenius's approach in 1892, the two mathematicians would publish a series of papers on topics closely related one to the other (Sylow theorem, composition series, solvable groups etc.). But unlike Hölder's, Frobenius's works did not become a shared reference in the Dickson network until 1901.

The variety of attitudes to Frobenius shows that the category of "abstract finite group theory" is not appropriate for identifying members of the Dickson network. For instance, Moore did not refer to Frobenius in the 1890s even though [Burnside 1896] pointed out that [Frobenius 1893] had already made use of the notion of the group of automorphisms of a group that [Moore 1894] had claimed to introduce. Several authors actually claimed independently to have abstractly identified the group of automorphisms of abelian groups of type $(1, 1, \dots, 1)$ (i.e., Frobenius, Hölder, Moore, Burnside, Le Vavasseur, and Miller), a problem that pointed to the traditional introduction of the general linear group in the legacies of Galois and Jordan as will be seen in greater detail later.

Moreover, that issues related to abstract groups circulated in Dickson's network did not imply a shared approach toward abstraction. Unlike Moore, other key authors followed Frobenius's works closely. But, on the one hand, Burnside's 1897 *Theory of groups of finite order* indicated the longstanding concerns for symbolic laws of combination which had circulated from Cambridge to other academic contexts in Great Britain and the United States. On the other hand, it was on Cantor's set theory that Séguier had grounded his 1904 *Théorie des groupes finis. Éléments de la théorie des groupes abstraits.*

National categories were no more relevant than theories for identifying the Dickson network. For instance, the works of the Americans Frank N. Cole and John W. Young were frequently referred to by Frobenius. Reciprocally, Miller appealed to Frobenius's works early on in the mid-1890s. In his first *Report on recent progress in the theory of the groups of finite order*, he put to the fore Frobenius's representation theory when he acknowledged the growing importance of linear groups [Miller 1898, p. 248]. But Dickson nevertheless mentioned Frobenius neither in his 1899 *Report on the recent progress in the theory of linear groups*, nor in his 1901 monograph. The situation did not change until 1901, when a review of Loewy criticized Dickson's restatement of some of Frobenius's results.

The category of local research school is not directly useful either. Most of the texts of the Dickson network were involved in the emerging "Chicago research school," whose role in the development of the "American mathematical research community" is analyzed in [Parshall & Rowe 1994, p. 261–455]. But while this school has been characterized by its "abstract and structural" approach to algebra, which was called a "characteristic of trendsetting German mathematics" [Parshall 2004, p. 264–265], Moore's Galois fields were collectively described as having given an "abstract form" to previous works by Serret, Jordan, and Mathieu. Moreover, a similar abstract formulation was often attributed to [Borel & Drach 1895].[60]

The growing importance of linear groups was another large-scale trend in the 1890s. Here, the important role many of the texts devoted to "Jordan's linear groups" was specific. In the early 1890s, linear groups usually designated the groups of binary or ternary unimodular fractional linear substitutions Klein and his followers had investigated (i.e., $PSL_2(p)$ and $PSL_3(p)$). Even though Klein's linear groups would still play an important role at the turn of the century [Wiman 1900], some collective interest in Jordan's *general* linear groups in Galois fields emerged by that time. A telling example is the adoption of the term "*special* linear groups" for designating what used to be "linear groups" in the early 1890s.

The problem of the status of n-ary linear groups was nevertheless not an issue any more. Among others, Weber's *Lehrbuch* introduced homogeneous linear groups of n variables by appealing to the analytic form of n-ary linear substitutions. But it nevertheless only stated a few general properties before focusing on special groups such as $PSL_2(p)$. On the contrary, Dickson's 1896 thesis followed Jordan in introducing $GL_n(p)$ as the maximal group in which an abelian group of type $(1, 1, \dots, 1)$ would be a normal subgroup. The second part of the thesis was then devoted to generalizations of Jordan's results from F_p to $GF(p^n)$.

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60 Compare for instance the introductions of [Dickson 1896] and [Schottenfels 1900].

The label linear groups was thus far from pointing to a unified category. Recall that various ways of dealing with linear substitutions had parallel circulations until the constitution of linear algebra as a discipline in the 1930s [Brechenmacher 2010]. Amongst these, the most influential approach was based on Frobenius's 1877–1879 presentation of the theory of bilinear forms. This approach appealed to symbolic methods and to computations of invariants by determinants such as Weierstrass's elementary divisors [Hawkins 1977]. It incorporated the notion of matrix in the 1890s, and played a key role in Frobenius's representation theory.

But the main protagonists of the Dickson network shared an alternative approach based on Jordan's reduction of a linear substitution to its canonical form.[61] This collective attitude has to be regarded as an important specific feature of the Dickson network. Jordan's canonical form theorem had indeed almost disappeared from the public scene since it had been strongly criticized by Kronecker a few years after it had been stated [Brechenmacher 2007a]. Kronecker not only rejected the formal generality of Jordan's linear groups, but also criticized the non-effectiveness of the canonical reduction, which required the determination of the roots of arbitrary algebraic equations. Moreover, Frobenius not only presented Jordan's canonical form as a corollary of Weierstrass's elementary divisor theorem, but also insisted that the validity of Jordan's form was limited to the case when one would allow the use of "irrationals" such as "Galois's imaginary numbers" [Frobenius 1879, p. 544]. In contrast, the reformulation Kronecker had given to Weierstrass's theorem in 1874 was based on a rational method of computations of invariants in any "domain of rationality" (i.e., the invariant factors of matrices in a principal ideal domain).

During the 1880s and 1890s, Jordan's canonical form had an underground circulation in the works of authors such as Poincaré or Élie Cartan, where it was neither considered as a theorem nor attributed to Jordan [Brechenmacher 2012b]. On the contrary, it circulated in plain sight at the turn of the century.[62] Much work would be devoted to making some procedures of matrix decomposition explicit that had never been considered as mathematical methods *per se* until then ([Burnside 1899], [Dickson 1901], [Autonne 1905], [Séguier 1907]). Moreover, Séguier and Dickson would both publicly challenge the traditional structure of

61 This theorem had been stated at first for linear substitutions in Galois fields [Jordan 1868a; 1870], and later for linear transformations (operating implicitly on an algebraically closed field) [Jordan 1870]. See [Brechenmacher 2006; 2007a].

62 See [Moore 1898], [Maschke 1898] for the case of periodic substitutions, and [Burnside 1899], [Dickson 1901], [Séguier 1902; 1907] for the general case.

the theory of bilinear forms ([Séguier 1907], [Dickson 1924/1928]). Later on in the 1930s, decompositions to canonical forms would lay the ground for expositions of the theory of matrices, such as the ones of Cyrus Colton Mac Duffee—a student of Dickson.

In a word, the Dickson network can be considered as a space of circulation of key algebraic practices of the *Traité*. The use of the terminology "practice" here aims at highlighting the fact that reducing a substitution to its canonical form was not limited to a computational process. Unlike the static nature of the invariants of the Frobenius theory, this approach was based on dynamic decompositions of the analytic representations of matrices (or "Tableaux" as the French used to say at the time). Moreover, Kronecker's criticisms of canonical forms in 1874 resorted to issues involving the nature of the essence of mathematics, which the latter had laid on the special *objects* of investigations of arithmetic (forms, especially) as opposed to the general *relations* shown by groups in algebra.

Jordan's canonical form did indeed embody the method of reduction we have seen to be specific to Jordan's Galois. It especially resorted to the unscrewing into the two forms of actions of cycles $(i \quad gi)$ and $(i \quad i+a)$ which it assimilated to issues involving n variables. The canonical form was first stated for reducing primitive groups on p^2 letters to composition series [Jordan 1868a], a problem to which the end of Galois's second memoir had been devoted [Galois 1846, p. 436–444]. It was then generalized to n variables in *Livre II* and was used in *Livre IV* for reducing n-ary linear groups on the *model* of the groups of order $n = p^2$. Later on, Jordan would appeal frequently to reductions of substitutions (in $GF(p^n)$ or $\mathbb{C}$) in his works on groups, differential equations, algebraic forms, etc.

The extent of the space of circulation of algebraic practices such as Jordan's was usually not directly the consequence of the efficiency of the underlying procedures.[63] Such spaces of circulation can actually be understood as shared algebraic cultures. But the collective dimension attached to such an algebraic culture is nevertheless difficult to identify precisely. On the one hand, there is no evidence that any sociological dimension lay behind the Dickson network. Consider for instance two authors such as de Séguier, a Jesuit abbot aristocrat, and Ida May Schottenfels, one of the first women to graduate in mathematics at Chicago [Fenster & Parshall 1990].

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63 For instance, Georg Scheffers [1891], despite his close reading of [Poincaré 1884], did not adopt the practice of reduction the latter had dealt with for investigating continuous groups (i.e., Lie groups and Lie algebras). This practice would nevertheless play a key role in Cartan's groundbreaking approach to Lie algebras as well as in Dickson's works on this topic.

On the other hand, the 1896 discussion between Miller and Le Vavasseur about groups of operations and Galois imaginaries is one amongst the many examples that point to existing spaces of circulations between actors who did not have key positions in the main centers of production of mathematics (even though mathematics would play a key role at Chicago, recall that the university had opened its doors in 1892).[64] It is therefore difficult to determine the respective roles played by shared perspectives on the *Traité* on the one hand, and preexisting spaces of circulation on the other hand.

In any case, specifying the Dickson network's space of circulation would require further investigations on algebra and number theory at the turn of the 20th century.[65] The question of the time-period during which the Dickson network functioned will also be left open in this paper. To begin with, the fact that the expression "champs de Galois" was used for a long time in France in parallel to the use of the term "corps fini" should be studied further.[66] Second, the linear algebraic identity of the network is associated with other developments over the course of 19th century. For instance, some of the procedures of decomposition underlying Jordan's canonical form circulated from Cauchy's "calcul des Tableaux" to Cambridge in the 1840s, were incorporated into Cayley and Sylvester's matrices in the 1850s, and circulated with matrices to the U.S.A. where they would meet again with the "Tableaux" in the Dickson network [Brechenmacher 2010].

In the next sections, I will focus on the following, narrower, question: how did Jordan's specific reference to Galois circulate to Dickson between

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64 In 1896, Le Vavasseur had published a note in which he had used Galois imaginaries to express the "group of isomorphisms of the finite abelian group generated by independent operators each of period a prime number p" (i.e., the linear group). This note had prompted a quick response by Miller and the discussion between the two went on with two other notes. As has been seen before, the issue at stake had a long background in the context of the problem of the number of values of functions; Burnside, Moore, and Dickson would also tackle it.

65 Very little is known about the complex situation in France [Goldstein 1999]. Such issues are at the core of the collective ANR project CaaFÉ.

66 After 1905 the intertextual relationships seemed to change as well as the topics studied. On the one hand, the use of the reference to Galois field would be more widely used in the U.S.A. On the other hand, the works of Dickson as well as the ones of Séguier would focus on the invariants of quadratic forms and their geometric interpretations.
The network seems nevertheless to have kept some identity on the long run as is illustrated by the fact that in the 1950s it was through the Americans A.H. Clifford and G.B. Preston that M. Teissier and P. Dubreil learned about the works of Séguier and how they were related to the theory of demi-groups they were developing [Dubreil 1981, p. 59].

1893 and 1896? The small-scale query under consideration will nevertheless have to be investigated in the perspective of the larger scale implied by the references to the works of Hermite, Mathieu, Serret, and Jordan in the 1850s-1860s.

2.2. *Every Galois field is a Galois field (1893)*

Let us now take a closer look at Moore's lecture at the 1893 *Mathematical Congress of the World's Columbian Exposition* [Parshall & Rowe 1994, p. 296–330]. The issues the author tackled were those associated with the continuation of the lists of simple groups that had been established successively by [Hölder 1892] up to order 200 and by [Cole 1892] up to the order 500.[67] For the purpose of continuing the list up to 600, Cole had put to the fore a simple group of order 504 (i.e., $PSL_2(2^3)$). Moore showed that Cole's group—as well as a simple group of order 360 he had introduced in 1892—belonged to a "new doubly infinite" system of simple groups (i.e., $PSL_2(p^n)$) [Parshall 2004, p. 264]. This new class was introduced as a generalization of the class of groups of unimodular binary linear fractional substitutions mod p (i.e., $PSL_2(p)$):[68]

$$i' = \frac{ai+b}{ci+d} \text{ with } ad - bc \equiv 1.$$

The extension of the system of indices from p elements to p^n elements was based on the introduction of the notion of a "field" as a "system of symbols" defined by "abstract operational identities" of addition and multiplication. Moore then pointed out that Galois had "discovered an important generalization" of the field of "incongruous classes" of integers mod p, i.e., the "Galois field" of incongruous classes of polynomials F modulo p and modulo a given irreducible integral polynomial f of degree n.

A first version of Moore's lecture was published in 1893. A second revised version would be published in 1896 in the proceedings of the congress. In the first version, Moore had noted that: "it should be remarked further that every field of order s is in fact abstractly considered as a Galois field of order s" [Moore 1893, p. 75]. Moore provided neither any proof nor any further details about this remark until the second version he completed in autumn 1895 [Moore 1896, p. 242]. But in the meantime, Burnside dealt in 1894 with exactly the same issue of the

67 Except for the orders 360 and 432 which Frobenius dealt with in 1893.

68 $PSL_2(p^n)$, p prime.

simplicity of $\mathrm{PSL}_2(p^n)$. Drach gave an abstract definition to "Galois imaginaries" [Borel & Drach 1895, p. 343–349]. Moreover, [Weber 1893b; 1895–1896] and [Hilbert 1894] claimed to lay new ground on Galois's theory of equations by appealing to Dedekind's concept of Körper.[69]

What was particular to Moore's congress paper? The focus of the first version was on the proof of the simplicity of $\mathrm{PSL}_2(p^n)$ on the model of the case of $\mathrm{PSL}_2(p)$ treated in [Klein & Fricke 1890, p. 419–450]. Alongside with [Hölder 1889a], the textbook of Klein and Fricke was actually the main bibliographic reference of Moore's paper. Not only had Cole authored the references to the simple groups investigated by Jordan [Moore 1893, p. 74], but most other references had been taken from the Klein-Fricke textbook.It is likely that Moore had not read [Gierster 1881] closely, and had not read [Serret 1859; 1866] at all. Moore even suggested that both mathematicians dealt only with the case $n = 1$, while in fact they used Galois imaginaries in some parts of their works [Moore 1893, p. 76].

Moreover, even though the relevant works of [Mathieu 1860, p. 38], [Mathieu 1861a, p. 261] on linear fractional substitutions and number-theoretic imaginaries were identified precisely by [Gierster 1881, p. 330], Moore did not mention Mathieu until 1895 when he would add a last-minute note to the revised version of his paper [Moore 1896, p. 242]. Mathieu had nevertheless investigated various aspects of $\mathrm{PSL}_2(p^n)$, and had already introduced Cole's group of order 504 (with no concern about the issue of simplicity).[70]

Had Moore built his 1893 lecture on the four pages Klein and Fricke had devoted to Galois imaginaries? Actually, his use of the expression Galois theory indicates that Moore had certainly read Jordan's *Livre I.* Moreover, the formulation he gave of Galois fields pointed to the extensive development of [Serret 1866]. Indeed, both [Klein & Fricke 1890] and [Jordan 1870] were faithful to Galois's original presentation in focusing on the fact that $\mathrm{GF}(p^n) \sim F_p(j)$. In contrast, and as has been seen in the previous section, Serret had developed an arithmetic approach to $\mathrm{GF}(p^n)$ as $F_p(X)/(f(x))$.

69 On Galois theory in Weber's *Lehrbuch*, see [Kiernan 1971, p. 137–141]. For a comparative study with [Weber 1893b], see [Corry 1996, p. 34–45].

70 In 1861, Mathieu had used the threefold transitive group $\mathrm{PSL}_2(p^n)$ for introducing a five fold transitive group on 12 letters. He had also announced the existence of a five fold transitive group on 24 letters which he would eventually introduce in 1873 (i.e., the Mathieu groups M_{12} and M_{24}).

Galois's (or Jordan's or Klein's and Fricke's) presentation was the one that was actually helpful for the group-theoretical purpose of Moore's paper. But Moore turned Galois upside down. On the one hand, what he designated as a Galois field was Serret's concrete function field representation. On the other hand, the notion of abstract field was actually close to Galois's initial presentation. The statement that a finite field can be abstractly considered as a Galois field actually expressed the identity between two points of views on the same object, as had already been done by [Serret 1866, p. 179–181]. It was quite close to stating that every Galois field (in the sense of Galois) is the abstract form of a Galois field (in the sense of Serret).

The relation of abstract fields to number-theoretic imaginaries was analogous to the relation between classes of abstract simple groups and the representation of a given simple group. On the one hand, because irreducible polynomials mod p "do exist" as Moore claimed, Serret's approach provided a construction of a field of p^n elements [Moore 1893, p. 75], i.e., "an existence proof" of the abstract field [Moore 1896, p. 212]. On the other hand, the notion of abstract field was a normal interpretation of Galois's 1830 *Note* in the context of the considerations on the symbolic laws of complex numbers and associative algebras that had been developed since the 1870s ([Hawkins 1971, p. 244–256], [Parshall 1989, p. 226–261]).[71] Klein-Fricke had indeed considered Galois imaginaries as complex numbers $aj^{n-1} + bj^{n-2} + \cdots + 1$. Commutative systems of hypercomplex numbers were typically investigated by the consideration of the minimal polynomial of the system. In the case of finite fields, the minimal polynomial was of the form $x^{p^{n-1}} - 1 \equiv 0$ as Moore would prove in 1896.

In short, Moore had stated that finite abstract fields can be represented as function fields. In contrast, this statement had no relation with Galois fields in the sense of field extensions and Galois groups. In the framework of Weber's presentation of Dedekind's Galois, Moore's "Galois fields" were both "Endlicher Körper" and "Congruenz Körper" but they were not "Galois'sche Körper."

Recall that the introduction of abstract Galois fields was not the initial aim of the 1893 lecture. But as a result, Moore nevertheless established a direct relation between [Serret 1866] and [Klein & Fricke 1890]. In doing so, he had jumped over more than twenty years of the development

71 In the tradition of investigations on associative algebras, [Borel & Drach 1895, p. 343–350] and [Moore 1895a] both provided tables of compositions of number-theoretic imaginaries.

of mathematics. In the framework of Kronecker's 1882 *Grundzüge*, Serret's "fonctions modulaires" should have been understood as "modular systems" on "domains of rationality" [Goldstein & Schappacher 2007b, p. 83]. For instance, [Hölder 1893] resorted to Kronecker's framework to formulate Galois imaginaries. But Jordan seems to have been an alternative reference to Kronecker in the 1890s as is illustrated by the parallel evolution of the works of Moore (and later Dickson) and Burnside who would both consider successively $\mathrm{PSL}_2(p)$, $\mathrm{PSL}_3(p)$, $\mathrm{PSL}_m(p)$, and eventually $\mathrm{GL}_n(p)$.

But unlike Burnside who mastered the relevant references to the works of Joseph Gierster, Serret, Jordan, and Mathieu, Moore seems to have been lost in a fog of old French works. What Mathieu had done exactly on Galois fields was especially problematic.[72] When Moore referred to the latter for the first time, he promised he would devote a subsequent paper to "point[ing] out the exact point of contact [of his works] with Mathieu's results" [Moore 1895a, p. 38]. But no such paper was ever published and Moore eventually settled for the addition of a short allusive note to the revised edition of the congress paper.

As will be seen in greater detail later, the main problem of Dickson' s thesis was to specify the relations between the works of Jordan and Mathieu on linear groups in Galois fields. Dickson's close reading of Jordan's *Traité* thus eventually resulted from the investigation of the collective dimensions Moore's Galois fields had accidentally bumped into in 1893.[73] We have seen that Moore had indeed resorted to Klein's mediation of a longstanding French tradition. But as for the circulation of either linear fractional substitutions or number-theoretic imaginaries, Serret's *Cours* played a much more important role than Jordan's *Traité*. Moreover, Moore and Dickson initially appealed to the *Traité* as an element of continuity between the works of Mathieu and those of Klein and his followers. This situation provides the opportunity to question further how specific the Jordan-Galois relation had been considered to be, in the context of the institutionalization of group theory.

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72 The 1893 version of the congress lecture was supposed to be followed by a more complete publication in *Mathematische Annalen*. But this did not happen and Moore published instead a paper on triple systems.

73 One of the main results of Dickson's thesis was to generalize Moore's doubly-infinite system of simple groups to the triply-infinite system $\mathrm{SL}_m(p^n)/Z$ with Z the center of $\mathrm{SL}_m(p^n)$ and $(m,n,p) \neq (2,1,2)$ or $(2,1,3)$ [Parshall 2004, p. 265].

2.3. *Jordan's Galois in a continuous line of development*

It must first be pointed out that the crucial role Hölder gave to composition series had not resorted to the specific features of Jordan's Galois. Factor groups had been presented as fundamental notions of "pure group theory," i.e., as autonomous from their "applications" to the "algebra" of equations [Hölder 1889a, p. 28]. Such a distinction was actually traditional. Apart from Jordan's *Traité*, a clear-cut separation between substitutions and equations had indeed structured most presentations of the theory of substitutions since those of Betti and Serret.

Hölder thus presented Jordan's composition series as an element of continuity between the longterm development of the theory of substitutions and the works of [Kronecker 1882], [Netto 1882], [Klein 1884], and [Dyck 1882]. Moreover, he actually appealed to the relation Jordan-Galois to reorganize lines that had diverged in the 1880s. Hölder began his paper by pointing to those aspects of the works of Kronecker or Klein to which he would appeal for the legacies of Abel and Jordan. Pure group theory was developed by following Walther von Dyck or Klein. Galois theory was based on Kronecker's *Grundzüge*.

In the 1890s, a recurrent discourse praised the "unity" and the "unifying force" of group theory. On the one hand, the landscape nevertheless remained fragmented, as is illustrated by the various approaches to groups presented at the 1893 Chicago congress [Parshall & Rowe 1994, p. 309–331]. But on the other hand, the process of institutionalization of finite group theory gave the actors a free hand in dealing with the legacies of prominent authors of the 1880s such as Kronecker, Klein, and Lie.[74] Shared references concerning the roles of earlier actors such as Jordan and Galois were adopted on a global scale: to Galois was attributed the distinction between simple and compound groups, to Jordan the generalization to composition series. Even Frobenius, who had rarely referred to Galois until then, eventually attributed the classification of simple groups of order less than 60 to Galois [Frobenius 1893, p. 337].

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74 For other examples of synthesis between the works of Kronecker, Klein and Jordan's linear groups and congruences, see [Bolza 1891], [Borel & Drach 1895], [Vogt 1895], [Pierpont 1897].

The structure of this Galois-Jordan relation had been initially put to the fore by Jordan himself [Jordan 1870, p. vii]. As we said before, this Galois-Jordan relation presented the *Traité* as an element of continuity with previous works on substitutions. Several aspects of the *Traité* were indeed in such a continuity, such as the descriptions of specific substitution groups.[75]

But when Jordan dealt with Galois's second application, the classes of groups involved were always in relation with one another through the method of reduction. On the contrary, both Hölder and Frobenius aimed at a systematic determination of groups of given orders, e.g. the groups of orders composed of distinct prime factors [Frobenius 1893], or groups of square free order [Hölder 1895b]. Groups of more complex orders were *composed* of such special groups [Hölder 1895a]. When they investigated solvable groups, they appealed to the Sylow theorems and not to Jordan's method of *reduction* [Frobenius 1893, p. 339]. Following Hölder, Moore introduced his 1893 paper by presenting the composition series of "Jordan's decompositions of certain linear groups" as an important tool for finding simple groups "by the principle that the quotient-group of any two consecutive groups in the series of composition of any group is a simple group" [Moore 1893, p. 74].

In a word, even though Jordan's specific method of reduction was initially attached to Galois's second application, in the mid-1860s, it was no longer perceived that way in the 1890s. On the contrary, it was considered as an element of continuity with other lines of development such as Cayley and Hölder's abstract groups or the issue of the determination of groups of a given of order, which pointed to both the legacies of the problem of the number of values of functions and to local approaches such as the path involving combinatorial tactics that had developed in Great Britain [Ehrhardt 2007, p. 329–352]. For instance, the Galois groups, i.e., the groups of the modular equations for transformations of order 5, 7, and 11, were the simple groups of orders 60, 168, and 660. They fell into the class of groups of orders $\frac{1}{2}p(p^2-1)$ $(p>3)$. In 1893, Moore aimed to generalize them to the doubly infinite system $\frac{1}{2}p^n(p^{2n}-1)$ $(p>2,\ (p,n)\neq(3,1))$, or $p^n(p^{2n}-1)$ $(p=2,\ n>1)$.

It is therefore necessary to establish a clear distinction between, on the one hand, the particular features of Jordan's approach to Galois's second application, and on the other the inscription of the *Traité* in the collective

75 In the 1890s Jordan would be presented as a specialist of primitive groups on the model of the descriptions that were given of the "contributions" of contemporary actors by relating them to specific kinds of groups [Frobenius 1893, p. 337], [Miller 1898, p. 238].

genealogies of group theory. Jordan's *Traité* had two facets that did not have the same background and were not to have the same circulation.

2.4. *A specific image of Galois in Jordan: linear groups in Galois fields*

As we have seen, Moore first considered Jordan's *Traité* as an element of continuity in a line connecting Klein to Galois through Serret, Mathieu, and Gierster. A second step was the discontinuous circulation of the specific features of the image of Galois presented by Jordan.[76]

In 1894–1895, Moore published two papers closely related to his 1893 lecture. The first connected the groups of automorphisms of an abelian group of order 2^3 and of type $(1,1,1)$ to the simple linear group of 168 elements, i.e., $\mathrm{PSL}_3(F_2)$ [Moore 1894, p. 65]. As was already the case with Galois fields, Moore's approach can be understood as shedding new light on older works. We have indeed seen that linear groups had been presented as originating from abelian groups of type $(1,1,\dots,1)$ in Jordan's *Traité*. This traditional dimension of the problem might actually be the reason why, about nine months earlier, Burnside proved the more general result that the group of automorphisms of the abelian group of p^n elements of type $(1,1,\dots,1)$ is isomorphic to $\mathrm{GL}_n(p)$ [Burnside 1894, p. 139].[77]

Exactly the same theorem constituted the core of Moore's 1895 "Concerning Jordan's Linear Groups." This paper was presented as a demonstration of the efficiency of Galois fields in group theory; it concluded with tables of primitive elements of Galois imaginaries that had been computed by Moore's students. Amongst them, Dickson might have been already in charge of investigating the works of Mathieu. He had indeed identified that a group of substitutions on p^n letters introduced in [Mathieu 1861b] was isomorphic to $\mathrm{GL}_n(p)$. Moore concluded that "this seems to be the

76 Indeed, even though Dickson had met with Jordan in person during his one-year student trip in Europe, such had been also the case of other actors who had not developed any specific approach to the *Traité*, such as Miller.

77 This notion would be investigated independently by [Burnside 1896]. Neither Moore nor Burnside referred to one another at that time and it is unclear if their works were independent or were actually competing. The introduction of [Burnside 1896] seems to have aimed at contradicting [Moore 1894; 1895a] in claiming that the notion of the group of automorphisms of a group was not a new concept. In 1896 Moore sent to the London mathematical society a paper on the abstract definition of the symmetric group. Burnside introduced the paper he published on the same topic by claiming he had asked the Council of the society permission to withdraw his communication given the "more complete" results stated by Moore. Burnside would not refer to either Moore or Dickson in his 1897 treatise and the other way round with [Dickson 1901].

source from which Mr Jordan's linear groups were drawn" [Moore 1895a, p. 39].

Dickson's thesis would then especially investigate the relations between the works of Mathieu and Jordan. It ended with a proof that $GL_n(p)$ is isomorphic to the Betti-Mathieu group, i.e., the set of all "quantics" (polynomials) of an analytic form φ as follows that represent a substitution on $GF(p^{mn})$ (considered as a vector space over $GF(p^n)$):

$$\varphi(i) = \sum_{k=0}^{n-1} a_k i^{p^k}, \text{ for each } a_k \in GF(p^n).$$

As a result, Dickson's investigations raised some new interest in Mathieu's works on multiply transitive groups on Galois fields. These groups indeed provided classes of simple groups and it was through their investigations that the notion of Galois field circulated to the works of Miller and, from there, to [Séguier 1902; 1904a;b] and [Frobenius 1902; 1904].

Moreover, Dickson's very close reading of Jordan's *Traité* resulted in a flood of papers that systematically generalized results from linear substitutions on F_p to $GF(p^n)$ [Parshall 2004, p. 265].[78] In Dickson's 1901 monograph on linear groups, the theorem on the Betti-Mathieu group was the hinge between the first section on Galois fields, based on Hermite's 1863 approach on "substitution quantics," i.e., the investigation of the analytic representation of substitutions of less than 11 variables, and the second section on Jordan's n-ary linear groups. This new synthesis between the approaches of Hermite and Jordan acted as an impulse for the development of the Dickson network both within the framework of the Chicago school and for other close readers of the *Traité*.

2.5. *Jordan's linear groups and the Galois of the* Traité

We shall now question how Jordan's *Traité* could have supported the discontinuous circulation of specific practices attached to the analytic forms of n-ary linear substitutions from Paris in the late 1860s to Chicago or London in the 1890s. More precisely, this section aims at investigating the structure of the parts of the book that referred to Galois. As we shall see, the image of Galois in the *Traité* was structured by a *chain* of successive *generalizations* of *special model* cases, which ended with the method of reduction of general linear groups.

78 The very close reading of Jordan by Dickson is illustrated by the latter's adoption of terminologies which had already been much criticized such as the one of "abelian group" for what Hermann Weyl would designate as "symplectic groups."

The "Théorie de Galois" in *Livre I* was modeled closely on the cyclotomy of the indexing of the primitive roots of Gauss's "binomial congruence". When they returned in *Livre II*, Galois imaginaries played the role of a *model* case for later *generalizations* of the problem of the analytic representation of substitutions. As we saw above, the origin of the linear group was presented as a generalization of the special type of substitution underlying the indexing methods of *Livre I*, i.e., (*) $(i \quad i+1)$ or (**) $(i \quad gi)$. Moreover, Jordan's first theorem on linear groups stated that all linear substitutions can be expressed as products of substitutions of the forms (*) (transvections) and (**) (dilatations) [Jordan 1870, p. 93].

I shall not detail here the "general" part of *Livre III. Des irrationelles*, i.e., what would be nowadays considered as Galois Theory.[79] This was indeed not the main issue at stake in the line of development I follow in this section, and I shall therefore focus for now on the sole chapter of application of *Livre III* that Jordan related to Galois, namely "algebraic applications." Jordan had first considered the "commutative groups" associated with Abel's equations, whose roots are rational functions of one of them. Primitive abelian equations actually corresponded to cyclic groups, and Jordan quickly focused on the binomial equations $x^n = 1$ and the associated cyclotomic equation of degree $n = p^\alpha$ (p an odd prime). All the roots can then be expressed by a primitive root $\xi, \xi^2, \xi^3, \ldots, \xi^{p^{\alpha-1}}$. The group of the equation is thus cyclic and generated by $(i \quad i+1)$. But the roots can also be reordered by the use of a primitive root g mod p^α, i.e., by the following sequence corresponding to $(i \quad gi)$, $\xi_0 = \xi$, $\xi_1 = \xi^g$, $\xi_2 = \xi^{g^2}, \ldots, \xi_{p^\alpha-1} = \xi^{g^{p^{\alpha-1}}}$.

Second, "Galois equations" were introduced as generalizing Abel's in three different ways.[80] First, they were irreducible equations of prime degree p all of whose roots could be expressed rationally by two of them, an obvious generalization of the equations considered by Abel. Second, their groups were constituted of substitutions of the form $|i \quad ai + \alpha|$, i.e., those originating from the cycles of abelian equations. Third, a special case of

79 For a comparison between the first chapter of Livre *III* and Galois's *Mémoire*, see [Ehrhardt 2007, p. 393–431].

80 Jordan's "Galois equations" would be criticized by Netto in 1882 because of the confusion with the Galois resolvent. After Netto, these equations would be traditionally identified as "metacyclic equations," i.e., as generalizations of Gauss's cyclic equations.

Galois equation was given by $x^p - A = 0$, i.e., an obvious generalization of binomial equations.[81]

Galois equations could thus be understood as the result of a chain of generalizations based on the relations between number-theoretic imaginaries, cyclic groups, and linear groups. But from the standpoint of *Livre III*, the chain could now be considered the other way round. Indeed, the relation between abelian and Galois equations provided an *application* of the reduction of a group by the "adjunction of irrationals to the [associated] equation." Given a Galois equation, let φ_1 be a function of the roots invariant by $|i \quad i + \alpha|$. Recall that such substitutions form a normal subgroup of the group $|i \quad ai + \alpha|$ (origin of the linear group). Let then φ_1 be adjoined to the Galois equation: the group of the equation is then reduced to a cyclic group and the equation itself into an abelian equation; as for the group of the equation in φ_1, it is composed of substitutions $|i \quad ai|$ and is then a commutative group too. The initial Galois equation has eventually been reduced to two abelian equations and its linear groups to two commutative simple groups.

But the general theory of *Livre III* was itself a special model case for the next step of generalization. *Livre IV* opened with two theorems, the first stating that abelian equations of prime degree are solvable by radicals, the second, that "an equation is solvable by radicals if and only if its solution can be reduced to the one of a sequence of abelian equations of prime degrees" [Jordan 1870, p. 386]. The reduction of Galois equations into abelian equations had thus incidentally proved Galois's *criterion*. But Jordan did not state the *criterion* explicitly. The special case of the reduction of linear groups to commutative groups did not aim at imitating Galois's *criterion*, but instead at the following theorem, which Jordan designated as "the criterion of solvability":

> THÉORÈME IX. — Pour qu'un groupe L soit résoluble, il faut et il suffit qu'on puisse former une suite de groupes $1, F, G, H, \ldots, L$ se terminant par L, et jouissant des propriétés suivantes : 1° chacun de ces groupes est contenu dans le suivant, et permutable aux substitutions de L; 2° deux quelconque de ses substitutions sont échangeables entre elles, aux substitutions près du groupe précédent. [Jordan 1870, p. 395].

81 Equations $x^p - A = 0$ provide an example of the criterion stated by Galois for solvable equations of a prime degree that their roots should all be rational functions of two of them: consider a root of the equation x_0 and a pth roots of unity y: $x_1 = yx_0$, $x_2 = y^2x_0$, ..., $x_{p-1} = y^{p-1}x_0$. It is then obvious that any root can be expressed rationally by two of the roots. For instance, $x_r = x_1^r x_0^{1-r}$.

This theorem concluded the general part of Livre *IV*. Jordan contrasted it with the following result he attributed to Galois:

> THÉORÈME IV. — Pour qu'une équation soit résoluble par radicaux, il faut et il suffit que son groupe puisse être considéré comme dérivant d'une échelle de substitutions $1, a, b, \dots, f, g$, telles: 1° que chacune d'elles soit permutable au groupe dérivé des précédentes; 2° que la première de ses puissances successives qui sont contenues dans le dit groupe soit de degré premier. [Jordan 1870, p. 389].

No such theorem can be found in Galois's works, and *Théorème IV* actually seems to be Jordan's interpretation of Galois's discussion of the reduction of a group through the adjunction of roots to an equation [Galois 1846, p. 427]. But while *Theorem IV* resorted to substitutions and composition series (with quotient groups that are abelian and simple, i.e., F_p^*), *Theorem IX* focused on groups and concerned any chain of normal subgroups with quotient groups that are abelian. Jordan claimed his theorem laid the ground for a method by which one would "rise progressively to the knowledge of [solvable] groups," i.e., the problem to which all the rest of the treatise would be devoted.[82] By the use of this method, "each new step toward the solution will make the field of research narrower. Such a simplification would not take place if one took as a starting point Theorem IV as Galois did" [Jordan 1870, p. 396].[83]

Let us now come to some conclusions about the structure of the view of Galois provided by the *Traité*. Recall that Jordan introduced the linear group as a *generalization* of the *special* case of the cyclic substitutions associated with number-theoretic imaginaries. Later on, when the notion of a group of an equation had been introduced, the origin of the linear group could be considered as a *model* for the *generalization* of cyclotomic equations to Galois equations. *Special cases* were both *models* for the *general* theory and *applications* of it. Each link in the resulting chain of generalizations was providing a "higher point of view" toward the previous links. In Livre *IV*, the relation between linear substitutions $(i \quad ai + b)$ on the one hand, and the two forms of representations of cycles $(i \quad i + 1)$ and $(i \quad gi)$ on the other hand, would eventually provide a *model* for *Theorem IV*.

But the theorem Jordan attributed to Galois was not the last step of the chain of generalizations. The last link consisted in turning substitutions

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82 Jordan distinguished between three types of problems: A. The reduction from maximal solvable transitive groups to maximal solvable primitive groups and thereby to B. Maximal solvable groups in $\mathrm{GL}_n(p)$, which included the particular cases of C. Maximal solvable groups in $\mathrm{Sp}_{2n}(p)$ or $O^+_{2n}(2)$ and $O^-_{2n+1}(2)$. See [Dieudonné 1962].

83 See also [Jordan 1867a, p. 270]

into groups and *Theorem IV* into *Theorem IX*. In a sense, *Theorem IX* crystallized the chain structure of Galois in the *Traité*'s. Its great generality was also legitimized by its application to an object of investigation. As Jordan claimed, *special* solvable equations were indeed considered as *subclasses* of *general* ones:

> Supposons que nous ayons formé, pour un degré donné, le tableau de tous les groupes résolubles et transitifs les plus généraux. Chacun d'eux caractérisera un type distinct d'équations irréductibles résolubles par radicaux. Les groupes résolubles et transitifs, non généraux, caractériseront des types d'équations plus spéciaux, et contenus dans les précédents comme cas particuliers. [Jordan 1870, p. 396].

The chain of generalizations which structured the *Traité* could thus be reversed in turning special model cases into applications. For instance, *Livre II*'s origin of the linear group now appeared as a crucial step for the determination of solvable transitive groups. After having reduced the problem from solvable transitive groups to solvable primitive groups, Jordan indeed showed that a minimal normal subgroup A of a solvable primitive group G is commutative and isomorphic to sums of cyclic groups (i.e., of type $(1, 1, \ldots, 1)$). But G is actually acting on A by linear substitutions: it therefore corresponds to the general linear group, originating from A.

Most of *Livre IV* was actually devoted to the relations between the general linear group and its *special* subgroups, such as the symplectic group. *Theorem IX* thus eventually rendered legitimate a linear n-variable approach to traditional issues in the framework of the problem of the number of values of functions, such as enumerating groups of given order [Jordan 1870, p. 386], a claim Jordan had already made in his thesis in 1860 with no reference to Galois or to the problem of the solvability of equations.

Conclusion

We have seen that Jordan started to work on Galois after he had introduced the general linear group in 1860. It was at least partly to legitimize his general approach to substitutions that he then commented on the *Second memoir*. After 1864, Jordan transferred the essential character that his thesis had attributed to the relations of Poinsot's theory of order to the reduction of groups. The idea that the theory of order had a specific dimension was originally based on the connections between algebra and number theory provided by cyclotomy [Boucard 2011, p. 71–99]. But Poinsot also characterized the theory of order as having a relation to algebra analogous to the relations between Gauss's higher arithmetic and usual arithmetic, or that between *analysis situs* and geometry. The issues Jordan tackled between 1864 and 1868 remained very coherent with this framework. In addition to the topics related to Galois, i.e., the groups of algebraic equations, the classification of the irrationals, and higher congruences, Jordan published memoirs on the symmetries of polyhedra, crystallography, the "groups of motions" of solid bodies, and the *analysis situs* of the deformation of surfaces.

The core of the specificity of Jordan's method of reduction was the essential nature it attributed to the *relations* between classes of objects (groups, linear groups, primitive groups, etc.). These classes formed chains from the general to the special such as the one that runs through the architecture of the *Traité*. We have seen that Jordan, unlike Serret, had focused on the relational nature of the Galois's imaginaries, i.e., on the cyclic groups $GF(p^n)^*$. These groups were not only the smallest in the chain of reduction, but the reduction of linear groups into two forms of cyclic groups also provided a model for the chain reduction itself. In the opening of the notice he wrote in 1881 for his application to the *Académie*, Jordan insisted that mathematics is not limited to magnitudes or quantities but that it actually deals with "order" as well as "situation." Referring to Poinsot, Jordan claimed that besides "ordinary Algebra" one finds the "*Algèbre supérieure*" that is based on "the theory of order and of combinations" [Jordan 1881, p. 7–8].

As we have seen, the attribution of an essential nature to relations underlying classes of objects was not a perspective Jordan had taken directly from Galois. It was inscribed in a line of developments involving Poinsot and Gauss especially. Echoes of the longterm dimension of the *cyclotomic relations* underlying higher congruences still resounded at the turn of the century, for example when Paul Bachmann considered that "The theory of congruences bases itself substantially upon the fundamental concept of mathematics, which is already the foundation of Poinsot's method, the concept of group" [Bachmann 1892–1905, trans. Miller, 1903, p. 89].

We have seen also that Jordan's image of Galois was not as connected to equations as Jordan had claimed it was. First, the 1860 thesis had originally been presented explicitly as more general than its application to equations. Second, Jordan actually applied his method of reduction of linear substitutions to several other contexts, e.g. differential equations, bilinear and quadratic forms etc. Moreover, we have seen that the later circulation of

Jordan's Galois in Dickson network was not connected to the solvability of equations.

Let us conclude this section by returning to Chicago. The introduction of Galois fields in 1893 might have appeared somewhat chaotic at first sight. But, on the one hand, Moore's approach was supported by a long tradition dealing with substitutions and number-theoretic imaginaries. On the other hand, Moore's move was coherent with the contemporary reorganizations of the legacies of Klein and Kronecker in finite group theory. In that same year of 1893, Weber appealed to Dedekind's Körper to lay new grounds for "Galois's theory of general equations." In the 1896 edition of his congress paper, Moore noted the equivalence of the terms "Field" and "Endlicher Körper." But we saw that the algebraic number aspect of Galois theory had not played any role in Moore's approach. Moreover, the notion of Galois field did not have the same evolution as that of Körper. Moore repeatedly insisted that the "purely abstract form" of Galois fields "would seem to fit best for immediate use wherever it can with advantage be introduced" [Moore 1896, p. 212], i.e., the investigation of "Jordan's linear groups" [Moore 1895a, p. 38]. As a result, he attributed to Jordan a more important role than Hölder had in 1889.

When Hölder reorganized the legacies of Kronecker and Klein on behalf of Jordan-Galois, the Galois-theoretical part of his paper had remained faithful to Kronecker. But Moore's Galois field blurred Hölder's distinction between pure group theory and algebra. When he eventually referred to Kronecker in 1897 by appealing to [Molk 1885], Moore presented modular systems as a "concrete purely arithmetic phrasing" of abstract Galois fields [Moore 1898, p.281]. In 1898, the bibliography of Dickson's *Report* included the works of Schönemann (as well as the ones of Pellet in the 1880s), thereby illustrating the efforts that had been done for making the collective dimensions of both Galois fields and linear groups precise. But Dickson's *Report* insisted on the autonomy of abstract Galois fields in linear group theory in connection with number theory.

Linear groups in Galois fields eventually reorganized lines of development in a no less radical way than Weber and Hilbert did when they celebrated Dedekind's approach. References to Galois played a key role in the reorganizations based on the notions of field and Körper. Both had jumped over Kronecker on behalf of two *alter egos*, Jordan and Dedekind. In Moore's 1893 congress paper, Dickson's *1898 Report*, or the latter's 1901 *Linear Groups*, the reference to Jordan-Galois played a role analogous to the reference to Dedekind-Galois in Weber's 1893 "Die allgemeinen Grundlagen der Galois'schen Gleichungstheorie," the 1895 *Lehrbuch*, or

Hilbert's 1897 *Zahlbericht.* In the legacy of the essence of the theory of order, Galois fields came to represent an abstract algebraic alternative to Weber-Hilbert's arithmetic-algebraic Körper.

On the contrary, authors such as Kronecker, Pellet, or Bachmann, who stressed the number-theoretic nature of the problem of Galois's theory of equations, actually rejected Jordan's Galois. As we shall see in the next section of this paper, the references to Galois that circulated in the 1870s and 1880s in connection with Jordan's *Traité* had little relation to cyclotomy, number-theoretic imaginaries, or general linear groups: they mainly referred to the second *Traité*'s Galois that could be found in *Livre III. Des irrationelles.*

3. THE GALOIS OF *LIVRE III* (1868–1890)

We shall now turn our attention to the two groups of texts in which one could find most references to Jordan and Galois between 1870 and 1890, i.e., the Klein and Kronecker networks. As we shall see, almost no reference could be found in these groups to Jordan's image of Galois that we have seen in the two previous sections. On the contrary, most texts referred to a second Galois of the *Traité*, one developed by Jordan after 1867.

This second Galois of Jordan was the result of a series of notes and memoirs that Jordan published between 1867 and 1870. Amongst these, some concerned general sections of the forthcoming *Traité*, such as the commentaries on Galois's "Mémoire" that laid the ground for the general section of *Livre III.* In his first commentaries, Jordan aimed at presenting the adjunction of a root "to an equation" in the framework of his 1864 theorem on the chain reduction of solvable groups [Jordan 1865, p. 774]. But in 1866, Jordan was more ambitious in arguing that his theorem could be "very useful for the classification of algebraic irrationals." This, he claimed, was because the number of links in the reduction "gives a very distinct definition of the *degree of irrationality* of the roots of a given equation" [Jordan 1866, p. 1064].

Other papers, as well as most of Jordan's correspondence between 1867 and 1870, were devoted to special equations such as modular equations or the equation of the 27 lines on a cubic.[84] These equations were presented

[84] Jordan especially corresponded with Brioschi and Cremona about the relations between the equation of the 27 lines and the equation of the trisection of the periods of abelian functions.

as applications of *Livre III*'s "higher point of view on the classification and the transformation of irrationals" [Jordan 1870, p. V].

In *Livre III*, Jordan thus placed substitution groups in a framework completely different from that of the theory of order to which he had originally appealed. While no consideration of polyhedra, crystallography, *analysis situs*, or groups of motions appeared in the *Traité*, Jordan's *Livre III* attributed an important role to Alfred Clebsch's geometric approach and to Hermite's works on modular equations. As we shall see, Jordan's claim to contribute to the development of a higher point of view on irrational numbers/functions aimed to situate the *Traité* in a preexisting collective interpretation of Galois's works. Therefore *Livre III*'s notion of irrational not only played a key role in the early circulation of the *Traité*'s image of Galois but also raised obstacles to that circulation, for example with Kronecker's hostility to Jordan's focus on substitution groups.

3.1. *Kronecker's anti-Galois : equations with affects*

The identity of the group of texts under consideration here is close to what one could designate as Kronecker's school in reference to the role the latter had played in Berlin from the end of the 1870s to his death in 1891. Not only were the main authors of this network all former students of Kronecker, but most of the texts were published either in *Crelle*'s (i.e., Kronecker's) Journal or at the Academy of Berlin. The specific relation to Galois that circulated on the local level of the Kronecker network was indeed not shared at the larger scale of the *Grundzüge*'s influence. Making little reference to Galois and only to specific aspects of the latter's work was one of the characteristics of the group. Frobenius, for instance, would make almost no reference to Galois until the beginning of the 1890s. As for Adolf Kneser and Kurt Hensel, they would adopt their master's notions of Galois genus, Galois equation, and "equations with affects."

Since the early 1850s, Kronecker usually alluded to Galois by way of a generic reference to "Abel and Galois." But in his letter to Dirichlet of 31 January 1853, Kronecker claimed that it was impossible to fathom the "true nature" of solvable equations "from Galois's investigations. For Galois only addresses the first task to find the 'conditions of solvability', whereas Abel also takes into account the other one, 'to find all solvable equations'." [Petri & Schappacher 2004, p. 233]. A similar claim is made in the French translation of [Kronecker 1853] added to the second edition of Serret's *Algèbre* [Serret 1854, p. 561].

One will thus investigate equations of a given degree on the model of the explicit expressions Abel had given to the roots of the quintic. But instead of looking for algebraic functions of the coefficients as in the *special* case of solvability by radicals, the *general* problem consisted in finding the "most general" function by which the roots of any equations of a given degree could be expressed.

In 1853, Kronecker stated what is nowadays designated as the Kronecker-Weber theorem,[85] i.e., that the roots of abelian equations with integer coefficients can be expressed as rational functions of the roots of unity. The theorem was explicitly presented as aiming at separating the domains of algebra and of number theory in the investigation of the essence of the quantities associated with algebraic equations. Later on, Kronecker appealed to the complex multiplication of elliptic functions for solving the general quintic. In 1858, he showed that the modular equation of degree 7 contained a polynomial function of degree 7 in 7 variables, which took only 30 values instead of 7! for a general equation of degree 7. This property was called the affect of the equations. As has been seen earlier, Hermite considered in 1858–1859 that the affect of modular equations characterized a specific order or irrationality.

Kronecker's approach was therefore not very compatible with Jordan's focus on non-effective procedures on classes of groups and on equations of arbitrary degree. Jordan's well-known qualification of Kronecker's results on equations as "the envy and the despair" of other mathematicians might thus not have been the most challenging part of the *Traité*'s *Préface* to Kronecker:

> Nous aurions désiré tirer un plus grand parti que nous ne l'avons fait des travaux de cet illustre auteur sur les équations. Diverses causes nous en ont empêché : la nature tout arithmétique de ses méthodes, si différentes de la nôtre ; la difficulté de reconstituer intégralement une suite de démonstrations le plus souvent à peine indiquées; enfin l'espérance de voir grouper un jour en un corps de doctrine suivi et complet ces beaux théorèmes qui font maintenant l'envie et le désespoir des géomètres [Jordan 1870, p. VIII].

In 1874, a public controversy occurred between the two mathematicians on the status of substitution groups. This quarrel eventually turned into an opposition between the values Jordan and Kronecker attributed to arithmetic and algebra. Kronecker in particular argued that it was the duty of algebra to serve arithmetic [Brechenmacher 2007a].

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85 On the developments of Abel's approach by Malmsten (1847), Luther (1847), Kronecker, Sylow and Weber, see [Gårding & Skau 1994], [Petri & Schappacher 2004, p. 235–236], and [Edwards 2009].

Kronecker and Netto were nevertheless close readers of Jordan's *Traité*. But unlike Jordan's *Livre III*, Kronecker clearly separated the arithmetic foundational issues on algebraic quantities he dealt with in his 1882 *Grundzüge* from the substitutions to which Netto devoted a monograph in 1882. When they were dealing with substitutions, both Kronecker and Netto appealed to the traditional point of view of the problem of the number of values of functions, i.e, to substitutions acting on functions of n variables [Kronecker 1882, p. 34]. It was indeed to Cauchy that Netto attribued the first systematic investigation of the theory of substitutions. Moreover, the notion of Galois group of an equation was presented as a secondary notion as compared to Kronecker's "concrete" notions of Galois equation and affects.

As opposed to the formal nature he ascribed to classes of objects such as groups of substitutions, Kronecker aimed at building a concrete, i.e., effective, arithmetic theory of algebraic quantities. The emphasis was laid on Abel's considerations on rational functions with integer coefficients, from which Kronecker claimed he had extracted the notion of rational domain in 1853 [Kronecker 1882, p. 3]. Galois's approach was mostly discussed in relation to the notion of genus domain, i.e., the consideration of an algebraic function on the natural rational domain. This notion could be understood by appealing to Galois's "technical expression of adjunction" the author noted [Kronecker 1882, p. 8]. But for Kronecker, the "algebraic principles of Galois" had to be grounded in an arithmetic reformulation of the traditional notion of Galois resolvent [Kronecker 1882, p. 32].

Let $c_1, c_2, \dots, c_n$ be some quantities in a rational domain $(R', R,'' \dots)$ and consider an "irrational equation" $x^n - c_1x^{n-1} + c_2x^{n-2} - \cdots \pm c_n = 0$, whose roots $\xi_1, \xi_2, \dots, \xi_n$ are thus algebraic functions of the quantities R. Let $f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n)$ be the elementary symmetric functions. Then $\xi_1, \xi_2, \dots, \xi_n$ are the solutions of the n equations :

$$f_k(\xi_1, \xi_2, \dots, \xi_n) = c_k \qquad (k = 1, 2, \dots, n)$$

Consider $F(x) = 0$, the $n!$th degree resolvent obtained by elimination on the latter system, and let x be a function of n indeterminates $u_1, \dots, u_n$[86] $x = u_1x_1 + u_2x_2 + \cdots + u_nx_n$, then

$$F(x) = \prod(x - u_1\xi_{r_1} - u_2\xi_{r_2} - \cdots - u_n\xi_{r_n}),$$

86 The adjunction to a domain of such a quantity x, which is a linear function of n indeterminates, is Kronecker's way of obtaining what he called a "Galois genus" [Kronecker 1882, p.35]

where $r_1, \ldots, r_n$ are permutations on $1, 2, \ldots, n$. The function F, with $n!$ values, was the traditional Galois resolvent. But what Kronecker designated as the Galois equation was the expression of the resolvent as a function of x and of the symmetric functions $G(x, f_1, f_2, \ldots, f_n) = 0$. The coefficients of this equation of the $n!$th degree were therefore concretely given by entire functions of x and of indeterminates u_i with coefficients in the rational domain $(R', R,'' \ldots)$. The consideration of an irreducible factor g of the Galois equation on the latter domain,

$$g(x, u_1, \ldots, u_n) = \prod (x - u_{r_1}\xi_{r_1} - u_{r_2}\xi_{r_2} - \cdots - u_{r_n}\xi_{r_n})$$

was then equivalent to the consideration of "certain designated permutations" for which the function g would remain "unchanged":[87]

> This constitutes the tremendous importance of Galois's algebraic principle, to place at the basis of the investigation not the single equation but rather the system of equations which also defines the conjugate roots. Galois himself has recognized clearly that his new way of seeing algebraic equation makes it possible to abstract from each numerical equation the properties which alone are essential, and he has further expounded on this method which leads to true insight by showing how each special equation is characterized by a property which is independent of the values of the coefficients or roots, via its so-called "group of substitutions." However, it seems to me that Galois's theory may be given a further formal development by the slight modification here presented, by which the abstract substitutions and their groups are replaced by the concrete functions which are invariant under a given group of substitutions. [Kronecker 1882, p. 34].

Genus domains were then presented as domains of functions of n indeterminates. Specific domains corresponded to special equations, i.e., equations whose resolvents split up arithmetically in irreducible factors. These equations were designated as "equations with affects."

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87 Hierin liegt die grosse Bedeutung des *Galois*schen algebraischen Princips, anstatt einer einzigen Gleichung das die conjugirten Wurzeln gleichzeitig definirende *Gleichungssystem* der Untersuchung zu Grunde zu legen. *Galois* selbst hat es klar erkannt, dass seine neue Auffassung der algebraischen Gleichungen es möglich macht, von jeder Zahlengleichung, die für die algebraische Theorie einzig wesentlichen Eigenschaften, zu abstrahiren, und er hat diese zur wahren Erkenntniss führende Methode dadurch vollständig dargelegt, dass er gezeigt hat, wie jeder speciellen Gleichung eine von der Werth-Bedeutung der Coefficienten oder Wurzeln unabhängige Eigenschaft in der von ihm so genannten "Gruppe der Substitutionen" zukommt. Nur scheint mir, dass der *Galois*schen Theorie noch eine weiter *formale* Ausbildung durch die leichte hier eingeführte Modification zu geben ist, bei welcher an Stelle der abstracten Substitutionen und deren Gruppen die concreten bei einer Gruppe von Permutationen unveränderlichen Functionen behandelt werden.

Kronecker then distinguished between the approaches of Abel and Galois as in the two sides of the following expression:[88]

$$\overline{(C)} \qquad g(x_1, x_2, \ldots, x_n) = c_o f_k(\xi_1, \xi_2, \ldots, \xi_n) = c_k.$$

> To explain the fundamental difference between Galois's and Abel's treatment of algebraic equations one may start from the peculiar, "separated form" of the equations $\overline{(C)}$. In fact, Abel stays with those rational function $R', R,'' R','' \ldots$ given or to be found with respect to the particular equation, which figure on the right hand side of the system $\overline{(C)}$. Galois however—at least implicitly when he forms the group—abstracts from the particular equation at stake, only the theoretically relevant functions on the left hand side of the system $\overline{(C)}$. It is true that, precisely because of this total abstraction, Galois misses one of the most interesting problems which Abel found, and also treated, in the theory of algebraic equations. This is the problem to write down all the equations of a fixed class with respect to a given domain of rationality. I want to go into this in detail here since it brings out the arithmetic nature of algebraic questions. [Kronecker 1882, p. 38]

The corpus we have considered in this section has shown in particular that the circulation of Galois's theory of general equations between 1870–1890 was neither related in an obvious way to the reception of the *Traité* nor to group theory nor even to algebra. Actually, Netto and Kronecker carefully distinguished between groups and the arithmetic dimension underlying "Galois's algebraic principles." As we shall see in the next section, while irrationals would come with groups in Klein's Galois, this approach would nevertheless not be directly related to Jordan's *Traité*.

3.2. *Klein's Galois*

Both the works of Galois and Jordan have often been associated to Klein's approach on groups of transformations. But even though Klein

88 An die eigenthümliche "separirte Form" der Gleichungen $\overline{(C)}$ lässt sich die Darlegung des principiellen Unterschiedes der *Abel*schen und *Galois*chen Behandlung der algebraischen Gleichungen am besten anknüpfen. *Abel* bleibt nämlich bei den durch die specielle Gleichung gegebenen oder zu ermittelnden rationalen Functionen $R', R,'' R','' \ldots$ stehen, welche in dem System $\overline{(C)}$ die rechte Seite bilden, während *Galois*, wenigstens implicite durch die Aufstellung der Gruppe, von dem speciellen Gleichungsproblem die theoretisch allein wichtigen Functionen auf der linken Seite des Systems $\overline{(C)}$ abstrahirt. Freilich entgeht *Galois* eben durch diese vollständige Abstraction auch eines der interessantesten Probleme, welches *Abel* in der Theorie der algebraischen Gleichung findet und auch behandelt. Es ist das Problem der Aufstellung aller Gleichungen einer bestimmten Classe für einen gegebenen Rationalitäts-Bereich, und ich will dasselbe hier auch desshalb näher darlegen, weil dabei die arithmetische Natur algebraischer Fragen deutlich hervortritt.

and Lie had visited Jordan in Paris in 1870, and although both of them had read the *Traité* early on, Klein would nevertheless not appeal to Jordan's Galois before 1876–77 and he would actually never adopt it.

Two forms of references to Galois must be distinguished. The first is the kind of general reference that could be found in the so-called 1872 Erlangen program: "in Galois theory as it is presented for instance in *Serret*'s 'Cours d'Algèbre supérieure' or in *C. Jordan*'s 'Traité des substitutions', the real subject of investigation is group theory or substitution theory itself from which the theory of equations results as an application. Similarly, we require *a theory of* transformations, a theory of the groups producible by transformations of any given characteristics" [Klein 1893, p. 242]. Here, the reference to Galois theory aimed at legitimizing transformation groups as an autonomous subject of investigation, one which was developed by Lie. A second type of reference to Galois can be found in Klein's researches; as will be seen below, it pointed to Hermite's Galois.

Most of the papers of the Klein network were published in the *Mathematische Annalen*, i.e., Klein's journal. The main authors were Klein, Brioschi, Bachmann, Weber, and Heinrich Maschke, as well as most of Klein's doctoral students in the early 1880s, e.g. Dyck, Gierster, Ernst Fiedler, Georg Friedrich, Robert Fricke, Adolf Hurwitz, and Cole. But the Klein network also included more unexpected authors such as Georges Halphen and Poincaré. The latter's sole reference to "Gallois" (sic.) in the 1880s was indeed related to the use Klein had made of the Galois resolvent of the modular equations as will be seen in greater detail later [Poincaré 1883].

On the one hand, most of the references to Galois in the Klein network revolved around the "fundamental problem," which consisted in showing that the essence of the "irrational quantity" of the general quintic was essentially represented by the "icosahedron."[89] The latter could be considered as a polyhedron, a Riemann surface (through the projection of the sides of the polyhedron on the sphere in which it is inscribed), an algebraic form, an equation, a transformation group, or a substitution group. What Klein referred to as Galois theory after 1877 was involved in the discussion of the "algebraic character" of the fundamental problem:

> [We have seen in a previous chapter that] we can consider the solution of our fundamental equation, from a function-theory point of view, as a generalization of the elementary problem ; to extract the nth root from a magnitude Z.

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89 For more details on the mathematics involved, see [Serre 1980], and Slodowy's preface to [Klein 1993].

> The algebraic reflexions of the present chapter have then shown us that the irrationalities which are introduced by the equations of the dihedron, tetrahedron, and octahedron can be computed by repeated extractions of roots. *The ikosahedral irrationality, on the contrary, has maintained its individual importance.* Hence an extension of the ordinary theory of equations seem to be indicated. In the latter we are generally restricted to the investigation of those problems which admit solution by repeated extraction of roots. We shall now adjoin, as a further possible operation, the solution of the ikosahedral equation, and ask whether, among the problems which do not admit of solution by mere extraction of roots, there may not be some for which this can be effected by the help of the ikosahedral irrationality. [Klein 1884, p. 112], translated in [Klein 1888].

The explicit aim of connecting various parts of mathematics would play a key role in the structure of Klein's 1884 monograph. The latter especially aimed at linking "geometrical results of group-theory [...] with a definite region of recent mathematics, namely, with the *algebra of linear substitutions* and the corresponding *theory of invariants.* [The following chapters are destined to] effect the connection with the two other disciplines. These are *Riemann's theory of functions* and *Galois's theory of algebraic equations*" [Klein 1888, p. 61]. This aim echoed the transversal role elliptic and abelian functions had played between the 1830s and 1850s in the field of research of arithmetic algebraic analysis [Goldstein & Schappacher 2007a]. But various lines of development had diverged in the 1860s and Klein's approach was especially based on Clebsch's geometric approach to the theory of invariants.

3.2.1. *Klein's first Galois: resolvents and modular equations (1870–1875)*

While Jordan had laid the emphasis on equations and substitutions, Klein would focus on forms, invariants and covariants. The background of Clebsch's geometrical approach to invariant theory as well as the latter's joint work with Gordan on applications of abelian functions to geometry, were nevertheless not unknown to Jordan. On the contrary, *Livre III*'s "geometric applications" were especially based on the formulations (and reformulations) Clebsch had given to some problems of contact which involved substitutions of lines (e.g. the 27 lines on a cubic surface which had been discovered by Cayley and Salmon and investigated by Steiner, the double tangents of a plane quartic curve without multiple point, the 16 straight lines on a quartic surface having a double conic) and symmetries of configurations of points (e.g. Hesse's 9 inflection points on a plane cubic curve lying on twelve straight lines, the 16 singular points on the surface of Kummer).

In 1869, Jordan claimed that the substitution groups attached to the special equations of geometry supported "an investigation of the hidden properties of the given equation" [Jordan 1869b, p. 656]. Here the groups were nevertheless not usually introduced as Galois groups but by the invariance of some algebraic forms, such as the symplectic group $Sp_{2n}(p)$ introduced by Jordan in 1869 as leaving invariant a nondegenerate alternating bilinear form. But Jordan aimed at illustrating the efficiency of the theory of substitutions in showing how it could be used to deduce some (known) geometric connections between special configurations of lines [van der Waerden 1985, p. 126–131]. By successive adjunctions of roots, the group of the 28 double tangents ($Sp_6(2)$) could indeed be reduced successively to the groups of the 27 lines ($Sp_4(3)$) and of the 16 lines ($Sp_2(2)$).

In 1869, Jordan also investigated some cases of reduction of degree of equations on the model of Hermite's investigations on Galois modular equations. He discovered that the reduction of the 80th degree equation of the trisection of abelian functions of four periods led to a group he had recognized as the one of the equation of the 27 lines [Jordan 1869c, p. 865]. This result concluded the "geometric applications" of Livre *III* [Jordan 1870, p. 333], which Jordan connected to the "applications to transcendental functions" [Jordan 1870, p. 365]. When Cremona congratulated Jordan for his *Traité* in 1870, his letter was all about the theorem on the 27 lines. This theorem was indeed one of the main original results of Livre *III* and therefore played a key role for the legitimacy of the *Traité* as a whole as well as for its early reception [Brechenmacher 2012a].[90]

It was nevertheless not necessary to appeal to the notion of adjunction of roots for the decomposition of the groups of the geometric equations Jordan considered. As has been said before, these groups were usually introduced by geometric permutations or symmetries in relation to the invariance of an algebraic form. The reductions of the groups thus usually appealed to interplays between the consideration of fixed points (or lines) and factorizations of algebraic forms. But Jordan systematically associated equations with the substitution groups involved. A fixed point then corresponded to a factorization of the equation by the adjunction of a root and to the corresponding reduction of its Galois group.

By contrast, when Klein first mentioned the name of Galois in 1871, he aimed at laying the general theory of equations on geometrical grounds.

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90 A later echo of the fundamental role of this theorem can be seen in Dickson's *Linear Groups*: even though Dickson was not much interested in applications, he nevertheless ended his book by stating and proving Jordan's theorem.

He insisted particularly on the tension between the "abstraction" of Jordan's theory and its "applications," and proposed to discuss further the relations between the general substitution groups attached to equations and the groups of linear transformations introduced by *algebraic forms* that he and Lie had investigated in three previous publications [Rowe 1989]. As a result, Klein eventually avoided equations and focused on the invariance of algebraic forms. His Galois was therefore more in the legacy of the traditional approach to the problem of the number of values of functions than in the legacy of Jordan's *Livre III.*

Later, in 1875, when Klein started to investigate the connections between the icosahedron and the modular equation of degree 5, he referred to Galois in the line of development of the third application, i.e., in reference to the works of Betti, Hermite, Kronecker, and Brioschi. His initial aim had been to investigate the binary forms left invariant by linear transformations through a classification of all binary linear groups [Gray 2000, p. 83–87]. The classes of groups were attached to the regular polyhedra they were leaving invariant (i.e., the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups). Klein was thus following Jordan's investigations of groups of motions. But recall that Jordan had not related this topic to Galois. Klein actually followed Hermite in considering the Galois group of the modular equation of degree 5, which he showed to be isomorphic to the group of binary unimodular linear fractional transformations that left the icosahedron invariant.[91]

Given an nth degree equation, the $n!$ values of its Galois resolvent can be interpreted as $n!$ points in the projective space $\mathbb{CP}^{n-1}$. In *general,* a point of the projective sphere can thus be moved to $n!$ distinct points by the "linear transformations on the continuous space" associated to the resolvent. But *special* points can have smaller orbits, i.e., smaller systems of linear transformations corresponding to irreducible factors of the Galois resolvent. In the case of the icosahedron, a general point of the sphere has an orbit of $5!/2=60$ proper motions while the vertices of the icosahedron have an orbit of order 12, the midpoints of the faces an orbit of order 20, and the midpoint of the edge an orbit of order 30. Each of these orbits can be considered as the zeros of a homogeneous algebraic form of appropriate degree;

91 $PSL_2(5)$ is, indeed, isomorphic to the group of the 60 ikosahedral rotations which is itself the group of even permutations of five elements (the 60 motions permute the 5 octahedrons inscribed in the icosahedron). It must be pointed out that Kronecker's approach to the quintic was especially influential on Klein [Petri & Schappacher 2007].

this form is thus left invariant by the linear transformations corresponding to the orbit considered.

The problem of finding the roots of the initial equation is thus turned into the one of finding the possible values x_i of a given system of some n-ary homogeneous forms invariant for a finite linear group, i.e., the *Formenproblem* of the theory of invariants (and covariants) as developed especially by Gordan.[92] In this framework, one had to look for some relations between the generators of such a system of forms. In the case of the icosahedron, the system was generated by the forms f, H, T, corresponding to the orders 12, 20 and 30. Here, the three forms T^2, H^3, f^5 of order 60 were linearly dependent (if the vertices of the icosahedron are given, so are the faces and the edges): $T^2 + H^3 - 1728f^5 = 0$. In 1875, Klein connected the 60th degree "equation of the icosahedron" to the resolvent of the general quintic (i.e., corresponding to the alternating group of order 60 of the symmetric group of order 120). More precisely, he designated as the normal form of the resolvent the following rewriting of the equation of the icosahedron (with $z = \frac{H^3}{f^5}$): $T^2 - (1728 - z)f^5 = 0$. The 60th roots of the equation can be expressed as functions of one of the roots η by expressions such as $\frac{A\eta - B}{C\eta - D}$ $(AD - BC = 1)$ corresponding to the binary unimodular group of the modular equation of degree 5.

After he had pointed out the connection between the icosahedron, the general quintic and the modular equation in 1875, Klein developed a complex theory articulating geometry, groups, invariants, Riemann surfaces, and linear differential equations. In the legacies of Betti, Hermite, and Kronecker, he insisted in presenting his approach as a natural generalization of the notion of algebraic solution by radicals. Klein argued that all the interpretations of the theory of the icosahedron, including conformal mapping, can be made to run parallel to the case of solvability by radicals: the only difference between solving an equation by radicals and solving the quintic by the irrationality of the icosahedron, is that in the first case the groups involved are linear in one variable, whereas in the second, they are linear in two variables [Klein 1894, p. 69].

Klein's approach to Galois had a complex legacy. Some authors focused on the problem of establishing the "normal forms" Klein had presented as defining the "irrationalities" attached to a given equation, i.e., what is now designated as the inverse Galois problem.[93] Others, such as Moore, focused on the purely group-theoretical dimension of the special linear

92 See [Gray 2000], [Petri & Schappacher 2004].

93 See [Hölder 1899] and [Wiman 1900].

groups involved in the theory of the icosahedron. In the following I shall focus on the group-theoretical dimension of Klein's works.

3.2.2. *Klein's second reading of Jordan (1877–1879)*

As we shall see here, the classifications of binary linear groups given by Jordan and Klein between 1875 and 1877 highlight the different approaches of the two mathematicians. Since 1873, Jordan applied his works on linear substitutions to the monodromy groups of linear differential equations (in the complex plane). The problem of the algebraic integration of linear differential equations of nth order reduced to the determination of a system of n linearly independent integrals (i.e., a vector space of solutions). Given a solution of the differential equation, a closed circuit of the complex variable around a singular point would operate an n-ary linear substitution on the expression of the solution in the basis of the n integrals (i.e., a monodromy matrix).

As we saw earlier, this approach was developed by Puiseux in 1850 for the problem raised by the determination of multi-valued abelian functions of one variable in the complex plane. Recall that it was on the occasion of a response to Puiseux that Hermite first publicly referred to Galois in 1851. Moreover, Jordan devoted his second thesis in 1860 to Puiseux's approach. In 1870, the *Traité* presented monodromy groups as one of the general notions of the theory of substitutions, which were applied to transcendental functions, especially to elliptic and abelian functions.

In 1876, Jordan aimed at enumerating the binary linear groups that appeared as monodromy groups in Fuchs's problem of algebraic integration of linear differential equations around a singular point. But his classification was not only incomplete; it had also already been given by Klein in 1875.[94] The connection between Klein's 1875 classification and monodromy groups played a key role in the reorientation of the latter's works toward the invariance of modular functions under linear fractional transformations.

Recall that elliptic and abelian functions were introduced as the reciprocal functions of the integrals of special classes of differential equations. Works on more general classes of linear differential equations thus aimed repeatedly at generalizing elliptic functions. In 1877, Lazarus Fuchs had especially considered the inverse functions to the quotient $\omega = K/K'$ of the two periods of elliptic integrals, i.e., two independent solutions of a

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[94] As had been readily pointed out by Klein, the finite group of 168 elements of the modular equation of order 7 was missing, a mistake Jordan would correct before he would publish the complete version of his work.

differential equation.[95] K and K' were analytic functions of k (the modulus), and $k = f(\omega)$ was a single valued (modular) function of $\omega = x + iy$ for all positive x (i.e., on the upper half plane) [Gray 2000, p. 101–104]. But in the analogous situation with the periods J and J' of elliptic integrals of the second order,[96]

$$J = \int_0^1 \frac{k^2 x^2\, dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \text{ and } J' = \int_0^{\frac{1}{k}} \frac{k^2 x^2\, dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

one ceases to have single-valued functions. As Hermite emphasized, in both cases the modular functions were nevertheless invariant for binary unimodular linear fractional substitutions with integer coefficients. Analytic continuations of J and J' could thus be investigated by considering the action of $\mathrm{PSL}_2(Z)$ on the upper half plane.

The connection of this invariance interpretation of modular functions to Hermite's 1858–1859 works on the quintic was readily developed by [Dedekind 1877] into a presentation of the theory of modular functions almost independent of the framework of elliptic functions [Gray 2000, p. 107–115]. In parallel, [Gordan 1877] related the problem to Klein's classification of binary linear substitutions, to which he had given a new invariant-theoretical formulation [Gray 2000, p. 88]. On the other hand, [Jordan 1878] had extended his classification to ternary linear transformations with results on general linear groups.[97] As was emphasized by [Fuchs 1878, p. 1], three formulations of the classification of binary linear groups had thus been stated by 1877.

As a result, Klein's 1875 classification of transformation groups had been retrospectively connected to linear differential equations, and had thus met again with Jordan's substitution groups. But this second encounter took place in a context where Jordan's linear groups were not the only issue at stake. On the contrary, as discussed in Tom Archibald's paper in this volume, various lines development on linear differential equations would be reorganized with the developments of Poincaré's

95 Let $k^2 = 1/u$ and $K = \frac{1}{2}\sqrt{u}\eta_1$ and $K' = \frac{1}{2}\sqrt{u}\eta_2$, then η satisfies Legendre's equation $2u(u-1)\frac{d^2\eta}{du^2} + 2(u-1)\frac{d\eta}{du} + \frac{1}{2}\eta = 0$. A closed circuit on the neighborhood of the singular points 0, 1, and ∞ transforms a solution according to monodromy matrices $\begin{pmatrix} a & bi \\ ci & d \end{pmatrix}$ with $ad + bc = 1$.

96 The two periods J, J' are two independent integrals of $(1-k^2)\frac{d^2y}{dk^2} + \frac{1-k^2}{k}\frac{dy}{dk} + y = 0$.

97 Every linear group has an abelian subgroup which index does not exceed a bound depending only on the number of variables. As a lemma, Jordan had proved that every periodic linear substitution can be diagonalized, a result Moore and Mashke would generalize in 1898.

Fuchsian functions, Klein's automorphic functions, Lie's approach to partial differential equations through continuous groups, or Picard's first papers on differential Galois theory.

Even though the status attributed to groups changed radically during this time-period, this evolution did not imply a radical rupture with older approaches. For instance, the generalization of elliptic functions to automorphic functions could be understood as the extension of functions invariant under groups of translations (the double periodicity of elliptic functions) to functions invariant under binary linear fractional substitution groups. This generalization was thus in continuity with the traditional function-theoretic presentation of substitutions in the framework of the problem of the number of values of functions. The consideration of the algebraic solvability of linear differential equations as resorting to the "vast region of single-valued transcendental functions with linear transformations into themselves" [Klein 1884, p. 127] was quite coherent with the traditional definition of resolvents in the framework of algebraic solvability by radicals.

Jordan's approach was thus only one amongst many developments of a group theoretical approach to linear differential equations. Moreover, the actors involved referred to various parts of the *Traité*. For instance, Poincaré's success in developing Fuchsian functions theory was partly due to his appropriation of key algebraic practices of Jordan—such as the canonical form—that Klein would never use [Brechenmacher 2012b]. But Poincaré was nevertheless not much interested in Galois, whose name he would misspell on the rare occasion he would mention it in the 1880s.[98]

As for Klein, even though he resorted to some of the general concepts presented in the *Traité*, such as the notion of isomorphism [Klein 1879b, p. 254], he would not appeal to Jordan's Galois. On the contrary, the context of linear differential equations consolidated Klein's reference to the modular group in the legacy of Hermite's Galois [Klein 1877b, p. 505]. First, the group of the modular equation of degree 5 was at the center of the various interpretations of the icosahedron Klein developed in 1877 in connection with the general quintic and the invariance of binary forms [Gray 2000, p. 126–134]. Second, in 1878 Klein extended his approach to the two other Galois groups in connection to appropriate Riemann surfaces [Gray 2000, p. 152–161]. The group of the transformation of order 7 involved ternary linear substitutions (i.e., $PSL_2(F_7)$ is isomorphic to $PSL_3(F_2)$), ternary forms and higher plane curves such as the plane

98 See [Poincaré 1883] as well as [Perott 1887].

quartics and their 28 bitangents. After 1879 Klein would approach the group of the transformation of order 11 and Gierster would devote his thesis to modular groups of prime orders.

In his review of the first edition of Klein's 1884 *Vorlesungen*, Lampe highlighted the role the book devoted to "the theory of Galois groups." The designation still implicitly pointed to the three modular groups. But Klein also presented a general exposition of "Galois groups of equations" of the nth degree. However, this presentation was not based on Jordan's *Livre III* but on Kronecker's *Grundzüge* [Klein 1884, p. 86].

But unlike Kronecker, Klein attributed a fundamental status to the Galois groups and had adopted some terms that had been used by both Hermite and Jordan, such as "Galois's ideas," or "Galois's method" [Klein 1879a, p. 418]. The extensive and various uses of Galois's name that developed in connection to Klein's work circulated quickly. This situation seems to have caused echoes and reactions in other contexts. It is likely that both Kronecker's 1882 discussion on the relative roles of affects and Galois groups, and Picard's 1883 note on differential Galois equations, had partially aimed at responding to an increasing number of references to Galois in connection with Klein. In any case, even though most later presentations of Galois's theory of equations would be based on Kronecker's approach, they would nevertheless follow Klein in attributing to Galois groups a status Kronecker had denied them.

In this situation, bits and pieces of Jordan's *Livre III* played the role of interfaces between Klein and Kronecker's approaches. For instance, when Weber investigated some issues related to the double points of an algebraic curve (a quartic) in 1883, he would present his paper as having to do with the "Galois group of an equation of the 28th degree" on the model of *Livre III.* Later on in 1889, Hölder appealed to the Galois of Livre *III* to propose a synthesis of the works of Klein and Kronecker.

Conclusion

The global picture resulting from our collections of self-portraits with Galois is quite different from the role which has been often attributed to Jordan as the one who had unpacked the group-theoretical content of Galois's works. Questions in the history of algebra are certainly not limited to problems of origins or diffusions of abstract notions: we have seen several examples of local circulations of algebraic practices at various scales of the collective dimensions associated to complex networks of texts.

The specificity of Jordan's Galois was that it attributed an essential nature to a method of reduction of classes of groups. That method had been modeled on the analytic linear forms provided by the decomposition of letters into blocks under the action of imprimitive groups. Linear groups could therefore not be disconnected from the underlying Galois fields, which were actually considered as groups themselves: the linear form $(z \quad az + b)$ was unscrewed into the two forms of actions $(z \quad z + 1)$ and $(z \quad gz)$ of the cycles of $\mathrm{GF}(p^n)^*$. The general linear group then originated as the maximal group in which an elementary abelian group is a normal subgroup.

The method of reduction was the basis for several key results of Jordan, e.g., the introduction of the composition series of a group, the criterion of solvability, and the reduction of linear substitutions to a canonical form. But no explicit definition was given to the reduction itself as is illustrated by the crucial addition Hölder later made through the concept of quotient group. The reduction was presented as the "essence of the question," something that expressed its presence by making "visible" unexpected relations between various objects and classes of objects. Successive generalizations from special model cases were one of the modes of expression of such essential relations. Jordan developed a specific way to deal with the special and the general which he considered as successive links in a chain of reductions from the most general to the simplest.

But Jordan's method of reduction did not directly result from an appropriation of Galois's decomposition. On the contrary, the reduction of imprimitive groups to primitive groups was a point of contact between Jordan and Galois. The fact that Jordan had presented his reduction in the framework of Poinsot's theory of order suggests that Jordan and Galois shared a specific perspective on the roles played by *relations* in both algebra and number theory. Little is nevertheless known about the role played by the theory of order in the 19th century. Jordan's early works suggest that a specific approach to mathematics might have circulated within crystallography and mechanical investigations of motions of solid bodies (such as polyhedrons) [Brechenmacher 2012a]. Until the 1920s, Jordan and Galois were often referred to in connection to the "science of order" or the "algebra of order."[99] We have seen also that in the space of circulation of

99 At the turn of the 20th century in France, discourses on mathematics would often oppose the theory of order, considered as a part of Analysis, to algebraic-arithmetic approaches developed in Germany (see [Couturat 1898]). In 1922, Robert d'Adhémar would designate as the "Algebra of order" the conceptual approach to mathematics consisting in replacing computations by ideas.

the Dickson network, Jordan's linear groups supported an abstract notion of Galois field that focused on group-theoretic relations as opposed to the arithmetic-algebraic dimensions of the contemporary notion of *Körper*.

But Jordan's *Traité* was not limited to Jordan's Galois. We have seen that between 1867 and 1870, Jordan had inscribed his works into preexisting collective references to Galois. It was on this occasion that he had focused on Galois's fundamental theorem, which otherwise had not played any role in the reduction of linear groups. The groups associated with the process of adjunction of roots to an equation were presented as offering a higher point of view on the "classification and the transformation of irrationals." Indeed, we have seen that, between the 1830s and 1850s, Galois's works had been collectively considered as relating to the *special* "orders of irrationalities" introduced by general equations of a given degree. In this context, the emphasis had been lain on special objects of investigation such as the equations of the division of periods of abelian functions or modular equations.

What was then the connection between the "Théorie générale des irrationnelles" in which Jordan's *Livre III* had inscribed Galois's *Mémoire* and the "Théorie des équations" in which Serret's 1866 textbook had commented on Galois?

Although it is not the place here to discuss in greater detail the public dimensions of the figures of Galois and Abel, it must be pointed out that the association of Galois's and Abel's works with the "theory of equations" cannot be dissociated from the role these questions traditionally played in public discourse [Brechenmacher 201?b]. It is therefore compulsory to distinguish different scales in the collective dimensions of Galois's mathematical works. It was indeed mostly within discourses aiming at broader audiences than the Academy that Galois's works were presented within a longterm history of the solvability of equations by radicals. In contrast, the main characteristics of the emerging field of research of arithmetic algebraic analysis were rarely mentioned in a large public sphere, i.e., the focus on the notion of congruence, the acceptance of complex numbers, and the investigations on the algebraic-arithmetic properties of elliptic functions [Goldstein & Schappacher 2007a, p. 26].

Unlike the public dimensions of the solvability of equations by radicals, we have seen that on the more local level of networks of texts, Galois's works were not much commented on in connection to the general theory of equations. Betti, Kronecker, Hermite, and Klein had all insisted that the problem of expressing roots as algebraic functions of coefficients had to be replaced by the one of associating adequate analytical expressions to each "order of irrationalities" introduced by general equations of degrees higher than four. In Bertrand's 1867 report on the progress of mathematics, Galois's works were clearly inscribed in a twofold collective dimension [Bertrand 1867, p. 3–17]: first, the theory of equations as it was exposed in textbooks such as Serret's; second, the higher point of view of analysis on the nature of the algebraic and transcendental irrationals related to elliptic and abelian functions and their related special equations. A later echo of the twofold collective dimension of Galois's works was heard in 1898 when Paul Tannery discussed the works of Galois in connection to the approaches of Hermann Grassmann and William Rowan Hamilton on irrational numbers [Tannery 1898, p. 739].

Liouville's edition of Galois's works in the late 1840s was contemporary to the emergence of the *Algèbre supérieure* as an intermediate discipline between elementary arithmetic and algebra and the "higher" domain of analysis as it was taught at the *École polytechnique* [Ehrhardt 2007, p. 211–236]. The elementary and higher points of views were far from being disconnected from each other. For instance, the new proof of the impossibility of solving the quintic that Hermite had published in the *Nouvelles annales de mathématiques* in 1842 when he was still in high school was followed in 1844 by a memoir on the division of the periods of abelian functions. The paper on the quintic was introduced by Olry Terquem who also praised on this occasion both the classification of transcendental functions into species pursued by Liouville in 1837 and Wantzel's 1837 proof of impossibility of the antique problems of the duplication of the cube and of the trisection of an angle. In doing so, Terquem explicitly alluded to the higher point of view on equations provided by investigations on the nature of the "irrationals." Later publications on this topic by Wantzel and Victor-Amédée Lebesgue would be the first to greet Liouville's project to edit Galois's works.

The theory of equations was therefore no more an autonomous domain of research than algebra itself was an object-oriented discipline shared by a community of specialists [Brechenmacher & Ehrhardt 2010]. Recall that despite the fact that Galois's works were scarcely taken into account in the first edition of Serret's book, the figure of Galois was nevertheless already celebrated in the framework of a *longue durée* history of the "theory of equations." This historical perspective clearly aimed at delimiting the domain of algebra as it was taught. The crucial role attributed to the issue of solvability by radicals did not reflect contemporary research on equations.

Moreover, the organization and the status of Serret's *Algèbre* were implicitly built on the higher point of view of analysis. The issues related to Galois were first steps toward the higher point of view on transcendental irrationals introduced by the inverse integrals of differential equations, and which involved complex analysis, arithmetic considerations on congruences or quadratic forms, the algebraic theory of invariants, etc.

The interactions between Serret's textbooks and Hermite's researches especially illustrate the two levels of the *Algèbre supérieure* and of the field of arithmetic algebraic analysis. As a matter of fact, even in the 1866 two-volume edition of the *Cours*, Serret's considerations on linear substitutions were limited to the binary substitutions Hermite had investigated in connection to the modular equations. This approach was clearly much more suited to preparing the student for the theory of elliptic functions of [Briot & Bouquet 1859] than for Jordan's forthcoming *Traité*.

In the 1860s, there was thus already a twofold collective interpretation of the works of Galois. For this reason, Jordan's claim to deliver the commentaries on Galois that Liouville had promised is a telling illustration of the different autonomous developments that would tear apart the field of arithmetic algebraic analysis [Goldstein & Schappacher 2007b, p. 97]. In the name of Galois, Jordan did indeed reorganize various results that had grown into an autonomous theory in the 1850s. Previous works that had made precise reference to Galois, such as those of Hermite, thus fell into the global legacy of Galois, in the company of works that used to be disconnected from any reference to Galois, such as Cauchy's substitutions or Clebsch's problems of contacts.

A few years later, Klein's Galois emerged from an interpretation of Hermite's Galois in both the context of groups of geometric transformations and the traditional constructive invariant-functional approach of Galois resolvent. Klein did not appeal to Jordan's n-ary linear groups but focused on the special linear fractional groups attached to the modular equations of order 5, 7, and 11, i.e., the Galois groups. Because these groups appeared at the core of Klein's various interpretations of the nature of the irrationality of the icosahedron, Klein celebrated Galois for the introduction of a fully general notion of group in the special case of the theory of equations.

In the 1880s, the question of the status of the notion of group with regard to arithmetic, algebra, and analysis was much debated. Kronecker rejected Jordan's Galois and developed a constructive approach to the Galois resolvent in the framework of an influential comprehensive arithmetic theory of irrational quantities. His notion of equations with affect was nevertheless replaced by the notion of Galois groups of an equation.

At the turn of the century, the disciplinary issues related to the nature of Galois groups would often have national, and in fact nationalistic, overtones. In France, Galois would often be celebrated as the founder of the continuous groups involved in Analysis. For instance, while Drach's *Algèbre supérieure* had appealed to Kronecker for presenting the "famous theory created by Galois" as an extension of arithmetic, Tannery's *Préface* explicitly recalled that it was Analysis that provided a higher point of view on "the general irrationality" of which "the algebraic number is nothing more than a particular case" [Borel & Drach 1895, p. iv].

As for Jordan's presentation of Galois's *Mémoire*, it eventually did not seem to have been much followed, in contrast to those of Serret or Kronecker and also in contrast to other aspects of Jordan's book.[100] The central position of *Livre III* in the *Traité* nevertheless supported the claim Jordan had made in the *Préface* that his whole book was nothing more than a commentary on Galois. This claim played a role in the collective attribution of an *essential* ontological value to "Galois's ideas" or "Galois method." But the *Traité* did not need to be read in order to play such a role (at least not as whole). The book expressed Galois's mathematical greatness by its very existence, or more precisely by recurrent references to its existence.

100 It could nevertheless be studied directly in the *Traité*. For instance in 1913–1914, Georges Humbert gave to his lectures on the theory of substitutions at the *Collège de France* a structure very close to the one of the *Traité*.

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