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THE CONTRIBUTIONS OF HILBERT AND DEHN TO NON-ARCHIMEDEAN GEOMETRIES AND THEIR IMPACT ON THE ITALIAN SCHOOL

Cinzia Cerroni

Abstract. — In this paper we investigate the contribution of Dehn to the development of non-Archimedean geometries. We will see that it is possible to construct some models of non-Archimedean geometries in order to prove the independence of the continuity axiom and we will study the interrelations between Archimedes' axiom and Legendre's theorems. Some of these interrelations were also studied by Bonola, who was one of the very few Italian scholars to appreciate Dehn's work. We will see that, if Archimedes' axiom does not hold, the hypothesis on the existence and the number of parallel lines through a point is not related to the hypothesis on the sum of the inner angles of a triangle. Hilbert himself returned to this problem giving a very interesting model of a non-Archimedean geometry in which there are infinitely many lines parallel to a fixed line through a point while the sum of the inner angles of a triangle is equal to two right angles.

Résumé (Les contributions de Hilbert et de Dehn aux géométries non-archimédiennes et leur impact sur l'école italienne)

Cet article présente les contributions de Max Dehn au développement des géométries non archimédiennes. Un moyen pour montrer l'indépendance de l'axiome d'Archimède par rapport aux axiomes d'incidence et d'ordre est de construire des modèles de géométries non archimédiennes. Les travaux de Max Dehn dans ce champ concernent pour l'essentiel les relations entre l'axiome d'Archimède et les théorèmes de Legendre. Quelques-unes de ces liaisons ont

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été aussi étudiées par Bonola, un étudiant d'Enriques, qui est parmi les rares Italiens à avoir apprécié le travail de Dehn. Un des principaux résultats, lorsque l'axiome d'Archimède n'est pas satisfait, est que l'axiome des parallèles est indépendant de celui de la somme des angles internes d'un triangle. Hilbert lui-même revint sur ce problème en construisant un modèle de géométrie non archimédienne dans lequel il y a une infinité de droites passant par un point et parallèles à une droite donnée, alors que la somme des angles internes d'un triangle est égale à deux angles droits.

1. INTRODUCTION

The *Grundlagen der Geometrie* (1899) by David Hilbert triggered a new phase in geometrical research. The analysis of the interrelation and independence of the axioms gave rise to new geometries, the study of which acquired the same importance as that of classical, Euclidean geometry. This work is part of a research project on the creation of new geometries in the first half of the 20$^{\text{th}}$ century and on the analysis of their interrelations with algebra, which yielded its first result in a work [Cerroni 2004] on the study of non-Desarguesian geometries. The present paper aims to pursue this line of research, dealing with non-Archimedean geometries.

The starting point of research on non-Archimedean geometries was the investigation of the independence of Archimedes' axiom from other axioms.

As it is well known, Archimedes' axiom states that if A and B are two segments, with A smaller than B $(A < B)$, then there exists a positive integer n such that n times A is greater than B $(nA > B)$.

Giusepppe Veronese made the first attempt to construct a model of non-Archimedean geometry [1]. In *Fondamenti di geometria* [Veronese 1891], he constructed, in abstract manner, a geometry in which he postulated the existence of a segment which is infinitesimal with respect to another, and where the straight line of geometry is not equated with the continuous straight line of Dedekind. It is not our aim to develop in depth the study of Veronese's work, for which we refer to the literature [2] and to a forthcoming paper. So, we limit ourselves to sketch the contributions by

1 For a study of the emergence of non-Archimedean systems of magnitudes see [Ehrlich 2006]

2 For study of Veronese's non-Archimedean straight line see [Busulini 1969/70] and [Cantù 1999].

Veronese and we want to go directly to Dehn's contributions and to its influence on Italian geometers.

Veronese attacked the question about the existence of a segment which is infinitesimal in respect to another and so about the existence of a non-Archimedean geometry while analyzing the independence of Archimedes' axiom from the others:

> "[...] The question about the existence of a segment which is infinitesimal in respect to another is ancient; but neither the supporters neither the opponents have proved the possibility or the impossibility of this idea, because they did not put the question in a right way, complicating it with philosophical considerations unrelated with it. Instead, it has to be put in the same way than those about the parallel axiom and the space dimensions; that is, if all the axioms hold, is Archimedes' axiom a consequence of the others? Or in other words, let A and B be two segments $(A < B)$, does there exist a geometry in which in general it is not true that $An > B$, where n is a positive integer $1, 2, \dots n$?
>
> If one takes Dedekind's axiom as the continuity axiom, or if one maps the points of the line in to the real numbers, then the previous relation is a consequence of it. But I gave a new definition of the continuity axiom that does not contain Archimedes' axiom [...]" [3] [Veronese 1898, p. 79].

Therefore, the central problem for the author is to find a definition of the continuity axiom that does not contain Archimedes' axiom [4], that is to define a non-Archimedean continuous [5]. The geometric continuous is the way to define an abstract continuous independent from the numeric continuous. Veronese supposed, in a system of axioms and in his definition of continuity, the existence of infinite limited segments and so the existence

3 "[...] *La questione del segmento infinitesimo attuale è antica; ma né i sostenitori né gli oppositori di tale idea ne hanno mai provata la possibilità o la impossibilità geometrica, perché essi non hanno posta la questione in modo chiaro e determinato, complicandola talvolta con considerazioni filosofiche ad essa estranee. Essa invece va posta in modo analogo a quelle relative ai postulati delle parallele e delle dimensioni dello spazio; vale a dire dati tutti i postulati necessari per costruire la figura corrispondente al campo della nostra osservazione esterna e considerati come possibili tutti quei postulati che non contraddicono ai precedenti e non si contraddicono fra loro, il postulato di Archimede è esso conseguenza degli altri? O in altre parole è possibile una geometria nella quale dati due segmenti A e B $(A < B)$ non obbediscano in generale alla relazione $An > B$, essendo n un numero intero qualunque della serie $1, 2, \dots, n, \dots$? Se si dà il postulato della continuità nella forma proposta da Dedekind, o facendo corrispondere biunivocamente i punti della retta ai numeri reali ordinari, allora detta relazione si può considerare come un'immediata conseguenza di esso. Ma io diedi della continuità un'altra forma che pur mantenendo i caratteri del continuo rettilineo non racchiude quella di Archimede* [...]". All the translations are by the author of the paper.

4 The first work of Veronese on this topic is [Veronese 1890].

5 For a study of Veronese's non-Archimedean continuous see [Cantù 1999].

of segments which are infinitesimal in respect to another, that is the existence of non-Archimedean geometry. Moreover, the existence of infinite limited segments permits him to introduce a non-Archimedean system of numbers [6] [Veronese 1891].

The work was criticized by many (including Wilhelm Killing (1847-1923), Georg Cantor (1845-1918), Otto Stolz (1842-1905), and Arthur Moritz Schönflies (1853-1928)) and was disapproved by Giuseppe Peano, who ended his review with the words: "[...] the lack of accuracy and rigour of the whole book removes every value from it" [7] [Peano 1892a].

Here, we briefly refer to the international discussion [8], which covers the period from 1890 to 1907. It developed essentially around three arguments: 1) the relation between Dedekind's axiom of continuity and Archimedes' axiom and the existence of a non-Archimedean continuous; 2) the existence of infinite and infinitesimal segments and consequently the existence of a non-Archimedean system of numbers; 3) the relation between Archimedes' axiom and the principle of completeness.

The first and the second arguments [9] are analyzed and discussed prevalently by Bettazzi [1890; 1891; 1892], Cantor [1895; 1897], Killing [1885; 1895–96; 1897], Levi Civita [1893; 1898], Peano [1892a; 1892b; 1892c], Schönflies [1897a; 1897b], Stolz [1883; 1888; 1891], Veronese [1890; 1891; 1892; 1896; 1897; 1898] and Vivanti [1891a; 1891b]. In particular, in 1893 there was a turning-point in the discussion; Tullio Levi Civita (1873-1941) published his work "*Sugli infiniti ed infinitesimi attuali quali elementi analitici*" in which he constructed from the real numbers, in an analytical way, a number system whose numbers (the "*monosemii*") are the marks of Veronese's infinite and infinitesimal segments. Thus, Levi Civita's construction is completely analytic and it could be considered a first view of the so called non-standard analysis. At last, the third point has been made in 1906 by Schönflies [Schoenflies 1906] and solved by Hahn in 1907 [Hahn 1907].

The discussion about the possibility of the existence of infinite and infinitesimal segments ended with the publication of Hilbert's *Grundlagen*, in which there is an analytic construction of a non-Archimedean geometry. [10]

6 For study of Veronese's non-Archimedean system of numbers see [Cantù 1999].

7 "*La mancanza di precisione e di rigore di tutto il libro, tolgono ad esso ogni valore*".

8 For the national discussion see [Galuzzi 1980], [Manara 1986] and [Ehrlich 2006].

9 For a study of the contributions of Bettazzi, Cantor, Killing and Stolz see [Cantù 1999] and [Ehrlich 2006].

10 In 1904, H. Poincaré in a review of *Grundlagen* ([Poincaré 1904] admitted that Veronese was the first to construct a non-Archimedean geometry, but said that Veronese's geometry used Cantor's transfinite numbers and that Hilbert's one is more

This international discussion probably influenced Hilbert, who knew the work of Veronese and referred to it as "deep work" [11] [Hilbert 1899, p. 48].

Hilbert systematically studied the independence of Archimedes' axiom from the others by constructing an analytical model of a non-Archimedean geometry and he also analysed the relationship between Pascal's theorem [12] and Archimedes' axiom. On Hilbert's suggestion, Dehn studied the relationship between Legendre's theorems and Archimedes' axiom. This last analysis is based on the point of view of Hilbert's *Grundlagen* [13]. In fact, in the proofs of Legendre's theorems, that we can find in the literature (i. e. those of Euclid, those of Saccheri and those of Legendre himself) Archimedes' axiom is used, in a more or less explicit way. In the optic of Hilbert, and therefore of Dehn, it is meaningful to study whether these theorems really depend on this axiom.

We recall that Legendre's theorems [14] state that:

(1) The sum of the angles of a triangle is equal to or less than two right angles.

(2) If in a triangle the sum of the angles is equal to two right angles, it is so in every triangle.

Dehn showed that the first theorem is a consequence of Archimedes' axiom and that the second is independent of it. He also showed that if Archimedes' axiom does not hold, the theorem on the sum of the inner angles of a triangle is not equivalent to the parallel axiom. It might come as a surprise, but we will see that if Archimedes' axiom does not hold, the hypothesis on the existence and number of parallel lines through a point

simple and original; it aroused the anger of Veronese, who in the work "*La geometria non archimedea. Una questione di priorità*" [Veronese 1905] remarked that his tranfinite numbers are different from Cantor's and that Hilbert's non-Archimedean geometry is contained in his geometry.

11 See [Bottazzini 2001].

12 Pascal-Pappus's theorem states: let A_1, B_1, C_1 be three distinct points on a line l_1 and let A_2, B_2, C_2 be three distinct points on a line l_2, distinct from the line l_1. Then the points A_3, B_3 and C_3—intersections, respectively, between the lines through the points B_2C_1 and B_1C_2, the points A_1C_2 and A_2C_1, and the points A_1B_2 and A_2B_1—are collinear.

13 Hilbert's main goal in his *Grundlagen* is to show how to obtain from purely geometric axioms the so-called "coordinatizating algebra", i.e. to find an analytical model K^n of this geometry, where K is a generic algebraic structure (Field or Ring, etc.). This program is in many ways a prosecution of Staudt's program which aims to derive the axioms of real and complex numbers from the axioms of pure projective geometry. In the works of Veronese and of Levi Civita we find many suggestions in this direction.

14 These theorems are already in [Saccheri 1733], and in Italy they are called Saccheri's theorems. Dehn called these theorems Legendre's theorems.

is not related to the hypothesis on the sum of a triangle's inner angles. Hilbert was amazed by Dehn's result, and an elaboration of this result became part of Hilbert's lectures on the foundations of geometry in 1902.

We find Bonola's attempt to give a different proof of some of Dehn's results very interesting. He was following Saccheri's classical approach and thus obtained a much more intuitive and informal description of the links between Archimedes' axiom and the theory of parallels. Bonola, in his new elaboration of Dehn's result followed the point of view of his master Federigo Enriques, that is to prefer an elementary and intuitive model which is deducible directly from the geometrical properties.

Enriques entered the discussion on Non-Archimedean geometry only in 1907, when he published his contribution to the *Encyklopädie der mathematischen Wissenschaften.* In this work, however, he was able to refer to all the previous debate on this subject and to the most recent developments on it. Furthermore he placed Veronese's work on the same level as that of his contemporaries.

2. HILBERT AND NON-ARCHIMEDEAN GEOMETRIES

As is well known, Hilbert devoted Chapter II of his *Grundlagen der Geometrie* to proving the independence and non-contradictoriness of axioms. In particular, he proved the independence of Archimedes' axiom from the other ones. More precisely, he showed that Archimedes' axiom is not a consequence of axioms I (of incidence (connection)), II (of order), III (of parallelism), and IV (of congruence) [15] by exhibiting a geometry where Archimedes' axiom fails to be valid [Hilbert 1899].

Hilbert constructed a non-Archimedean number system on which he based an analytic geometry. In particular, he considered the set $\Omega(t)$ consisting of the algebraic functions of t obtained from the set of polynomials with rational coefficients in t by the five operations of addition, subtraction, multiplication, division, and the operation $\sqrt{1+\omega^2}$, where ω is a function derived from the previous five operations [Hilbert 1899, § 12, pp. 24-26].

The set $\Omega(t)$ is countable and $\Omega(t)$ can be regarded as the set of real-valued functions of a real variable defined at all but a finite number of points. Moreover, if c is a function in $\Omega(t)$, i.e. c is an algebraic function of t, it will vanish only on finitely many values of t. Therefore, c, for positive

15 Usually, axioms III are on congruence and axioms IV are on parallelism, like in the more recent editions of the *Grundlagen.*

large values of t, is either always positive or always negative. The usual operations are valid in $\Omega(t)$, and if a and b are two functions in $\Omega(t)$, a will be greater than b ($a > b$) or a less than b ($a < b$) if $a - b$ is always positive or always negative for positive large values of t, respectively.

Let n be a positive integer. Then n is less than t ($n < t$) since $n - t$ is always negative for large positive values of t. Consider the numbers 1 and t in $\Omega(t)$. Every multiple of 1 is always less than t, so $\Omega(t)$ is a non-Archimedean number system.

Hilbert constructed a 3-dimensional analytic geometry on this number system, as follows: (x, y, z), where $x, y, z \in \Omega(t)$ is a point; $ux+vy+wz+r = 0$, where $u, v, w, r \in \Omega(t)$ is a plane; a line is the intersection of two planes [Hilbert 1899, § 12, pp. 24-26]. It is easy to see that such a geometry is non-Archimedean; indeed, on the basis of the above, a line segment the length of which is n times that of the unit segment will never exceed a segment of length t on the same line.

In Chapter VI of *Grundlagen der Geometrie*, Hilbert studied the relation between Pascal's theorem and Archimedes' axiom [Hilbert 1899, §§ 31-34, pp. 71-76]. He assumed axioms I, II, III and Desargues' theorem [16] to be satisfied and therefore a calculus of segments without the axioms of congruence [17] [Hilbert 1899, §§ 24-26, pp. 55-63], proving the following main theorems:

> "Pascal's theorem can be proved on the basis of axioms I, II, III, V, i.e., it can be proved without the congruence axioms and with the help of Archimedes' axiom" [18] [Hilbert 1899, § 31, p. 71].

> "Pascal's theorem cannot be proved based on the axiom I, II, III, i.e. without using either the congruence axioms or Archimedes axiom" [19] [Hilbert 1899, § 31, p. 71].

16 Desargues' theorem states that if two triangles $a_1b_1c_1$, $a_2b_2c_2$ are in perspective from a point V, then the lines containing the opposite edges intersect in three collinear points, d_1, d_2, d_3.

17 In the *Grundlagen* Hilbert defined, using Desargues' theorem, a calculus of segments without the axioms of congruence and showed that all the rules of a field are satisfied, except the commutative law of multiplication [Hilbert 1899, §§ 24-26, pp. 55-63].

18 *Der Pascalsche Satz ist beweisbar auf Grund der Axiome I, II, III, V, d.h. unter Ausschliessung der Congruenzaxiome mit Zuhülfenahme des Archimedischen Axioms.*

19 *Der Pascalsche Satz ist nicht beweisbar auf Grund der Axiome I, II, III, d.h. unter Ausschliessung der Congruenzaxiome sowie des Archimedischen Axioms.*

To prove the first theorem, Hilbert first showed that a Desarguesian[20] number system fulfilling Archimedes' axiom satisfies the commutative law of multiplication [Hilbert 1899, §§ 31-32, pp. 71-73] and then that Pascal's theorem is valid in an analytic geometry over this number system[21] [Hilbert 1899, § 34, p. 75]; in order to prove the second theorem, he considered the set of all finite or infinite expressions $T = r_0 t^n + r_1 t^{n+1} + r_2 t^{n+2} + r_3 t^{n+3} + \cdots$, where t is a parameter, r_0 $(\neq 0)$ and r_i are rational numbers and n is any integer, as well as the set of all finite or infinite expressions $S = s^m T_0 + s^{m+1} T_1 + s^{m+2} T_2 + \cdots$ where T_0 $(\neq 0)$, T_i are expressions of the previous form T and m is any integer. He then defined a number system $\Omega(s,t)$, considering the set of all expressions of the form S together with the number zero, with the following calculus rules: the usual rule of addition for parameters t and s; the multiplication of parameters t and s is defined by $ts = 2 \cdot st$; addition and multiplication of two forms S and S' are defined component wise. He showed that the number system $\Omega(s,t)$ is Desarguesian but non-Archimedean [Hilbert 1899, § 33, pp. 73-75]. He thus showed that an analytic geometry on this number system does not satisfy Pascal's theorem and Archimedes' axiom [Hilbert 1899, § 34, p. 75].

3. DEHN AND NON-ARCHIMEDEAN GEOMETRIES

One of Hilbert's most prominent students was Max Dehn. Born in Hamburg in 1878, he received his doctorate in Göttingen at the age of twenty-one, under Hilbert's supervision, with the dissertation "*Die Legendre'schen Sätze über die Winkelsumme im Dreieck*" on the foundations of geometry [Dehn 1900a]. He obtained his *Habilitation* in Munich in 1901, solving the third of Hilbert's twenty-three problems [Dehn 1900b; 1901]; he was the first to solve one of Hilbert's problems. The third problem concerned the foundations of geometry. His solution showed that Archimedes' axiom was needed to prove that two tetrahedra have the same volume[22], if they have the same altitudes as well as bases of the same

20 In the *Grundlagen* Hilbert called a number system Desarguesian when all the laws of the real numbers are satisfied, except for the commutative multiplication law, and he showed that an analytic geometry on such a number system is Desarguesian and conversely [Hilbert 1899, §§ 28-29, pp. 66-70].

21 In the *Grundlagen* Hilbert defined, using Pascal's theorem, a calculus of segments and showed that all the laws of field are satisfied [Hilbert 1899, § 15, pp. 32-35].

22 We refer to [Hartshorne 2000, §§ 26-27, pp. 226-239] for the general definition of volume and for a study in depht of the problem.

area. Dehn was *Privatdozent* in Munich from 1901 until 1911 and became *Ordinarius* in Breslau in 1913. He moved to the University of Frankfurt in 1921 where he lectured until 1935. Dehn's lectures were especially stimulating because of the new ideas they contained, and under his direction a whole series of valuable dissertations were written. Dehn was well versed in ancient and modern history, and was especially interested in the conception and development of the fundamental insights of classical antiquity. He published several valuable essays on the relationship between Greek philosophy and mathematics. In 1922 the seminar on the History of Mathematics was founded, in Frankfurt, and Dehn was the driving force of this institution. The seminar on the History of Mathematics was held every semester until 1935. The rule of the seminar was to study the most important mathematical discoveries from all epochs in the original version. In 1939, since he was a Jew, he emigrated from Germany to Copenhagen and later to Trondheim in Norway, where he took over the post of a vacationing colleague at the Technical University until 1940. In early 1941, when German troops occupied Trondheim, Dehn emigrated to the United States. There he led an itinerant life until he found a position offering some satisfaction. First he spent one year and a half as Professor of Mathematics and Philosophy at the State University of Idaho at Pocatello. The next year he worked at the Illinois Institute of Technology in Chicago and then at St John's College in Annapolis, Maryland, where he was especially unhappy. Finally, in 1945, Dehn arrived at the final station in his life. This was Black Mountain College in North Carolina, where he stayed for the last seven years of his life, leaving only for short periods as guest lecturer in Madison, Wisconsin. He died in 1952 in Black Mountain, North Carolina [Gillispie 1970–1990; Siegel 1965].

Dehn contributed substantially to three different areas of mathematics, namely foundations of geometry, topology, and group theory.[23] From 1900 to 1906, he worked mainly on the foundations of geometry, to which he returned in 1922 with the article "*Über die Grundlagen der Geometrie und allgemeine Zahlsysteme*" [Dehn 1922]. In 1926, he published "*Die Grundlegung der Geometrie in historischer Entwicklung*" [Dehn & Pasch 1926], in which he focused on insight and ideas. To describe Dehn's character W. Magnus and R. Moufang cited this significant extract from one of his lectures in 1928 on "The spiritual profile of a mathematician".[24]

23 For a survey of the mathematical work of Max Dehn and for an evaluation of the secondary effects of Dehn's work see [Magnus & Moufang 1954] and [Magnus 1978/79].

24 *„Über die geistige Eigenart des Mathematikers"*.

> "The mathematician has from time to time the passion of the poet or of the conqueror, the rigor of thinking of a public official conscious of his responsibility or, more simply expressed, of a family head with worries, the experience and the resignation of an old sage; he is both revolutionary and conservative, totally sceptical and yet has the optimism of a believer" [25] [Magnus & Moufang 1954, p. 225].

Another evocative memory of Dehn is that of André Weil:

> "I have met two men in my life who make me think of Socrates: Max Dehn and Brice Parain. Both of them — like Socrates as we picture him from the accounts of his disciples — possessed a radiance which makes one naturally bow down before their memory: a quality, both intellectual and moral, that is perhaps best conveyed by the word "wisdom"; for holiness is another thing altogether. In comparison with the wise man, the saint is perhaps just a specialist — a specialist in holiness; whereas the wise man has no specialty. This is not to say, far from it, that Dehn was not a mathematician of great talent; he left behind a body of work of very high quality. But for such a man, truth is all one, and mathematics is but one of the mirrors in which it is reflected — perhaps more purely than it is elsewhere" [Weil 1992, p. 52].

On Hilbert's suggestion, Dehn, in his dissertation "*Die Legendre'schen Sätze über die Winkelsumme im Dreieck*", analysed the relationship between Legendre's theorems and Archimedes' axiom. In particular, he asked:

> "Can one prove Legendre's theorems without an axiom of continuity, i.e. without making use of the Archimedian axiom?" [26] [Dehn 1900a, p. 405]

To answer this question, Dehn first showed that Legendre's second theorem is only a consequence of the incidence, order and congruence axioms by proving, in a geometry where such axioms hold, the following more general theorem:

> "If the angle sum of one triangle is less than two right angles then this is true for every triangle.
>
> "If the angle sum of one triangle is equal to two right angles then it is so for every triangle.
>
> "If the angle sum of one triangle is greater than two right angles then the same holds for every triangle" [27] [Dehn 1900a, pp. 430-431].

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25 *„Der Mathematiker hat zuweilen die Leidenschaft des Dichters oder Eroberers, die Strenge in seinen Überlegungen wie ein verantwortungsbewußter Staatsmann oder, einfacher ausgedrückt, wie ein besorgter Hausvater, die Nachsicht und Resignation eines alten Weisen; er ist revolutionär und konservativ, ganz skeptisch und doch gläubig optimistisch."*

26 *"Kann man die Legendre'schen Sätze ohne irgend ein Stetigkeitspostulat beweisen, d.h. ohne vom Archimedischen Axiom Gebrauch zu machen?"*

27 *"Ist in irgend einem Dreieck die Winkelsumme kleiner als zwei rechte Winkel, so ist in jedem Dreick die Winkelsumme kleiner als zwei rechte Winkel.*

Note that the second statement is Legendre's second theorem.

Subsequently, Dehn showed that it is impossible to prove Legendre's first theorem with the incidence, the order, and the congruence axioms and without Archimedes' axiom by constructing a "Non-Legendrian geometry" in which there are infinitely many lines parallel to a fixed line through a point, Archimedes' axiom does not hold and the sum of the inner angles of a triangle is greater than two right-angles, and by constructing a "semi-Euclidean geometry" in which there are infinitely many lines parallel to a fixed line through a point, Archimedes' axiom does not hold but the sum of the inner angles of every triangle is still equal to two right angles [Dehn 1900a, pp. 431-438].

3.1. *Legendre's second theorem*

Dehn constructed a "Pseudogeometry" in which he defined a relation of pseudoparallelism and of pseudocongruence in order to show that Legendre's second theorem is valid in a geometry in which Archimedes' axiom does not hold [Dehn 1900a, pp. 410-411].

In particular, his aim is to immerse a geometry in which the axioms of connection hold into a "projective geometry", completing it by introducing "ideal" points and "ideal" lines.

He remarked that given two lines in a plane π, in which the axioms of connection hold, they may or may not intersect and that given, in the plane π, a point P outside a line r, there exists at least a line s through P which does not intersect r. Moreover, he noted that it is possible to project from a point the plane π into a plane π', so that the projections r' and s' of r and s are two lines that intersect in a point $Q' \in \pi$. In this way he could say that the two lines r and s intersect in an "ideal" point in π and that a line $t \in \pi$ passes through Q when his projection passes through Q'. In the same way, he called "ideal" line a line that passes through two not intersecting planes or a line through two "ideal" points, which does not lie on a "real" line. Then he showed that, if the axioms of congruence are valid for the "real" points and the "real" lines of π, all the perpendiculars to one fixed line m intersect in one "ideal" point, named pole of m and that the poles of every line intersecting in a "real" point O belong to one "ideal" line, named the polar line of O and vice versa [Dehn 1900a, pp. 406-409].

Ist in irgend einem Dreieck die Winkelsumme gleich zwei rechten Winkeln, so ist sie in jedem Dreick gleich zwei rechten Winkeln.
Ist in irgend einem Dreieck die Winkelsumme grösser als zwei rechte Winkel, so ist sie in jedem Dreick grösser als zwei rechte Winkel."

Therefore, he considered the plane geometry completed with "ideal" points and "ideal" lines and he fixed a base point O and the polar line t of O. He then stated:

> "The real points in our Pseudogeometry are all real and ideal points of the funding geometry, without the points on t.
> The real lines in our Pseudogeometry are all real and ideal lines of the funding geometry, without the line t"[28] [Dehn 1900a, p. 411].

Dehn then showed that in the constructed Pseudogeometry the axioms of connection and of order hold and he introduced the relation of pseudoparallelism as follows:

> "We call pseudoparallel two lines a and a' that intersect on the line t"[29] [Dehn 1900a, p. 411].

In this Pseudogeometry two segments AB and $A'B'$ are called *pseudocongruent* if one is mapped into the other one by a pseudoparallel transformation or if they are mapped by a pseudoparallel transformation into two segments OE and OF such that the line EF passes through the Pole of the bisector of the angle $\widehat{EOF}$. The angles of the pseudogeometry are only those which have real vertices and real sides, and two angles $\widehat{ABC}$ and $\widehat{A'B'C'}$ are called pseudocongruent (in symbols $\widehat{ABC} \approx \widehat{A'B'C'}$) if one is mapped into the other by a Pseudoparallel transformation or if they are mapped by a Pseudoparallel transformation into two angles $\widehat{EOF}$ and $\widehat{GOH}$ with O common vertex, and the congruence relation is the one of Euclidean plane geometry [Dehn 1900a, p. 413, p. 418, p. 419].

Dehn defined two triangles as pseudocongruent if they have respectively sides and angles pseudocongruent and stated the first theorem of pseudocongruence:

> "If two triangles have two sides and the angle inside them pseudocongruent respectively, then they have the other sides and the other angles pseudocongruent respectively"[30] [Dehn 1900a, p. 419].

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28 *„Wirkliche Punkte im Sinne unserer Pseudogeometrie sind alle wirklichen und idealen Punkte der zu Grunde gelegten Geometrie mit Ausnahme der Punkte auf t.*
Wirkliche Geraden im Sinne unserer Pseudogeometrie sind alle wirklichen und idealen Geraden der zu Grunde gelegten Geometrie mit Ausnahme der Geraden t".

29 *„Zwei Geraden a und a', welche sich auf der Geraden t schneiden, nennen wir pseudoparallel".*

30 *„Sind in zwei Dreiecken zwei Seiten und der eingeschlossene Winkel beziehungsweise pseudocongruent, dann sind auch die übrigen Stücke in den beiden Dreiecken beziehungweise pseudocongruent"*

He concluded that:

> "The elements of our pseudogeometry, except those of the line t, satisfy all the axioms of the ordinary Euclidean geometry, including the parallel axiom, and line t plays the role of the 'ideal' or 'line at infinity' [31] [Dehn 1900a, p. 422].

From this consideration it follows immediately that "the sum of the inner angles of any triangle is pseudocongruent to two right angles" [32] [Dehn 1900a, p. 422].

Dehn then introduced the symbols $\approx$ of pseudoequality, $\prec$ of pseudosmaller, and $\succ$ of pseudogreater, and showed the interrelation between the congruence and the pseudocongruence relations: "Let A and A_1, two real points on the line n through O and OA (or a), be congruent to the segment AA_1 (or a_1) and let B and B_1, two other points on the same line or on another line m through O and OB (or b), be congruent to BB_1 (or b_1), then we have: if $a \approx a_1$ or $a \prec a1$ or $a \succ a1$ then $b \approx b1$ or $b \prec b_1$ or $b \succ b_1$, respectively" [33] [Dehn 1900a, p. 426].

All these considerations were necessary to demonstrate Legendre's second theorem without Archimedes' axiom. In particular, as we said, Dehn showed that: if the sum of the inner angles of one triangle is less, equal or greater than two right angles, respectively, then so it is for every triangle [34].

We will now briefly outline Dehn's argument.

First, he noted that since every triangle can be divided into two right-angled triangles, drawing the perpendicular from one vertex to the opposite side (Fig. 1), the theorem can be proved just for right-angled triangles.

Since every right-angled triangle can be mapped into this by a congruent transformation, Dehn considered a right-angled triangle with O vertex of the right angle (AOB). He then observed that it is necessary to show that the sum of the inner angles of the triangle (AOB) is less than, equal to, or greater than two right-angles, in order to obtain the first, the second, or the third "hypothesis" of the theorem and consequently to obtain that the sum of the inner angles of every triangle is less than, equal to, or greater

31 *"Die Elemente unserer Pseudogeometrie mit Ausnahme der Elemente der Geraden t erfüllen alle Axiome der gewöhnlichen Euklidischen Geometrie einschliesslich des Parallelenaxioms, und die Gerade t spielt die Rolle der „idealen" oder „unendlich fernen" Geraden"*

32 *"Die Winkelsumme in irgend einem Dreieck ist pseudocongruent zwei Rechten"*

33 *„Seien A und A_1 wirckliche Punkte auf der Geraden n durch O und OA (oder a) der Strecke AA_1 (oder a_1) congruent, ferner B und B_1 zwei beliebige andere Punkte auf derselben oder einen anderen Geraden m durch O und OB (oder b) congruent BB_1 (oder b_1), dann behaupten wir: Ist $a \approx a_1$ oder $a \prec a_1$ oder $a \succ a_1$ dann ist bezüglich auch $b \approx b_1$ oder $b \prec b_1$ oder $b \succ b_1$ "*.

34 See p. 268, note 27.

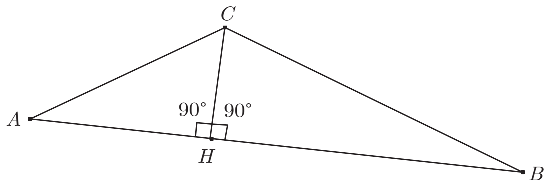

FIGURE 1.

than two right angles. Since the sum of the inner angles of a right-angled triangle is pseudocongruent to two right angles, and pseudocongruence and congruence are the same relation for the angle with a vertex in O, he showed that each angle $\widehat{OAB}$ and $\widehat{OBA}$ is pseudosmaller (or pseudoequal to or pseudogreater) than a part of the angle $\widehat{AOB}$ [Dehn 1900a, pp. 430-431].

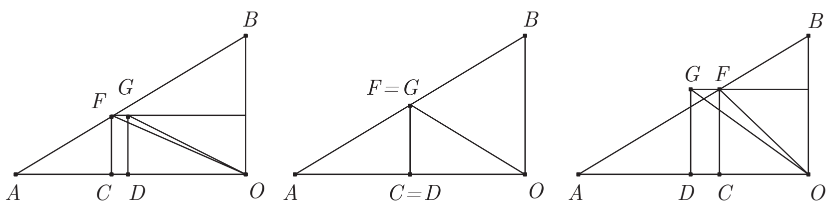

FIGURE 2.

In particular, he considered the point C that divides the side AO in two halves. He considered on OA the segment DO pseudocongruent to AC. D is thus between C and O (either D is on C or C is between D and O) and he drew lines through C and through D perpendicular to the side AO (see Fig. 2). Let F be the intersection point between the perpendicular through C and the side AB and G be the intersection point between the perpendicular through D and the line through F perpendicular to the side OB (see Fig. 2). Then DG is pseudocongruent to CF. The triangle (ACF) is congruent to the triangle (OCF), and therefore the angle $\widehat{OAB}$ is congruent to the angle $\widehat{DOF}$ and the triangle (ACF) is pseudocongruent to the triangle (GOD), and therefore the angle $\widehat{OAF}$ is pseudocongruent to the angle $\widehat{DOG}$. Also, the angle $\widehat{DOF}$ is $\prec$ (pseudosmaller) or $\approx$ (pseudoequal), or $\succ$ (pseudogreater) than the angle $\widehat{DOG}$. Therefore, the angle $\widehat{OAB}$ is mapped by a congruence transformation onto the angle $\widehat{AOF}$ and is pseudosmaller or pseudoequal or pseudogreater. The same holds for the angle $\widehat{OBA}$ [Dehn 1900a, p. 431].

3.2. *Non-Legendrian Geometry*

As is well known, Legendre's first theorem states that the sum of the inner angles of a triangle is less than or equal to two right angles. Dehn showed that Archimedes' axiom is necessary to prove this theorem, and analysed the relationships between Archimedes' axiom, the number of lines through a point parallel to a fixed line, and the sum of the inner angles of a triangle [Dehn 1900a, pp. 431-436]. It is known that, if Archimedes' axiom holds, there are the following relationships between the hypothesis on the existence and the number of parallel lines through a point and the sum of the inner angles of a triangle: if the sum of the inner angles of a triangle is greater than, equal to, or less than two right angles, then no line parallel to a fixed line passes through a point, there is exactly one line parallel to a fixed line and passing through a point, and there are infinitely many lines parallel to a fixed line through a point, respectively.

Dehn constructed a non-Archimedean geometry where there are infinitely many parallel lines through a point and where the sum of the inner angles of a triangle is greater than two right angles. He then stated:

> "Thus the improvability of Legendre's first theorem is shown; and also the hypothesis of the obtuse angle, as Saccheri called it, is not equivalent to the hypothesis of the finiteness of the line" [35] [Dehn 1900a, p. 432].

Like Hilbert, Dehn considered the following non-Archimedean number system: the set $\Omega(t)$ of the algebraic functions of t obtained by applying to t the five operations of addition, subtraction, multiplication, division, and the operation $\sqrt{1+\omega^2}$, where ω is a function obtained by the previous five operations. He then constructed an analytic geometry over the set $\Omega(t)$ as follows: the points are the pairs (x, y), with x, y in $\Omega(t)$, and the lines have the equations $ux + vy + w = 0$, with u, v, w in $\Omega(t)$ [Dehn 1900a, p. 432]. In the previous geometry all the axioms hold with the exclusion of Archimedes' axiom.

35 *"Damit ist dann die Unbeweisbarkeit des ersten Legendre'schen Satzes bewiesen und gezeigt, dass die Hypothese des stumpfen Winkels, wie sie Saccheri nennt, sich nicht deckt mit der Hypothese der Endlichkeit der Geraden".*

Dehn constructed, over this non-Archimedean plane, an "elliptic" or "Riemannian" geometry,[36] as follows. He took the imaginary conic

$$x^2 + y^2 + 1 = 0,$$

and considered as points and lines of the elliptic geometry all the points and lines of the non-Archimedean plane, together with the line at infinity with its points, and considered as congruences of his elliptic geometry the real transformations that establish the conic.[37] In this geometry, axioms I, IV, and the modified axiom II are valid. He then considered, as points of the new geometry, the points of the "elliptic" geometry (x, y) satisfying the following conditions:

$$-\frac{n}{t} < x < \frac{n}{t},$$
$$-\frac{n}{t} < y < \frac{n}{t},$$

where n is an integer. He considered as lines those lines the points of which satisfy the above conditions. The segments and angles are defined as in "elliptic" geometry. Therefore, if two segments or two angles are congruent in elliptic geometry, they will be congruent in this new geometry [Dehn 1900a, p. 433]. Dehn showed that every segment that is congruent to a segment of the limited zone is in the limited zone and that points of the new geometry correspond to points in the following rotations [Dehn 1900a, p. 434]:

$$x' = \frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y$$
$$y' = \frac{b}{\sqrt{a^2+b^2}}x + \frac{a}{\sqrt{a^2+b^2}}y$$

Dehn therefore showed that all the axioms are valid except Euclid's parallel axiom and Archimedes' axiom and that the sum of the inner angles of a triangle is greater than two right angles [Dehn 1900a, pp. 433-436]. Thus, as he wrote:

> "We have constructed a geometry where axioms I, II and IV hold, where through one point there exist infinitely many lines parallel to a fixed line but where nevertheless the angle sum of a triangle is greater than two right-angles.

36 *In dieser "Nicht-Archimedischen" Ebene construiren wir uns zunächst eine gewöhnliche „elliptische" oder „Riemannsche" Geometrie* [Dehn 1900a, p. 433]. (In this „non-Archimedean plane" we will construct first an usual „elliptic" or „Riemaniann" geometry.)

37 This construction in the Archimedean case was done by Klein [Klein 1871].

Archimedes' axiom does not hold. Thus it is shown that Legendre's first theorem cannot be proven without the help of Archimedes' axiom. Thus we can call the auxiliary geometry used in this proof, a "non-Legendrian" geometry [...]" [38] [Dehn 1900a, p. 436].

Further, Dehn showed that in the elliptic geometry constructed over the non-Archimedean plane the sum of the inner angles of a triangle is greater than two right angles:

> "If in a geometry where there are no parallel lines and where axioms I and IV and the correspondingly modified axioms II hold, the angle sum of a triangle is always greater than two right-angles" [39] [Dehn 1900a, p. 438].

Therefore, in the case of the non-existence of parallel lines a theorem analogous to the first Legendre theorem is valid.

3.3. *Semi-Euclidean Geometry*

Dehn continued his analysis of the relationship between the hypothesis on the sum of the inner angles of a triangle and the hypothesis on the existence and the number of parallel lines through a point by constructing another geometry. He considered the above non-Archimedean plane and constructed over it a new geometry as follows: the points of the new geometry are the points (x, y) of the non-Archimedean plane satisfying the following conditions,

$$-n < x < n,$$
$$-n < y < n,$$

where n is a positive integer and the lines are the lines of the non-Archimedean plane the points of which satisfy the conditions above. The segments and the angles are defined as in Euclidean geometry. Therefore, if two segments are congruent in Euclidean geometry then they will also be congruent in the new geometry [Dehn 1900a, p. 436]. Dehn further

38 *„Folglich haben wir eine Geometrie construirt, die allen Axiomen I, II, IV Genüge leistet, in der ferner durch jeden Punkt zu jeder Geraden unendlich viele Parallelen möglich sind, in der aber nichtsdestoweniger die Winkelsumme in jedem Dreieck grösser als zwei rechte Winkel ist. Das Archimedische Axiom gilt dann natürlich nicht.*
Damit ist die Unbeweisbarkeit des I. Legendre'sche Satzes ohne Zuhülfenahme des Archimedischen Axioms nachgewiesen.Die zu diesem Beweise benutzte Hülfsgeometrie können wir deshalb „Nicht-Legendre'sche" Geometrie nennen [...]".

39 *„Giebt es in einer Geometrie keine Parallelen und gelten in derselben alle Axiome der Gruppen I, IV, und die entsprechend modificirten der Gruppe II, dann ist die Winkelsumme stets grösser als zwei rechte Winkel."*

showed that every point of the new geometry is mapped into a point of it, under the following transformations:

$$\begin{aligned} x' &= x + a, \\ y' &= y + b, \\ x' &= \frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y \\ y' &= \frac{b}{\sqrt{a^2+b^2}}x + \frac{a}{\sqrt{a^2+b^2}}y \end{aligned}$$

and thus axioms I, II, and IV hold in this geometry [Dehn 1900a, p. 437]. Therefore, Legendre's first theorem is valid:

> "[...] Furthermore all theorems of the usual Euclidean geometry, as long as they involve only a bounded area (*ein beschränktes Raumstück*) are still valid [in this geometry]. The angle sum is equal to two right-angles in every triangle [...]" [40] [Dehn 1900a, p. 437].

However, it is easy to see that through a point there exist infinitely many lines parallel to a fixed line. To show this, Dehn considered the line through the points $(t, 0)$ and $(0, 1)$; this is a line of the new geometry, since it passes through the points $(0, 1)$ and $(1, \frac{t-1}{t})$, which are points of the new geometry, but intersects the x axis in a point that is not a point of the new geometry. He then considered the line through the points $(-t, 0)$ and $(0, 1)$; this line is a line of the new geometry which intersects the x axis in a point that is not a point of the new geometry. The two previous lines pass through the point $(1, 0)$ and are parallel to the x axis. Thus, Dehn showed that:

> "There are non-Archimedean geometries in which the parallel axiom is not valid but where the angle sum of a triangle is equal to two right-angles" [41] [Dehn 1900a, p. 438].

Dehn therefore constructed a geometry where the theorem on the sum of the inner angles of a triangle of Euclidean geometry holds, but where the parallel axiom does not; Dehn called this geometry "semi-Euclidean" geometry.

Dehn summed up the above results in the following diagram:

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40 *„Ferner haben aber auch sämmtliche Sätze der gewöhnlichen Euklidischen Geometrie, soweit sie mit einem "beschränkten" Raumstück zu thun haben, in derselben Gültigkeit. Die Winkelsumme ist in jedem Dreieck gleich zwei rechten Winkeln [...]".*

41 *„Es gibt Nicht-Archimedische Geometrien, in denen das Parallelenaxiom nicht gültig ist und dennoch die Winkelsumme in jedem Dreieck gleich zwei rechten Winkeln ist."*

The sum of the inner angles of a triangle is:	Lines through a fixed point and parallel to a given line: No parallel lines	One parallel line	Infinitely many parallel lines
$> 2R$	Elliptic Geometry	(impossible)	Non-Legendrian geometry
$= 2R$	(impossible)	Euclidean geometry	Semi-Euclidean geometry
$< 2R$	(impossible)	(impossible)	Hyperbolic geometry

These results were probably achieved by Dehn in 1899. Hilbert mentioned them in a letter to Hurwitz written on 5 November 1899 [quoted in Toepell 1986, p. 257]. Hilbert also summarized the results in detail in his conclusion to the French and English translations of the *Festschrift* [Hilbert 1899], and from the second edition of the *Grundlagen* on there are short remarks on Dehn's work at the end of Chapter III. [42]

4. HILBERT LECTURES ON THE FOUNDATIONS OF GEOMETRY (1902)

Hilbert gave some lectures on the foundations of geometry in the summer semester of 1902. There is an elaboration by August Adler (1863-1923) of these lectures [Hilbert 1902]. Inspired by Dehn's result, Hilbert constructed another model of "semi-Euclidean" geometry, emphasizing the fact that the theorem about the sum of the inner angles of a triangle is not equivalent to the parallel axiom:

> "Therefore the theorem of the sum of the inner angles of a triangle is not equivalent to the parallel axiom [...]" [Hilbert in Hallet & Ulrich 2004, p. 568].

It follows from these lectures that Hilbert was struck by this kind of geometry and consequently by Dehn's result, which he called a "remarkable geometry". [43]

Hilbert first constructed a non-Archimedean system of numbers, as follows. A number of the system is an expression:

$$\alpha = a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + \cdots$$

where t is a symbol and $a_0, a_1, a_2, \ldots, n$ are real positive or negative numbers. He showed the following properties:

42 See [Hallet & Ulrich 2004, pp. 527-529 & 601-602].

43 „*Merkwürdige Geometrie*" [Hilbert in Hallet & Ulrich 2004, p. 566].

(1) The four operations are valid and the exponentiation of 2 is defined:

$$\alpha^2 = a_0 t^{2n} + 2a_0 a_1 t^{2n+1} + \cdots;$$

(2) $\alpha > 0$ if and only if $a_0 > 0$ and $\alpha < 0$ if and only if $a_0 < 0$;
(3) from the second point it follows that:

$$\text{if } \alpha - \alpha' > 0 \text{ then } \alpha > \alpha',$$
$$\text{if } \alpha - \alpha' < 0 \text{ then } \alpha < \alpha'.$$

Then, from 2) and 3) it follows that the axiom of order is valid but Archimedes' axiom fails. For let us consider a number of the system $mt - a$, where a and m are positive real numbers. Then $mt - a = -at^0 + mt^1 < 0$ and so $mt < a$. Therefore, having fixed t and a, it is impossible to find m such that $mt > a$ [Hilbert in Hallet & Ulrich 2004, pp. 564-565].

To construct a semi-Euclidean geometry, Hilbert considered the "integer" numbers of the above system, that is the numbers of the type:

$$\alpha = a_0 t^n + a_1 t^{n+1} + \cdots ,$$

where $n \geq 0$, and then constructed over this number system an analytical geometry in which the points are the pairs (x, y), with x and y integer numbers, and the lines are the linear equations $ux + vy + w = 0$ through two points with integer coordinates.

Archimedes' axiom is not valid in this geometry, since it is not valid in the constructed number system and the axioms I (of connection), II (of order) and III (of congruence) are valid. Hilbert also showed, as follows, that the parallel axiom is not valid; he considered the two lines

$$tx + y = 1 \text{ and } - tx + y = 1.$$

These lines pass, respectively, through the points $(0, 1)$, $(1, 1 - t)$ and the points $(0, 1)$, $(1, 1 + t)$ which are points of the geometry. These lines intersect the x axis, respectively, in the points $\left(\frac{1}{t}, 0\right)$ and $\left(-\frac{1}{t}, 0\right)$ which are not points of our geometry. Therefore, the parallel axiom fails (see Fig. 3) [Hilbert in Hallet & Ulrich 2004, pp. 566–567].

On the contrary, the theorem on the sum of the inner angles of a triangle is valid in this geometry, as Hilbert remarked:

> "On the contrary the angle sum of every triangle of our geometry is $2R$" [44] [Hilbert in Hallet & Ulrich 2004, p. 567].

44 *„Dagegen ist die Winkelsumme eines jeden Dreieckes unserer Geometrie* $2R$ *"*.

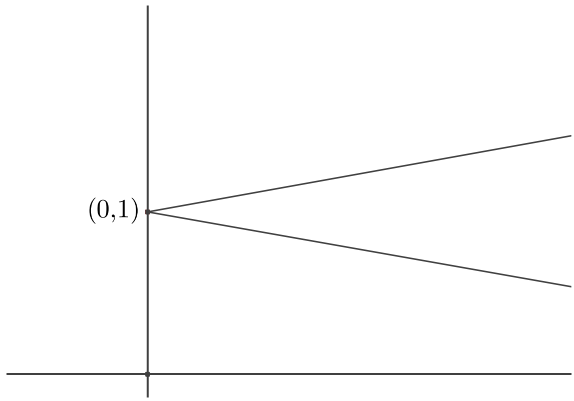

FIGURE 3.

To show this, Hilbert remarked that the angles of this geometry are defined by the rotation:

$$x' = \frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y,$$
$$y' = \frac{b}{\sqrt{a^2+b^2}}x + \frac{a}{\sqrt{a^2+b^2}}y,$$

and that two lines are parallel if one is mapped into the other by the following transformation:

$$x' = x + a,$$
$$y' = y + b.$$

Hilbert concluded thus:

> "Then the internal alternate angles are equal to each other, which arise when two parallel lines are intersected by a transversal line; therefore Euclid's proof of the sum of the inner angles of a triangle is valid" [45] [Hilbert in Hallet & Ulrich 2004, p. 567].

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45 *„Dann sind offenbar die Wechselwinkel einander gleich, welche beim Schnitte von 2 unserer „Parallelen" mit einer dritten Geraden entstehen; es gilt also auch der Euklidische Beweis für die Winkelsumme im Dreiecke".*

5. THE *PRINZIPIEN DER GEOMETRIE* OF ENRIQUES

In 1907 Federigo Enriques (1871-1946) published in the *Enzyclopaedie der mathematischen Wissenschaften* the article "*Principien der Geometrie*" [46]. The article was planned after 1892 and the author continued to work on it until its publication. In this article, Enriques was inspired by the ideas of Klein, finding in him an authoritative and precious interlocutor [47]. The article puts together the results on the foundations of geometry obtained until then; it is divided in seven chapters and the last of them is on non-Archimedean geometry. Moreover, the author distinguishes "the elementary orientation" [48], or questions directly deducible by the geometric properties from "the superior orientation" [49], which is needed to study in depth questions concerning the "theory of continuum", the "projective geometry" and so on. In particular:

> "After the presentation of these different orientations, the last chapter reports on the new developments, which, by abstraction from the common concept of continuum, have given rise to the construction of non-Archimedean geometries" [50] [Enriques 1907, p. 15].

Enriques' point of view on geometry emerges from the previous considerations: Geometry is a science about physical or intuitive facts and its results, logically established, must not be considered as a mature achievement until they can be understood intuitively [Enriques 1900].

In fact, the author emphasizes the difference between the so-called "elementary" questions and the "advanced" ones, evidencing the not immediate intuitive understanding of these last ones. Enriques, in the "elementary questions", devoted a paragraph to "continuity and Archimedes' axiom" in which he explained the postulates of continuity of Dedekind, Cantor and Weierstrass, as well as the relationships between these postulates and Archimedes' axiom. Moreover, he described Veronese's geometric model of non-Archimedean geometry and exposed, as Schoenflies [Schoenflies 1906] did, the difference between Veronese's and Dedekind's definitions of continuity:

46 A French translation has been published in 1911 in *Encyclopédie des sciences mathematiques* [Enriques 1911].

47 See [Bottazzini 2001].

48 "die elementare Richtung"

49 "die höhere Richtung"

50 *„Nach der Darstellung dieser verschiedenen Richtungen berichtet der letzte Abschnitt über die neuen Entwicklungen, die durch Abstraktion von dem gewöhnlichen Begriffe des Kontinuums zur Konstruction der nicht-Archimedischen Geometrien geführt haben."*

"The difference between Cantor and Dedekind's concept of continuity and Veronese's one can be presented as follows. Divide all the points of a segment OM in accordance with $2'$) [Dedekind's continuity] in two classes namely M' and M'' [that is in two non intersecting classes such that O belongs to the first class, M to the second, every point of the segment belongs to one of the two classes and at the end, every point of the first class belongs to a segment which joins O with a point of the second class.], then we can have four cases:

(1) M' has a last point, A', and M'' has a first point, A'' (we have a "jump");
(2) M' has a last point A', M'' does not have a first point;
(3) M' does not have a last point, M'' has a first point A'';
(4) neither M' has a last point, nor M'' has a first point (we have a "gap").

Dedekind's conception of continuity excludes the existence of jumps as well as of gaps. Veronese's conception excludes the existence of jumps, but that of "gaps" only under specific conditions"[51] [Enriques 1907, p. 38].

Note that Enriques gave to Veronese's conception of continuity the same importance than to Cantor's and Dedekind's axioms.

The last chapter of the *Prinzipien* is devoted, as already said, to non-Archimedean geometry. It is divided into six paragraphs: Introduction; The one dimension continuum of superior kind; Veronese's ideas; Non-Archimedean projective geometry; non-Archimedean Euclidean geometry; non-Archimedean development about the theory of parallels [Enriques 1907]. Enriques, in the second paragraph, reported on the results in non-Archimedean geometry obtained in the period from the study of the angle of contingency (that is the angle between a curve and its tangent or the angle between two curves) to the models of Veronese, Levi Civita and Hilbert [Enriques 1907, pp. 117-120]. In the third paragraph he exposed the ideas of Veronese on non-Archimedean geometry at more than one dimension [Enriques 1907, pp. 121-122]. In the fourth paragraph he spoke about the relationships between Pappus' and Desargues' theorems and Archimedes' axiom [Enriques 1907, pp. 122-124] and in the fifth paragraph he dealt with Hilbert's in depth study of non-Archimedean Euclidean geometry [Enriques 1907, pp. 124-126]. Finally, Enriques, in

51 *„Der Unterschied zwischen dem Cantor-Dedekindschen und dem Veroneseschen Stetigkeitsbegriff kann man auch folgendermaßen formulieren.*
Werden alle Punkte einer Strecke OM gemäß $2'$) in zwei Klassen M' und M'' geteilt, so sind folgende vier Fälle möglich:
1) M' hat einen letzen Punkt A', und M'' hat einen ersten A'' (es liegt ein Sprung vor)
2) M' hat einen letzten Punkt A', M'' keinen ersten;
3) M' hat keinen letzten Punkt A', M'' einen ersten A'';
4) weder hat M' einen letzten, noch M'' einen ersten Punkt (es liegt eine Lücke vor).
Die Dedekindsche Stetigkeit schließt nun sowohl Lücken wie Sprünge aus. Die Veronesesche schließt Sprünge immer aus, Lücken aber nur unter gewissen Bedingungen."

the last paragraph, exposed Dehn's results on the relationships between Saccheri's or Legendre's theorems and Archimedes' axiom [Enriques 1907, pp. 126-128].

It emerges, from the previous summary that Enriques dealt exhaustively with all the results obtained before 1907 on non-Archimedean geometry, but did surprisingly not quote the results [52] of his student Bonola [53] on Saccheri's theorem. Perhaps, Enriques considered Bonola's results on Legendre's theorem too elementary and not sufficiently original [Amaldi 1911, p. 149].

6. BONOLA'S RESEARCH ON SACCHERI'S THEOREM

Roberto Bonola was born in 1874 in Bologna and died there prematurely in 1911. He graduated in Mathematics in 1898 under the supervision of Enriques, who choose him as his assistant. In 1900 he became a teacher of mathematics in schools for girls, first in Petralia Sottana, then in Pavia, where he spent the best years of his short life. In 1902 he became assistant to the course of Calculus at the University of Pavia and in 1904 he gave lectures on the Foundations of Geometry. Moreover, from 1904 to 1907, he taught a mathematics course for Chemistry and Natural Science students. In 1909 he obtained the "*Libera Docenza*" of Projective Geometry and in 1910 he became Ordinary Professor on the *Regio Istituto Superiore di Magistero femminile* in Rome. He was seriously sick since 1900 and he died while he established in Rome [Amaldi 1911].

We quote Amaldi's description of Bonola's character:

> "[Bonola's activity] was cut off in the most pleasant and promising period, an activity nobly prepared to produce works in the meditation of a tireless work, in the fervent cult of the purest ideals. Roberto Bonola did not experience the impatience of quick success, he did not sacrifice to any kind of calculation, even the noblest, the free expression of his intellectual aspirations; but he picked up all the energies of his mind in an intense internal life, pursuing tirelessly his assiduous tendency to widen and raise the sphere of his thought" [54] [Amaldi 1911, p. 145].

52 Enriques did not quote either Bonola's result on Saccheri's theorem in the French translation [Enriques 1911].

53 Enriques quoted Bonola in the paragraph regarding the Parallel axiom [Enriques 1907, pp. 39-40]

54 *"Si troncava così, nell'ora più lieta e più promettente, una attività nobilmente preparata alle opere feconde nel raccoglimento di un lavoro indefesso, nel culto fervido di purissimi ideali. Roberto Bonola non conobbe le impazienze dei rapidi successi, non sacrificò ad alcuna specie di*

Bonola was among the very few Italians who were deeply interested in Dehn's work. At the time, he was working under the supervision of Enriques on non-Euclidean geometry from an historical point of view [Bonola 1906], which can be considered his main work:

> "The high level that he has been able to reach is proved by the interest that his historical and critical studies raised among the mathematicians of all the world, so that his main work has been translated into German, into English and also into Russian" [55] [Enriques in Viti 1911, p. 80].

Bonola was thus deeply interested in understanding the role of Archimedes' axiom in the proof of Saccheri's and Legendre's theorems. In this work [Bonola 1905], he demonstrated, in a direct way without the use of Archimedes' axiom, Saccheri's theorem on the sum of the inner angles of a triangle: [56]

> "This note aims at giving a direct and elementary proof of the result by Dehn, that is of the proof of Saccheri's theorem, without the use of Archimedes' axiom" [57] [Bonola 1905, p. 652].

Bonola shares his master's vision of geometry as a deeply intuitive discipline. Thus, only a direct proof could really satisfy our intuitive vision:

> "The way followed by Dehn to prove, without Archimedes' axiom, Saccheri's theorem is very elegant and logically complete. Geometrical intuition, however, needed a direct proof, that is a proof without those formal systems constructed on abstract concepts, that only formally satisfy the geometrical properties" [58] [Bonola 1905, p. 652].

His starting point was the research of Father Saccheri [Saccheri 1733] on Euclid's fifth axiom. He considered the birectangular isosceles quadrilateral $ABCD$ ($\hat{B} = \hat{D} = 1$ right angle and $AB = CD$), that is now called the

calcolo, fosse pure elevato, il libero estrinsecarsi delle sue aspirazioni intellettuali; ma tutte raccolse le energie dello spirito in una intensa vita interiore, perseguendo infaticatamente l'assidua sua tendenza ad allargare e ad innalzare la sfera del suo pensiero."

55 *"L'altezza che Egli ha saputo raggiungere è provata dall'interesse che I suoi studi storico-critici destarono presso i matematici di tutto il mondo, onde il suo trattato principale è tradotto in tedesco, in inglese e fin anco in Russo."*

56 Bonola called Saccheri's theorem the second theorem of Legendre.

57 *"La presente nota ha per iscopo una dimostrazione diretta ed elementare del risultato di Dehn, cioè del teorema di Saccheri, indipendentemente dall'ipotesi archimedea."*

58 *"Il metodo seguito da Dehn, per dimostrare indipendentemente dal Postulato di Archimede il teorema di Saccheri, è molto elegante e soddisfa pienamente dal punto di vista logico. Il senso geometrico (intuitivo) per essere soddisfatto, richiede però una dimostrazione diretta, una dimostrazione cioè che bandisca l'uso di quei sistemi convenzionali, edificati su concetti astratti, che solo formalmente soddisfano alle proprietà geometriche destinate a surrogare."*

Saccheri quadrilateral and distinguished the following three cases [Bonola 1905, p. 650]:

(1) Hypothesis of the right angle, such that: $\hat{A} = \hat{C} = 1$ right angle.
(2) Hypothesis of the acute angle, such that: $\hat{A} = \hat{C} = 1$ acute angle.
(3) Hypothesis of the obtuse angle, such that: $\hat{A} = \hat{C} = 1$ obtuse angle.

He then demonstrated Saccheri's theorem: "If one of the three previous hypotheses is valid in a Saccheri quadrilateral, this hypothesis is valid in every Saccheri quadrilateral" without using Archimedes' axiom, and since Saccheri's theorem on the sum of the inner angles of a triangle is a consequence of this theorem, the aim is achieved [Bonola 1905].

To prove Saccheri's theorem, Bonola considered a plane in which the axioms of connection, order, and congruence are satisfied, distinguishing two cases: "the closed line" and "the open line". We will briefly outline Bonola's argument.

6.1. *The closed line*

Bonola first noticed that the hypothesis of "the closed line" is equivalent to that of "every two lines intersecting in one point" and that following the method of the *Grundlagen* it is possible to prove that every two right angles are equal and thus to prove the congruence theorems of triangles. Since in this plane the theorem "all lines perpendicular to a given line are concurrent" is valid, the polarity between points and lines is defined (every given line corresponds to the point of intersection of the perpendicular lines to the given line and vice versa), he called two corresponding elements "pole and polar line". Two lines perpendicular to a line are called "conjugate lines", and two points that divide the line in two equal parts are called "conjugate points". He then defined two segments as supplementary if their sum is equal to the whole line, and he named "half-line" each of the two equal parts in which a line is divided by the "conjugate points". In this plane, a triangle such that every vertex is a pole of the opposite side has every angle equal to a right angle and every side equal to a half-line. Such a triangle was named PQR. [Bonola 1905, pp. 652–653].

In this geometry, Bonola showed the following theorems:

> "If in a triangle ABC the side AB is smaller than the half-line and the two adjacent angles to this side are both obtuse or one right and the other one obtuse,

then the two sides *AC* and *BC* are both greater than the half-line"[59] [Bonola 1905, p. 654].

"A triangle the sides of which are smaller than the half-line has two acute angles"[60] [Bonola 1905, p. 654].

"A triangle *ABC* the sides of which are smaller than the half-line, belongs to a triangular zone *PQR*"[61] [Bonola 1905, p. 654].

Bonola called a triangle the sides of which are smaller than the half-line a *normal triangle*. Moreover, he noticed that in the normal triangle Euclid's prop. XVI[62] is valid: "Every external angle of a normal triangle is greater than every opposite inner angle", since Euclid's proof is valid in the case of the normal triangle. Therefore, as Bonola noticed, the following consequences are valid:

"In every normal triangle the sum of two angles is smaller than two right angles" [Euclide d'Alexandrie 1994, XVII].

"In every normal triangle, the greatest angle is opposite to the greatest side [Euclide d'Alexandrie 1994, XVIII] and vice versa" [Euclide d'Alexandrie 1994, XIX].

"In every normal triangle one side is smaller than the sum of the other sides" [Euclide d'Alexandrie 1994, XX].

"If two normal triangles have two equal sides, respectively, and the included angles are not equal, then the greatest angle is opposite to the greatest side [Euclide d'Alexandrie 1994, XXIV] and vice versa" [Euclide d'Alexandrie 1994, XXV].

After these considerations Bonola showed that Saccheri's theorem (in the case of the hypothesis of the obtuse angle) is valid for a normal triangle and, since every non-normal triangle can be decomposed in a finite number of normal triangles, Saccheri's theorem is valid [Bonola 1905, pp. 655-656].

He first demonstrated the following theorem:

"Saccheri's hypothesis of the obtuse angle is valid in every Saccheri quadrilateral"[63] [Bonola 1905, p. 655].

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59 *"Se in un triangolo ABC il lato AB è minore della semiretta e i due angoli adiacenti a questo lato sono entrambi ottusi ovvero uno retto e l'altro ottuso, i due lati AC, BC sono entrambi maggiori della semiretta."*

60 *"Un triangolo coi lati minori della semiretta ha due angoli acuti."*

61 *"Un triangolo ABC, coi lati minori di mezza retta, appartiene ad una regione triangolare PQR."*

62 Here and in the following pages, we refer to propositions of the first book of Euclid's *Elements*.

63 *"In ogni quadrilatero birettangolo isoscele (convesso) vale l'ipotesi dell'angolo ottuso di Saccheri."*

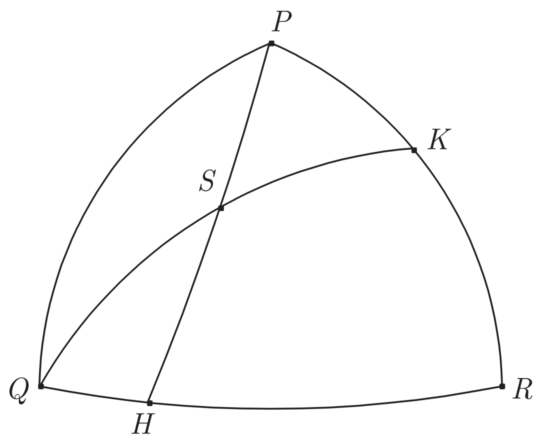

FIGURE 4.

He fixed a point S inside the triangle PQR and projected this point from the vertex P and Q on the opposite sides, obtaining points H and K. The quadrilateral $RKSH$ has three right angles; the fourth angle, with vertex in S, is obtuse, since it is outside the right normal triangle PSK (see Fig. 4). Moreover, $PS > PK$ and $QS > QH$ and since the segments PH, PR, QK, and QR are respectively equal to the half-line, $SH < KR$ and $SK < HR$. He also considered the quadrilateral $SHRK$ as half of a birectangular isosceles quadrilateral with vertex H, S and the symmetric points $H'S'$ with respect to the line PR. The quadrilateral $SHH'S'$ has the angles $\hat{S}$ and $\hat{S}'$ obtuse, and the hypothesis of the obtuse angle is therefore valid [Bonola 1905, p. 655].

Moreover, since $SK < HR$, $SS' < HH'$, then: "In every Saccheri quadrilateral the side adjacent to the two right angles is greater than the opposite side [Saccheri 1733, III]" [64] [Bonola 1905, p. 655].

After this consideration, Bonola supposed that the basis HH' of the quadrilateral is less than the half-line (see Fig. 5). Thus, the quadrilateral is contained in the triangle PQR and the diagonal HS' is less than the half-line; comparing the normal triangle HSS' and $S'HH'$, $HS = H'S'$, $HS' = HS'$, and $SS' < HH'$.

64 "*In ogni quadrilatero birettangolo isoscele il lato comune ai due angoli retti è maggiore del lato opposto* [Saccheri 1733, III]".

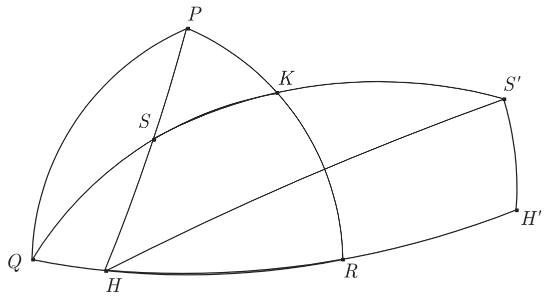

FIGURE 5.

Therefore, $\widehat{SHS'} < \widehat{HS'H'}$ and summing on the other side the angle $\widehat{S'HH'}$ we obtain

$$\widehat{SHS'} + \widehat{S'HH'} < \widehat{HS'H'} + \widehat{S'HH'}$$

Since the sum of the first side is a right angle, Bonola showed that:

> "The sum of the two acute angles of every normal right triangle is greater than a right angle and then the sum of the three angles is greater than two right angles" [65] [Bonola 1905, p. 656].

Since every normal triangle is sum of two right normal triangles, the theorem is proved.

6.2. *The open line*

Bonola proceeded by demonstrating Saccheri's theorem on the Saccheri quadrilateral and concluded by remarking that Saccheri's theorem on the sum of the inner angles of a triangle is a consequence of the following Saccheri theorems, since Saccheri's proofs do not depend on Archimedes' axiom:

> "The sum of the acute angles of every right triangle is equal to, less than, or greater than one right angle if the hypothesis of the right angle, the acute angle, and the obtuse angle respectively is valid" [Saccheri 1733, Prop. IX].
>
> "If there exists a triangle in which the sum of the inner angles is equal to, less than, or greater than two right angles, then the hypothesis of the right angle, the acute angle, and the obtuse angle respectively is valid" [Saccheri 1733, Prop. XV].

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65 *"In ogni triangolo rettangolo normale, la somma dei due angoli acuti è maggiore di un angolo retto, e conseguentemente la somma dei tre angoli maggiore di due angoli retti".*

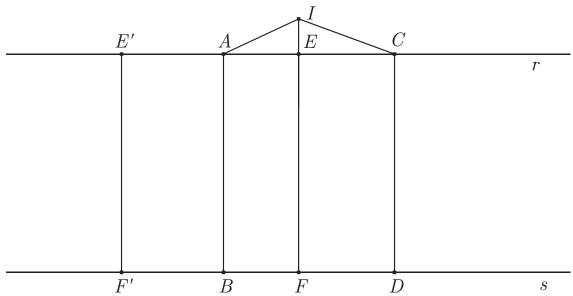

FIGURE 6.

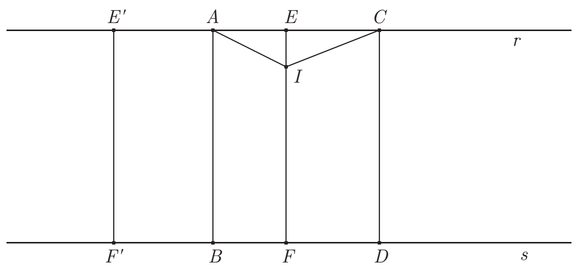

FIGURE 7.

In this case all the theorems before the 29th proposition of Euclid's *Elements* are valid and following Hilbert's method in the *Grundlagen* it is possible to prove that two right angles are always equal. Bonola considered the figure constituted by a line $BD = s$, by two segments AB and CD equal and perpendicular to s, and by the line $AC = r$. Since AB and CD are in the same side respect to s, the lines s and r do not intersect. Moreover, the two angles $\widehat{BAC}$ and $\widehat{DCA}$ are equal. He considered on r the points E (between A and C) and E' (not between A and C) and called F and F' the intersection points between s and the perpendicular through E and E' to s, respectively (see Fig. 6 and Fig. 7). He then showed:

"1°) *If* $\begin{cases} EF = AB \\ \quad or \\ E'F' = AB \end{cases}$ *then the angles* $\widehat{BAC}$, $\widehat{DCA}$ *are right angles.*

2°) *If* $\begin{cases} EF < AB \\ \quad or \\ E'F' > AB \end{cases}$ *then the angles* $\widehat{BAC}$, $\widehat{DCA}$ *are acute angles.*

3°) *If* $\begin{cases} EF > AB \\ \quad or \\ E'F' < AB \end{cases}$ *then the angles* $\widehat{BAC}$, $\widehat{DCA}$ *are obtuse angles*" [Bonola 1905, p. 657].

In particular, in case 1°), from the hypothesis $EF = AB$ it follows that $\widehat{BAE} = \widehat{FEA}$ and $\widehat{FEC} = \widehat{DCE}$. Therefore, $\widehat{FEA} = \widehat{FEC}$ and since they are adjacent they will be right angles. Therefore, the angles $\widehat{BAC}$, $\widehat{DCA}$ are between right angles. The same holds for the hypothesis $E'F' = AB$ [Bonola 1905, p. 657].

In case 2°), let $EF < AB$. Bonola extended the segment EF until $FI = AB$ and joined A and C to I. The following relations are valid: $\widehat{BAI} = \widehat{FIA}$, $\widehat{FIC} = \widehat{DCI}$. Moreover, from the theorem about the outside angle of a triangle, $\widehat{FIA} + \widehat{FIC} < \widehat{FEA} + \widehat{FEC}$ = two right angles. Bonola now noticed that from the figure (See Fig. 6) it follows that $\widehat{BAC} + \widehat{DCA} < \widehat{BAI} + \widehat{DCI}$ and from this relation and the previous one it follows that $\widehat{BAC} + \widehat{DCA} < \widehat{FIA} + \widehat{FIC}$ < two right angles. Therefore, $\widehat{BAC}$ < one right angle. Similar considerations are valid for the hypothesis $E'F' > AB$ [Bonola 1905, p. 658].

In case 3°), let $EF > AB$. He took $FI = AB$ on the segment EF and joined A and C to I. The following relations are valid: $\widehat{BAI} = \widehat{FIA}$, $\widehat{FIC} = \widehat{DCI}$. Moreover, from the theorem about the outside angle of a triangle, $\widehat{FIA} + \widehat{FIC} > \widehat{FEA} + \widehat{FEC}$ = two right angles. Bonola now noticed that from the figure (See Fig. 7) it follows that $\widehat{BAC} + \widehat{DCA} > \widehat{BAI} + \widehat{DCI}$ and from this relation and the previous one it follows that $\widehat{BAC} + \widehat{DCA} > \widehat{FIA} + \widehat{FIC}$ > two right angles. Therefore, $\widehat{BAC}$ > one right angle. Similar considerations are valid for the hypothesis $E'F' > AB$. [Bonola 1905, p. 659].

Bonola used a *reductio ad absurdum* in order to show that the inverse of the previous theorem is valid, i.e.:

"1°) If $\widehat{BAC} = \widehat{DCA}$ = one right angle, then $E'F' = AB = EF$.

2°) If $\widehat{BAC} = \widehat{DCA}$ < one right angle, then $E'F' > AB > EF$.

3°) If $\widehat{BAC} = \widehat{DCA}$ > one right angle, then $E'F' < AB < EF$" [Bonola 1905, p. 660].

He also considered the middle points M and N of the segments AC and BD and thus obtained the following proposition, as an immediate consequence of the previous result:

"1°) If $\widehat{BAC} = \widehat{DCA}$ = one right angle, then $MN = AB$.

2°) If $\widehat{BAC} = \widehat{DCA}$ < one right angle, then $MN < AB$.

3$^{\text{o}}$) If $\widehat{BAC} = \widehat{DCA} >$ one right angle, then $MN > AB$" [Bonola 1905, p. 660].

Finally, by *reductio ad absurdum*, bonola demonstrated the following proposition:

"1$^{\text{o}}$) If $\widehat{BAC} = \widehat{DCA} =$ one right angle, then $\left.\begin{matrix}\widehat{FEM}\\ \widehat{F'E'M}\end{matrix}\right\} =$ one right angle.

2$^{\text{o}}$) If $\widehat{BAC} = \widehat{DCA} <$ one right angle, then $\left.\begin{matrix}\widehat{FEM}\\ \widehat{F'E'M}\end{matrix}\right\} <$ one right angle.

3$^{\text{o}}$) If $\widehat{BAC} = \widehat{DCA} >$ one right angle, then $\left.\begin{matrix}\widehat{FEM}\\ \widehat{F'E'M}\end{matrix}\right\} >$ one right angle" [Bonola 1905, pp. 660-661].

Bonola then considered a point P on the line MN, not between M and N (these considerations are valid also if P is between M and N), the line RP perpendicular to the line MN, the line RK perpendicular to the line s in K, and the point H of intersection between RK and r (see Fig. 8).

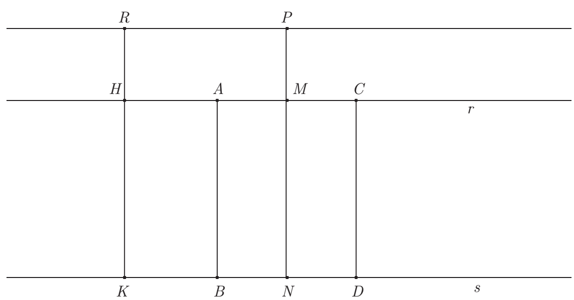

FIGURE 8.

"1$^{\text{o}}$) If $\widehat{BAM} =$ one right angle, then $\left.\begin{matrix}\widehat{KHM}\\ \widehat{KRP}\end{matrix}\right\} =$ one right angle.

2$^{\text{o}}$) If $\widehat{BAM} <$ one right angle, then $\left.\begin{matrix}\widehat{KHM}\\ \widehat{KRP}\end{matrix}\right\} <$ one right angle.

3$^{\text{o}}$) If $\widehat{BAM} >$ one right angle, then $\left.\begin{matrix}\widehat{KHM}\\ \widehat{KRP}\end{matrix}\right\} >$ one right angle" [Bonola 1905, p. 661].

This is Saccheri's theorem on the Saccheri quadrilateral.

7. CONCLUSIONS

In this paper we have shown that the point of view of Hilbert's *Grundlagen* has been completed by Max Dehn with regard to Archimedes' axiom. Dehn was not primarily interested in finding minimal sets of axioms or in separating the postulates of a given discipline into sets of weaker ones and then proving their independence and completeness. He was interested in finding solid and simple foundations for a theory, in particular for projective geometry:

> "The aim of the foundations of projective geometry corresponds to that of metric geometry: to transform the projective relations (collineations) into algebraic relations"[66] [Dehn & Pasch 1926, pp. 213-214].

Dehn was thus interested precisely in the mutual interrelations between algebraic structures and geometric relations. This kind of approach was to become a real "research program", for Dehn inspired many students. In particular, two students, Ruth Moufang and Walter Wagner, worked on the foundations of geometry in following Dehn's approach. Moufang constructed non-Desarguesian planes by studying the geometric properties through algebraic properties, and conversely [Moufang 1933]; Wagner answered the question, posed by Dehn [Dehn 1922], if there exist incidence theorems which do not imply the validity of Pappus' theorem but do imply the theorem of Desargues without being derivable from it [Wagner 1937].

The study of the logical relationships/interdependence between the axioms in the *Grundlagen* was the starting point of new research in geometry. Showing the independence of one axiom from the others proceeds by exhibiting a geometric model that satisfies all the axioms except the "independent" one. The "non-geometries" were thus born. For example, the "non-Desarguesian" geometries derived from the study of the independence of Desargues' theorem from the plane axioms, and the "non-Archimedean" geometries from that of the independence of Archimedes' axiom from the other ones. The study of these new geometries also led to a "modern" method of classification of algebraic and geometric structures.

In that context the study of non-Archimedean geometries, which was pursued by Veronese in Italy and by Hilbert and Dehn in Germany, was

66 *„Das Ziel bei der Begründung der projectiven Geometrie ist ganz entsprechend wie bei der metrischen Geometrie das, die projectiven Beziehungen (die Kollineationen) in algebraische Beziehungen zu verwandeln."*

tightly linked with the study of algebraic structures (like p-adic numbers), also much used in number theory, algebraic geometry, and physics. [67]

A comparison between Veronese's approach and of Dehn's is far beyond the scope of this paper. We may note that Veronese's pioneering work did not give rise to a real mathematical school, but to a lasting debate on the subject of non-Archimedean geometry, also involving Italian geometers [68]. The controversy between Veronese and Peano about the geometry of n-dimensional space and about non-Archimedean geometry is well known [69].

Concerning the use of intuition in geometry, we cannot say that the point of view of Enriques and Veronese are the same: Enriques admits completely the scientific validity of Hilbert's method [Enriques 1907] and adds to it (essentially for didactical reasons) a search for the psychological and physiological origin of the axioms. For Veronese instead, the use of intuition is strongly connected to an empiricist conception for which:

> "Geometry is the most exact experimental science, because the objects outside thought, that we need for the formulation of axioms, are replaced in our mind by abstract forms, and therefore the truths of the objects can be demonstrated by combining forms independently obtained from what happens outside" [70] [Veronese 1891, p. 8].

Consequently, the abstraction can be used only to extend the empirically determined axioms.

Bonola's work was more strongly influenced by Dehn's and Hilbert's approaches than by that of Veronese. In fact, since in his research he was principally concerned with non-Euclidean geometries, he was above all interested in studying the relationships between Archimedes' axiom and Saccheri's theorem. Besides, it was important for him to obtain Dehn's result in a more elementary fashion, much in the Enriquesian sense of using axioms and methods for didactic purposes: "in order to establish in a simple and luminous way the observation of Dehn, concerning the independence of Saccheri's theorem from Archimedes' axiom" [71] [Amaldi 1911, p. 149].

67 In fact, there exists a new approach to quantum gravity which is based on non-Archimedean geometry, in particular on a geometry on the p-adic. See for example [Vladimirov et al. 1994].

68 See Introduction.

69 See [Galuzzi 1980], [Freguglia 1998], [Manara 1986].

70 *"La scienza sperimentale più esatta è la geometria, perché gli oggetti fuori del pensiero, che servono alla determinazione degli assiomi, vengono sostituiti nella nostra mente da forme astratte, e quindi le verità degli oggetti si dimostrano colla combinazione delle forme già ottenute indipendentemente da ciò che succede fuori."*

71 *"A stabilire per via semplice e luminosa la osservazione del Dehn, relativa all'indipendenza dal postulato di Archimede del teorema fondamentale del Saccheri."*

In this sense, Bonola followed Enriques' point of view:

> "Remaining in the field of Geometry, it must not be forgotten that such science is science about physical or intuitive facts that we wish to consider. Logical formalism must be conceived not as an aim *per se* but as an apt means to develop and to advance intuitive faculties. The same results, logically established, must not be considered a mature achievement until they can be intuitively understood. But in the principles, intuitive evidence must shine luminously [...]" [72] [Enriques 1900, p. 12].

It follows from the previous considerations that Enriques (and his student Bonola) considered the foundations of geometry as a part of elementary mathematics, while the approach of Hilbert and Dehn is founded on the study of the interrelations between algebraic properties and geometric properties. For the latter, the question of the foundations is a fundamental part of research in modern mathematics. [73] In this sense Italian Mathematicians [74], in the subsequent years, were not able to make a place for themselves in the international developments on foundations of mathematics which mainly followed the way indicated by Hilbert and Dehn.

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72 *"Restando nel campo della Geometria, non bisogna dimenticare che tale scienza è scienza di fatti fisici o intuitivi che vogliano considerarsi. Il formalismo logico deve essere concepito, non come un fine da raggiungere ma come un mezzo atto a svolgere e ad avanzare le facoltà intuitive. Gli stessi risultati più lontani, logicamente stabiliti, non debbono considerarsi un acquisto maturo, fino a che non possano essere intuitivamente compresi. Ma nei principi l'evidenza intuitiva deve risplendere luminosa [...]".*

73 The links between the different approaches to the foundations of geometry of Italians and Hilbert have been investigated by Bottazzini [Bottazzini 2001] and Brigaglia [Avellone et al. 2002].

74 Without having gone deeper into the works of Levi-Civita [Levi-Civita 1893; 1898], Levi [Levi 1905] and Predella [Predella 1911; 1912], we assume that their work, at least as regards the role of Archimedes' axiom, cannot be considered the expression and implementation of a common research program.

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