POSITIVE THINKING CONCEPTIONS OF NEGATIVE QUANTITIES IN THE NETHERLANDS AND THE RECEPTION OF LACROIX'S ALGEBRA TEXTBOOK

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ABSTRACT. — The beginning of the 19th century witnessed the emergence of several new approaches to negative numbers. New notions of rigour made the 18th century conceptions of negative quantities unacceptable. This paper discusses theories of negative numbers emerging in the Netherlands in the early 19th century. Dutch mathematicians then opted for a different approach than that of their contemporaries, in Germany or France. The Dutch translation (1821) of Lacroix's Élémens d'algèbre illustrates the 'Dutch' notion of rigour.

RÉSUMÉ. — PENSER POSITIVEMENT. CONCEPTION DES NOMBRES NÉGATIFS AUX PAYS-BAS ET RÉCEPTION DU TRAITÉ D'ALGÈBRE DE LACROIX. — Au début du XIX^e siècle, des attitudes nouvelles par rapport aux nombres négatifs émergent. La notion de rigueur en mathématique se renouvelle, rendant inacceptables les approches qui s'étaient développées au XVIII^e siècle. Cet article présente les théories des nombres négatifs qui avaient cours aux Pays-bas au début du XIX^e siècle. Les mathématiciens néerlandais optaient pour une conception différente de celles de leurs contemporains en Allemagne et en France. La traduction néerlandaise des *Elémens d'algèbre* de Lacroix illustrera cette approche « néerlandaise » de la notion de rigueur.

Mathematical theories developed around 1800 were not readily accepted, and mathematics in the various European countries met with rather different epistemological approaches. Different institutional contexts, where mathematical knowledge was pursued in France and Germany for instance, gave rise to different notions of rigour [Schubring 1996]. Sylvestre F. Lacroix's textbooks provide examples to test this assertion, as they were very popular in France and kept in line with the ideas of

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rigour which prevailed there at the time. Moreover, they were translated into several European languages and during the process of translation the texts were adapted to fit other standards of rigour. This paper discusses the Dutch translation of Lacroix's algebra textbook and reveals which ideas on negative numbers then prevailed in the Netherlands. A map of the European background and 18th century Dutch works on negative numbers come first in order.

1. EUROPEAN BACKGROUND

During the 18th century many continental mathematicians began viewing algebra as a universal language. While the British chose a more careful approach to algebra and adopted the geometrically inspired Newtonian theory of fluxions, Leibnizian calculus, on the other hand, is often the exponent of the continental faith in results obtained from algebraic manipulations. The proof of the rule $\mathrm{d}y/\mathrm{d}x = (\mathrm{d}y/\mathrm{d}z)(\mathrm{d}z/\mathrm{d}x)$, for instance, relies heavily on the notation which is chosen (cf. [Grabiner 1990] and [Ferraro 1998]). Negative quantities in those days were regarded as "less than nothing", metaphorically linked to debt, as opposed to possession. By the end of the 18th century mathematicians generally agreed that this view of negative quantities was no longer befitting.¹

The British university curriculum stimulated a stark focus on foundations, by regarding mathematics as the paradigm of human reasoning. Mathematicians worried that algebra somehow fell short of geometry as a logic [Pycior 1981]. Rethinking algebra (or treating it anew from the field and concentrating on geometry) was the obvious thing to do in Britain [Pycior 1997, pp. 309–310]. Francis Maseres, in his 1758 dissertation [Maseres 1758], denounced the way negative quantities were defined in algebra. By the end of the 18th century Maseres and William Frend seriously attacked the use of negative quantities and proposed to cast algebra as a general arithmetic in the strictest sense [Frend 1796; Maseres

¹ According to Schubring the problem with accepting negative numbers had to do with mathematicians moving from the concept of quantity, as something feasible, of a physical nature, e.g. an object or its weight, or trajectory, to a more theoretical concept of number [Schubring 1986, pp. 7–10]. 'Feeling' the need to accept this abstraction or reject it and cling to some sense-related interpretation might have been the main stake in the different 19th-century attempts to rigorize mathematics in general and algebra in particular.

1800]. There, only operations which are meaningful in ordinary arithmetic are admitted and a negative solution to an equation is regarded as a solution to another problem. Many British scholars did not abandon algebra in such a radical way. Frend and Maseres were not a part of the establishment at the time (partly due to their religion, *cf.* [Pycior 1987]), but their work did help to set the stage for a rethinking of algebra as a whole [Pycior 1997, pp. 307–316]. The work by George Peacock [Peacock 1830, 1833] and its reception by Augustus de Morgan, however severely critized, even ridiculed at first [Pycior 1982a,b], introduced to the British a more abstract approach to algebra in which the manipulation of symbols obeyed a set of arbitrary laws – that, of course, 'happened' to be the laws for elementary arithmetic.

Mathematics played a marginal role at the French universities. By the 1750s, however, military education played a serious role and attracted able students. In military education mathematics was a subject taken very seriously, but in general teachers did not want to frighten students away by giving them difficult problems, and such were foundations of negative numbers [Schubring 1996, pp. 365–366]. Apart from the military schools there existed in France a broad intellectual élite whose reception of negative numbers was more ambiguous. On the one hand, the $Encyclop\acute{e}die$ contained an article by d'Alembert who thought that negative quantities – as quantities less than zero – were absurd (since 1:-1=-1:1, which meant that the larger was to the smaller in the same proportion as the smaller to the larger). On the other hand, the $Encyclop\acute{e}die$ sported an article that simply defined negative quantities as less than zero.

The point of view expressed by d'Alembert is interesting in that it influenced French mathematicians then and later. According to d'Alembert a proportion such as the above only makes sense in absolute values; the signs do not affect the quantities themselves, but only the 'state' they are in, which is not conceivable in an abstract sense, but only in relation to a specific problem. For example, someone owing another person -3 écus was intelligible, but "-3 pris abstraitement ne présente à l'esprit aucune idée". The establishment in 1794 of the École Polytechnique and the French project of making mathematical knowledge more elementary and appealing to the beginner encouraged rethinking mathematics from a

 $^{^2}$ Literally from the $\it Encyclop\'edie, v. 11, p. 73. Quoted from [Schubring 1986, p. 9].$

different perspective. This is how a new concept of negative numbers was fostered [Schubring 1982, 1988].

In his posthumous treatise La langue des calculs (1798) E.B. de Condillac claimed that algebra was the result of a number of abstractions severed from simple counting and believed that d'Alembert was wrong in denying the existence of negative quantities [Condillac 1981]. For Condillac, algebra served as the fundament of mathematics [Schubring 1986, pp. 10–12]. D'Alembert nevertheless found a follower in Lazare Carnot, who reinforced the rejection of negatives and his ideas about negative quantities were to become very influential. He denied to algebra the fundamental role it played in Condillac's view, restricting it to a mere translation of geometrical propositions [Carnot 1801]. Subtraction as a general algebraic operation was thereby unacceptable: a-b is meaningful only in case a > b. Being convinced that geometry predominated algebra he replaced the notion of negative quantity by the direct and inverse lines which were the basic notions of the geometrical framework which he proposed, instead of the algebraic theory of negative quantities, in his Géométrie de position [Carnot 1803]. Reckoning with negative numbers is an operation which Carnot considered not rigorous and thus he made algebra subordinate to geometry by basing (among other things) the theory of negative quantities on geometrical constructions [Dhombres 1997, pp. 514–519].

Carnot's ideas found their way into the textbooks of S.F. Lacroix. His textbook on elementary algebra (itself a new edition of a much older textbook by Clairaut), was one of the most popular elementary textbooks published in late 18th century France. After having read Carnot's *Géométrie de position* Lacroix revised his textbook—which he did many times—to fit this new point of view [Schubring 1996, pp. 368—369]. In this revised edition he avoided the use of negative numbers. When a problem (an equation) resulted in a negative solution, Lacroix simply re-formulated the initial problem (this topic will be further elaborated on in section 4). This made solving equations a highly complicated and very annoying business.

Unlike France, the Germanic countries had no significant system of military schools. Almost all scientific activity took place at the universities, where mathematics was represented by particular professorships within a philosophical faculty. During the second half of the 18th century these professors began to view mathematics as an interesting subject on its own accord, and started performing research. The foundations of mathematics became a very popular subject [Schubring 1991]. Euler had regarded negative (abstract) numbers as existing, though he did not pay much attention to them. He simply viewed multiplication with a negative quantity as changing the quantities' 'state' and, when solving equations, he simply mentioned – or did not mention – negative solutions, depending on the domain of his problem [Euler 1773, vol. 1, p. 21; vol. 2, pp. 72–82].³

Negative numbers became mathematised in a different fashion by German professors than by their French colleagues. Based on the philosophical notion of opposition, quantities were conceived as provided not only with a quantitative, but also with a qualitative (positive or negative) attribute. Thus, for every quantity a there was also a quantity \overline{a} of the same nature, but of an opposed quality, such that $a + \overline{a} = 0$. Matthias Metternich, in 1811, published a German translation of the algebra textbook by Lacroix [Lacroix 1811] - mandatory literature since at the time his hometown Mainz was within French territory. He unabashedly adapted Lacroix's text to the German point of view, correcting Lacroix with footnotes, omitting or adding several paragraphs on the theory of negative quantities [Schubring 1996, pp. 367–369]. During the early 19th century, the research paradigm extended to the gymnasia, where able and properly trained mathematicians taught. Gymnasium teachers contributed their part to the development of foundations of negative numbers [Förstemann 1817] as well as to the advancement of many other fields: they worked in a context that valued formal mental training and thus strove for clarity in the foundations. This led to a fairly wide acceptance of mathematical rigorisation in the German states: engineers like A.L. Crelle and J.A. Eytelwein felt the need for a rigorous theory in mathematics as much as did the university professors, while in France two contrasting views (the technicians versus the theoretically minded mathematicians) competed until the late 19th century [Schubring 1996, pp. 371–373].

During the first half of the 19th century algebra evolved from a technical subject concerned with solving equations to a much wider

 $^{^3}$ The Dutch translation of the algebra textbook is (at least at this point) in accordance with the German version of the *Vollständige Anleitung zur Algebra* from 1771 to which I don't have easy access to verify the page numbers.

field of knowledge from which various algebras emerged, with Boole, Hamilton, Peacock, Gauß and Galois as the most noteworthy contributors [Grattan-Guinness 1997, pp. 409–435]. These new theories, though they were intended to treat specific objects of some kind or other, and not meant to be mere structures defined by arbitrary laws [Grattan-Guinness 1997, p. 434], were regarded by many as pointless philosophical play. Consequently their reception went slowly [Pycior 1982b]. In particular, for many mathematicians, symbolical algebra could not serve as a foundation for negative numbers. Lacroix's algebra textbook, for example, was to be in use until the late 1870s in France. The German mathematician Diesterweg published in 1831 a treatise on negative numbers giving many problems in which negative numbers were involved. Diesterweg interpreted them all in the then current style of 'German' philosophy [Diesterweg 1831]. In the Netherlands other views prevailed.

2. NEGATIVE NUMBERS IN THE NETHERLANDS

Simon Stevin (1548–1620) was one of the people behind the mathematics courses at Leyden university that started as early as 1600. These were very practical courses, focussing on surveying and fortress building, and they shaped the Dutch mathematics curriculum up to the 19th century [Alberts et al. 1999, p. 372; 398]. Dutch mathematicians during the 18th century were mainly interested in physical and applied research [Bockstaele 1978]. They were quite capable of reading foreign work, though their main task was the teaching of mathematics: the practical engineering mathematics as well as Euclidean geometry, and the latter to all students. Most publications by university professors then, were textbooks emphasising the application of mathematics to various situations [Alberts et al. 1999]. From the 1750s onwards the Netherlands also witnessed the emergence of several scientific societies; research undertaken within these institutions was mostly of a very practical nature.

Many foreign books were translated throughout the 18th century. Who took the initiative for these translations is not always clear. Their purpose, however, is more apparent: book printing and selling was a very competitive business in the Netherlands at the time, and printers sought gains from publishing. It was rare that someone published a book at his own costs. Cases are known of publishers having a network of

translators. Since Government policy was not restrictive—compared to the rest of Europe—opportunities were nearly infinite [Cerutti 1998]. Most of the translations were from textbooks (which was probably also the best way of making a profit), put into the vernacular by engineers or schoolteachers. Everal of these translations reveal links to the Hamburg Kunstrechnungsliebende Gesellschaft, an 'amateur' Society of teachers and engineers active since 1690 [Wettengel 1990]. French was the second language in the Netherlands and textbooks were sometimes directly published in French, and used in engineering education; most noteworthy are the textbooks by Étienne Bézout which were used at the charitable institutions of "Renswoude" (founded in 1756) for training orphans in technical trades [Booy 1985, p. 276]. "Renswoude" would become a model of technical training, at least in the opinion of people seeking to establish a more theoretical military education by the end of the 18th century [Janssen 1989, pp. 147–148].

Most of the translators then were practically minded people, who sought no foundational problems in algebra: mathematical theories were destined to be applied and, as long as the application worked well, there was no reason to look into the details. For example, A.B. Strabbe (1741–1805) who translated the textbooks of A.C. Clairaut added to his translation of Clairaut's geometry the trigonometry by Th. Simpson [Clairaut 1760_{a,b}]. These two texts were stylistically so distinct that they show the translator's indifference towards the foundations of mathematics.

This practical point of view also had didactical consequences which might have influenced the mathematicians' view on negative numbers. Surveyors' textbooks treated algebra by means of recipes. It was a didactical habit harking back to the 16th century, that presented mathematics as a set of rules to be applied to obtain correct results [Kool 1999, pp. 243–244]. Negative quantities, and the multiplication of two negative numbers, slipped (for example) into applications of the rule of false from the 16th century onwards [Smeur 1978]. The most popular Dutch algebra textbook of the 18th century treated the rule of signs in this tradition: it was a rule one had to follow to get the correct result and it was illustrated by a

⁴ Some of the most notable textbooks to have been translated are: [Hammond 1759], [Hiddinga 1735], [Halcken 1767], [Crummel 1776], [Clairaut 1760a], [Clairaut 1760b], [Simpson 1770], [Martin 1763], [Wolff 1738], [Wolff 1740], [Wolff 1745], [Mauduit 1763], [Euler 1773].

few numerical examples [Venema 1714]. Similar examples may be found in other textbooks. For instance, Pibo Steenstra (ca. 1730–1780) included some algebra in his geometry textbook of 1763, where he first taught the pupil to distinguish the algebraic magnitude from the coefficient: in 3ab, for example, 3 was the coefficient and ab was the algebraic magnitude. Then the pupil was taught how to add comparable algebraic expressions of the same sign, by writing the sum of the coefficients in front of the common magnitude, and putting the sign (plus or minus) before it. Then he taught the pupil what to do if the signs in front of the coefficients were not all the same:

"First find the sums of the positive and negative expressions, the way it was done before, and subtract the smallest coefficient from the largest. Put the sign of the largest of the two coefficients in front of the result, and the entire result in front of the algebraic magnitude." ⁵

Some examples, such as 3x - 4x - 2x + 5x = 2x, helped to clarify the procedure. Since algebra meant solving equations and nothing more, the principles behind the computational rules for negative numbers could remain hidden in the recipes and a few words on the interpretation of a negative solution sufficed. Higher level textbooks by internationally oriented authors such as J.J. Blassière (1736–1795), W.J.'s Gravesande (1688–1742) and J.F. Hennert (1733–1813) did not go beyond elementary reckoning either, they more or less accepted negative numbers as self-evident. For Dutch textbook authors then, negative numbers were quite acceptable whereas foundational problems were, in most cases, hidden within the didactical approach sketched above.

⁵ [Steenstra 1763, pp. 140–141]. Literally: "Zoek eerst de Som van de affirmative en van de negative yder in 't byzonder, volg. 't I. Geval, en trekt de kleinste coëfficiënt van de grootste, stelt voor de rest het teeken van de grootste coëfficiënt, en agter dezelve de grootheid." All translations from the Dutch are my own unless otherwise stated.

⁶ Imaginary numbers were quite another story. Imaginary numbers are not to be found in Dutch mathematical texts during the 18th century, except for one case. In his algebra textbook J.J. Blassière devoted a few pages to imaginary numbers. He noted that some equations of degree two resulted in two impossible solutions. These solutions, impossible as they might have seemed, could nevertheless be used in reckoning. Blassière's problem was that these solutions did not seem to obey the law $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, since $-1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$. He 'solved' this by stating that the imaginaries seemed to have another way of reckoning, and gave some examples. After these examples he quickly closed the section, because: "le nombre d'exemples que nous venons de donner, étant plus que suffisant dans une matière

By the end of the 18th century algebra was viewed as a very promising theory to provide a rigorous foundation of calculus [Beckers 1999]. It was also considered to be a very valuable piece of reckoning equipment among engineers, and the confidence in its results was considered well-founded. Generally it was not taught to university students, but most engineers were acquainted with the subject, as might be illustrated by the fact that algebra was a subject in the very popular engineering courses of Pibo Steenstra quoted above [Steenstra 1763].

At the turn of the 19th century, this situation started to change. It became common then for middle class parents to found or financially support mathematical institutes in their home towns, where they sent their children to learn mathematics: no longer did they learn the old set of rules, but 'real' mathematics because it was considered a mind forming discipline. Since the 1780s practically every Dutch city of some significance had such an institution within its walls [Beckers 1998a]. In 1815 elementary algebra and geometry became compulsory subjects at gymnasia and universities. The formal training of students' minds being highly valued [Smid 1997], a serious rethinking of the foundational aspects of mathematics was natural.

Even at the Military Academy and in the training of engineers, where mathematics had always been an important subject, mathematics began to play an even more serious role [Goudswaard 1981]. At the Military Academy, founded in 1814, the level of precision in the teaching of mathematics became a bone of contention between the mathematics teacher, Jacob de Gelder (1765–1848), who favoured rigour for its own sake, and his superior officer, who wanted that more practical subjects be taught. Although De Gelder was finally fired, the Government revised the Academy's mathematics curriculum in a way which carried out his views [Janssen 1989, pp. 307–340]. Formal training permeated Dutch mathematics education [Smid 1998] and the production of mathematical textbooks with a solid treatment of mathematics (and its foundations) became a very respectable enterprise: De Gelder was appointed professor of mathematics at Leyden university and built his carreer on his own

qui est plutôt curieuse qu'utile, nous renvoyons ceux, qui ont envie de le voir traiter plus amplement, au *Traité d'Algèbre de Maclaurin*, Part II. dans les § 73. &c.". See: [Blassière 1770, p. 145]. Early Dutch attempts to obtain rigor in calculus can be traced back to a lack of 'faith' in complex numbers [Beckers 1999, p. 229].

highly regarded textbooks [Beckers 1996, pp. 288–290].

In general, little or no research in mathematics was conducted at the time and teaching was the main duty of mathematicians even at the university. The Royal Institute ("Koninklijk Instituut"), founded in 1808 as a counterpart of the Institut de France, would not become a true scientific institute in the modern sense. Although it gathered important Dutch scientists and was held in some esteem by their contemporaries, the main task of the institute was to advise the Government. Due to a lack of financing it never managed to professionalise research and all its members had duties outside the practice of science. It did, however, encourage the publication of scientific results through its series [Gerritsen 1997, pp. 15–56].

The Dutch Wiskundig Genootschap (Mathematical Society, founded in 1778 as an Amateur Society resembling the Hamburg Gesellschaft) began playing a serious scientific role from the 1810s on: by 1820 every respectable mathematician was a member of the Society [Baayen 1978]. Within the Society foundational issues were discussed, not so much to advocate a certain approach, but rather to emphasize the unity of mathematical knowledge. The following may serve to illustrate this attitude. In an 1817 paper in the Society's journal one of the members deduces the Taylor series of some goniometric functions. He refers to a much older paper by Euler and mentions that he will deduce the results differently. Re-proving a known fact was useful, according to him:

"... because if we do so it does not only prove Euler's results once again, but it will also show the younger mathematician that in all conclusions of mathematical reasoning one will encounter a beautiful harmony, even if the principles from which our reasonings start are so different from each other." 7

Arbitrary definition, however, was inconceivable to Dutch mathematicians. Of course, mathematics was regarded as a perfect way of sound reasoning: that was why it was taught. By doing mathematics one was performing a sensible abstraction from reality, trying to solve a certain class

⁷ [Huguenin 1817, p. 1]; literally: "wijl het, in zulk een geval, niet alleen ter bekrachtiging van de Eulersche besluiten verstrekt; maar ook tevens dient, om den jongen beöeffenaren der Wiskunde te toonen, dat men in de gevolgen der wiskundige waarheden, steeds de fraaiste overeenkomst ontmoet; hoe zeer ook de beginselen, waar uit men afgegaan is, van elkander verschillen."

of practical problems. This did not, however, stand in the way of 'purifying' mathematics: the mathematical way of reasoning was considered vital to reach sound conclusions, so foundations of mathematics were important. But the objects of mathematical reasoning were abstractions, and not arbitrarily chosen. We may find these views expressed in the writings of Leyden professor of mathematics Pieter Nieuwland [Nieuwland 1789, p. 79] and in the textbooks and lectures of his 'successor' Jacob de Gelder [Gelder 1806a, pp. 29–35]. Both of them resented the 18th-century rules and especially the didactic form in which the rules were presented which intimated that a merchant might be doing something entirely different than an engineer or a physicist while they all solved similar equations. To them, unity in the mathematical sciences was part of its beauty: the logical structure facilitated studying, and also taught a person to think properly. Yet, although they preferred to present it as an abstract subject, mathematics was nevertheless about something feasible, and not arbitrary.

Theoretical concepts therefore, had to be introduced in a given 'logical' order: Jacob de Gelder's books being an exponent of this view,⁸ the same attitude may be found in the algebra textbooks by the teacher P.J. Baudet (1778–1845) [Baudet 1820].⁹ A. van Bemmelen (1763–1822) wrote the first algebra textbook for the gymnasium level in which he praised the generality of algebra for finding all solutions to (practical) problems—even those solutions one would not have expected such as the negative ones [Bemmelen 1817, vol. 2 pp. 35–38].

A debate over arbitrariness in definitions took place in 1835, when De Gelder and his colleague J.F.L. Schröder (1774–1845), of German origin but nonetheless professor at Utrecht university, clashed over the foundations of geometry. Schröder had a more abstract approach to the

⁸ Most notably: [Gelder 1806b] which was enthusiastically received by the Dutch Mathematical Society, [Gelder 1810], [Gelder 1811], [Gelder 1823] and [Gelder 1826]. In his lecture [Gelder 1806a] De Gelder emphasises the usefulnes of mathematical (sound) reasoning as well as the practical aspects that are attached to it. His teaching is very theoretical and emphasises proof and understanding, but for his definitions he requires only that they be clear and recognizable [Beckers 1996, pp. 288–290].

⁹ In [Baudet 1824, pp. 63, 179–181] Baudet even warns his students never to do mathematics only for the sake of knowing things: one had to apply one's knowledge, or one would act like a miser. This is something De Gelder himself would never have said, though he and Baudet agreed that mathematics treated abstractions by stepping away from reality, and therefore definitions, somehow, had to be identifiable and follow in a 'logical' order.

teaching of mathematics and represented the 'German school' within the Netherlands. His most important textbook [Schröder 1831] appeared rather late in his career, while he was already teaching since the 1810s. He was a follower of the German philosopher Jacob Fries. ¹⁰ The controversy showed how unnatural Schröder's views were in the eyes of many Dutch mathematicians [Beckers 1998a]. In the next section, the major Dutch contributions to a suitable theory for treating negative quantities will be discussed. Special attention will be devoted to the proof of the rule " $(-a) \times (-b) = ab$ ", which was then the most obvious contentious rule.

3. DUTCH CONCERN WITH THE FOUNDATIONS OF NEGATIVE QUANTITIES

Jacobus P. Tholen (1764–1824) was awarded a doctorate under supervision of the famous Amsterdam professor Jan H. van Swinden (1746–1823) in 1784 and later in 1797 became professor of mathematics at Francker university. The first subject of his thesis was the question why multiplication of two negative numbers yields a positive number. This is the earliest serious Dutch concern with this subject, though Tholen's supervisor had already shown his concern for foundational questions [Swinden 1770]. One might expect that the view of one of the pupils of such a distinguished mathematician would be(come) somewhat influential.

Tholen started his dissertation by stating that a multiplication $a \times b$ could best be viewed as finding the number x that satisfied x:a=b:1. Inasmuch as it interprets the construction of x geometrically—Euclidean geometry being considered the epitome of rigour at the time—this definition was as sound as one might wish for. Tholen began to look into the nature of negative numbers by interpreting a negative quantity geometrically: one simply imagined a line to have a point "zero" somewhere, and points on the line at one side of this point are positive, while those on the other side are negative. Thinking more algebraically, Tholen used an arithmetical sequence to define his negative quantities. Considering

$$\dots$$
, $a - 3b$, $a - 2b$, $a - b$, a , $a + b$, $a + 2b$, $a + 3b$, \dots

 $^{^{10}}$ On J.F. Fries (1773–1843) and his influence on German geometry, see [Gregory 1983].

he took a = b and noted that a - b is equal to zero. He also stated that a - 2b is negative as it is equal to -b, which indicates that the next term in the sequence must be zero. In the same way -nb indicates that the zero term can be reached in n steps.

From these examples, Tholen concluded that negative numbers are thus real objects which can be manipulated algebraically, and to which he could apply his definition of multiplication. The geometrical interpretation of multiplication (probably perceived in the way that Descartes constructed a line equal to the product of two given lines in his *Géométrie*) demonstrated as a fact that the product of two negative numbers is a positive number [Tholen 1784, pp. 3–7]. Tholen thus accepted the existence of negative quantities.

His colleague Van Swinden might have shared his views at the time, or not. Anyway, he expressed a different opinion circa 1800:

"It is very inaccurate to say that a negative number is less than 0, which is what many authors claim. A negative number is a positive number, but in another sense, and therefore relative." ¹¹

It seems that Van Swinden opted for something close to d'Alembert's notion of negative numbers, in which quantities are always positive, but negative numbers could make sense as long as they express some physical value.

At the turn of the century questions such as those subjects were considered important, and were much reflected upon and intensely discussed. Van Swinden, who was involved in the business of introducing the metric system as well as in the European triangulation project (initiated by the work of Méchain and Delambre) might have been spurred by his French contacts to adopt this point of view. Yet Van Swinden was not the only man in the Netherlands to consider negative numbers worthy of attention, and even a middle class magazine such as the *Nieuwe Vaderlandsche Bibliotheek* (New Dutch Library), which had a relatively large and generally educated readership, paid attention to a new foundation of negative quantities. The anonymous author of a paper published in this journal was convinced that negative quantities made good sense and he conceived

 $^{^{11}}$ [Swinden *Notes*, p. 270]; literally: "Zeer onnaauwkeurig is de uitdrukking van veele schrijvers dat een negatiev Getal minder is dan 0. Een negatiev getal is een Positiev in een andere zin & dus relatiev."

 $^{^{12}}$ There are other examples of serious mathematical questions being discussed in

them as line segments standing at one side of zero. Interpreting (though not formally defining) addition and subtraction on a line segment he made it plausible to his readers that -(-a) = a [Anonymous 1798]. How this anonymous author tackled the problem of defining negative quantities is neither important, nor particularly noteworthy from a mathematical point of view. The fact that he did so, however, and that he published his thoughts in this journal illustrates a widely spread concern for negative quantities.¹³

In 1815, Jacob de Gelder published a treatise on negative quantities. His publication included a letter of support signed by three distinguished members of the Dutch Royal Institute of Arts and Sciences who welcomed De Gelder's book (243 pages in all with 5 figure sheets, not counting the preface) as a genuinely thorough text, well worth reading as it sheds light on a subject which often was found difficult by beginners. This book remained in good standing among Dutch mathematicians, even ten years after its publication. ¹⁴ Since his work received much praise and widespread attention, De Gelder became an influential personality of the Dutch mathematical scene. I will now discuss his contribution in detail.

In his introduction De Gelder explicitly announced that he wants to solve d'Alembert's paradox which had been tackled by authors who, in his view, only complicated matters by giving twisted philosophical explanations to the existence of negative quantities. In contrast, de Gelder wanted to introduce negative quantities as clearly and simply as possible, for he believed that they constituted an important part of algebra, which

middle class journals addressing a general audience. For example, Euler's ideas on the convergence of series were discussed in the *Vaderlandsche Letteroefeningen* of 1763 [Beckers 1999, p. 227].

¹³ Perhaps the phrasing "foundation of negative quantities" is a bit misleading here. As the reader might have noted the authors themselves were paying attention to these foundations, but what they were actually after was a proof for the signs rule. The word "foundation" (Dutch equivalents: "grond" or "grondslag") was mostly used in this respect, and not in establishing a logical foundation for negative quantities.

¹⁴ The praising review, in which Utrecht mathematician Richard van Rees (1797–1875) expressed the hope that a French translation might appear soon, may be found in [Rees 1825]; 'Belgian' mathematician/statistician A. Quetelet [1826] wrote about De Gelder: "Il avait déjà donné ses preuves de cette sagacité dans un Essai sur la nature des quantités positives et négatives en algèbre et sur leur interprétation géométrique (proeve over den waren aard van den positiven en negatieven toestand der grootheden etc.) Malheureusement, cet ouvrage, dont il n'existe aucune traduction, n'est pas aussi connu qu'il mériterait de l'être."

had to be made clear for every beginner [Gelder 1815, p. v]. He wrote d'Alembert's paradox as 8: -6 = -4:3.

De Gelder made it plain to his readers that a mathematical explanation is not a physical explanation. Whereas a physicist could be content as long as some handy principle explained his observations, a mathematician had to start from principles which had to be made evident and true [Gelder 1815, p. 15]. Therefore he might start with simple counting. Once one could count, numbers could easily be represented on a line. On the line there was a relation <, defined as: $a < b \iff a$ is to the left of b. In concrete terms, adding an item to a stack is much like going one step to the right on a line. The operation of addition was thus defined as the movement "going to the right". Analogously, going one step to the left was identified with taking one item away from a stack, and subtraction was defined in terms of "going to the left". Thus a - b had been defined for all b < a (when b and a are natural numbers). Multiplication was perceived as a short cut for adding the same number repeatedly. Division was perceived analogously.

Now De Gelder imagined the following scenario: there he is in some point a and he wants to consider a-b (b < a). This is represented by taking away b times one from a. In doing so, one would eventually end up with zero (namely in the case b=a). De Gelder then introduces the negative quantities by simply stating that there is no reason to stop these motions at zero. Counting backwards, one would come successively to 'points' one, two, three etc. to the left of zero. This led him to see the value of (a-b), for all a and b in the three distinct cases: a > b, a = b and a < b, where the middle case stands as a transition phase between the two other cases [Gelder 1815, pp. 22–25]. Considering this kind of counting, he remarked:

"So if one writes 8-5=+3 and 7-10=-3, the first expression means: the number 8 has a surplus of three units with respect to 5; and the second: the number 7 fails three units short with respect to 10. The differences +3 and -3 therefore are abstract numbers, that do not essentially differ from each other, neither in their nature nor in their quality: the signs + and - in front of the numbers only denote two opposite states of the formula a-b. No numbers can therefore be in a positive or negative state,

unless they can be perceived as values of the formula a - b." 15

Negative quantities were also meaningful and 'natural': the merchant expressed his loss or profit in a natural number – as a mathematician one could call a profit positive, and simply attribute a negative value to a loss. Without hesitation, people understood the expression: 700 years before our era. Of course negative numbers or quantities were not "less than zero" – that is only a matter of speaking which could (and should) be made more precise. Analogously, $\frac{1}{2} \times \frac{1}{2}$ is not a multiplication in the original sense of the word, but a division, though it is commonly called multiplication as well. No problems arose there, not even in expressions such as $a^{\frac{1}{3}}$ or a^{-4} . These expressions had simply developed among the linguistic tools of the mathematician: we knew what they meant, even if they were not exactly fitting our original notion of the operation, they could nevertheless be conceived as such [De Gelder 1815, pp. 28–32]:

"No philosopher we know of, ever opposed to the meaning of the words and symbols discussed, nor to the very peculiar expression $a^0 = 1$. This is very natural, since all these expressions, no matter how peculiar they may appear, are captured within the general expressions $a \times b$ and a^n , and they follow from them as naturally as the notions of positive and negative derive from the expression a - b." ¹⁶

This is noteworthy: for though De Gelder wanted to introduce negative quantities, he stated that it is 'natural' to use such negative quantities and to find a reasonable meaning for the operations already defined. In de Gelder's mind, negative numbers, as well as rationals, already existed.

 $^{^{15}}$ [Gelder 1815, p. 25]; literally: "Wanneer men dus schrijft 8-5=+3 en 7-10=-3, zegt de eene uitdrukking: het getal 8 heeft 3 éénheden over, om het tot op 3 te verminderen; en de tweede: er ontbreken aan het getal 7 drie éénheden om hetzelve tot op 10 te brengen. De verschillen +3 en -3, zijn derhalve slechts abstracte getallen, welke in aard en hoedanigheid niet onderscheiden zijn: de teekens + en - voor dezelve geplaatst, duiden slechts twee tegenovergestelde toestanden der formule a-b. Geene getallen kunnen derhalve zich in eenen positieven of negatieven toestand vertoonen, ten zij zij als waarden van de formule a-b beschouwd worden." - italics are as in the original.

 $^{^{16}}$ [Gelder 1815, pp. 32–33]; literally: "Geen wijsgeer, zoo veel ons bekend is, heeft zich tegen deze beteekenis van woorden en teekens, bij onderlinge overeenkomst, vastgesteld, immer verzet, noch zich aan de zeer oneigenlijke uitdrukking $a^0=1$ gestooten; en dit is ook zeer natuurlijk; omdat alle deze eigenlijke en oneigenlijke wijze van zeggen, in de algemeene formulen $a\times b$ en a^n , in dezelver algemeenste uitgestrektheid genomen, liggen opgesloten en uit dezelve even zoo natuurlijk voortvloeijen, als, uit de algemeenheid der formule a-b, het begrip van positief en negatief".

They only needed to be understood correctly, while expressions such as "less than zero" had to be given a meaning which is easily grasped. For he agreed with d'Alembert and Carnot that this last expression could not in itself be understood. It was mostly 'a matter of speech' which—in itself—was harmless, but had nevertheless to be explained precisely. He then embarked on the venture to formalise negatives by building on the ideas he had already introduced.

His formalisation started with the idea of counting, and counting backwards on a line, with a given point zero as a starting point. This immediately yielded the concept of negative numbers for, according to De Gelder, there was no reason to end counting backwards from 0: one could easily extend this procedure (by the Euclidean postulate) and go on counting -1, -2, -3, etc. These numbers he called "negative" to indicate that they were to the left of zero. In fact, the terms "positive" and "negative" were considered more general than the action-related 'to the left and/or right'. Should the line be vertical rather than horizontal one could indicate a point 0 and a positive side to it as well. Negative numbers were simply 'on the other side' of some point zero. The relation < (and its counterpart >) were meaningful for negatives as well as for positives: an expression like -2 > -3 had to be understood as -2 is closer to the chosen point zero than (and stands to the right of) -3 [Gelder 1815, p. 36]. Addition and subtraction remain possible, but now a subtraction like a - b is no longer restricted to cases where b < a. Multiplication with a negative number he interprets as multiplication with the corresponding positive number and changing the sign "which gives it a negative meaning" ("in een negatieve zin genomen"). This entails the rule $-a \times -b = +ab$. De Gelder was aware of the difficulty attached to the signs rule that he so casually stated at this point. Two proofs are given later in his book. The one discussed in some detail below illustrates De Gelder's awareness.

The mistake that d'Alembert and Carnot had made—according to De Gelder—was that they regarded negatives as negative numbers in too strict a sense. To De Gelder there existed no such things as 'abstract numbers' and, in this regard, he had a blind spot for what Carnot was striving to achieve: De Gelder's numbers always were abstractions of something feasible. So without making any distinction between the theoretical concept of number and the concept of quantity, De Gelder

once again emphasised the necessity for some reference point called zero: looking at a-b was the key to his solution. He formulated six axioms (he uses the Dutch term "axioma's") in which he considers the relation > between positive numbers:

Axiom 1: $a > b \Longrightarrow a + c > b + c$, Axiom 2: $b > c \Longrightarrow a - b < a - c$, Axiom 3: a > b and $c > d \Longrightarrow a + c > b + d$, Axiom 4: $a > b \Longrightarrow ac > bc$, Axiom 5: a > b and $c > d \Longrightarrow ac > bd$, Axiom 6: $a > b \Longrightarrow a/c > b/c$.

These axioms were regarded as generally accepted truths: they were illustrated with examples, and De Gelder referred to the definition of > in order to make these axioms plausible. By choosing to state these 'rules' as axioms De Gelder made clear that he believed that the foundations of mathematics ought to be made explicit—even as they are "generally accepted truths". The first three axioms, De Gelder stated, were obviously true as well for the negative numbers, a fact which he illustrated by a few examples and a short more general reference to the various definitions. The other three axioms changed if negative numbers were allowed: the inequality might turn around then. This, he argued, was logical too, since multiplication with a negative number is like multiplication with the absolute value, to which one changes the sign of the product [Gelder 1815, pp. 53–72].

At this point, De Gelder wanted to show that his theory of negative numbers would also work in geometry. For this reason he introduced the following construction: a perpendicular pair of lines, with the point of intersection called the zero point for both lines. Points to the right or on the upper side were called positive, the ones on the other side negative. Since he needed to implement his idea of viewing negative quantities as a 'state' of a-b he introduced what he called the distance ("afstand") of a point A on one of the lines (or on a line parallel to one of the lines) from zero (a reference point on that line), defined as its coordinate. This meant that points to the left of point zero had a negative distance to zero, and this is what he needed to multiply negative quantities. He noted that given such a distance from zero and given the line upon which a point laid upon, the point was thus uniquely determined.

Since negative quantities had, according to De Gelder, something to do with the 'state' of a-b, he wanted to simulate a 'changing state' by making points and lines move from a positive state to a negative state passing through zero. Algebraic and geometric reasonings had to lead to the same conclusions, so one of the things De Gelder wanted to show was that the signs rule worked in the geometrical context just described [Gelder 1815, pp. 74–116]. For this purpose he looked at two perpendicular lines AB and AD (see the figure below, which is taken from De Gelder's book); point A is the zero point of the two lines, with points E and E0 lying on the positive side of E1. On E2 the same of E3 a moving line E4 are kept the same; E5 is the intersection point of E5 and E6. Now E6 is a square with positive sides. Sliding the lines E6 and E7 makes it possible to construct a square larger than E6 and E7 makes it possible to construct a square larger than E6 and E7 makes it possible to construct a square larger than E6 and E7 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 makes it possible to construct a square larger than E6 ma

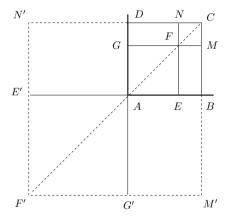


Figure 1

As he wants to study negative quantities, the idea is to let BC and DC slide back to the left and down, respectively. They first would pass through the state of the original square AEFG. With respect to ABCD, he can apply the gnomon theorem:

$$AEFG = ABCD - CNFM - DNFG - BMFE.$$

But then he keeps on sliding EN and GM (and thus their intersection point F as well) until they reach the position E'N' and G'M' intersecting at F'. That the above described movement yields this result is obvious, according to De Gelder, since:

"Once we have started to transform a geometrical object along a certain path, it will follow its original course and we can't assume any arbitrary jumps or unnatural changes in its form or in the laws it is obeying." 17

Since, during the displacement, F passed through point A, and thus crossed the zero point, the state of a-b might have been affected. Indeed, according to his definition, the square AE'F'G' (which is the result of the described transformation of AEFG) has negative sides, that is: the distances of E' and G' to A are negative. Therefore AE'F'G' is the product of two negative numbers.

That this gnonom theorem still holds is obvious to De Gelder, so it was a matter of determining what had happened to the various terms during the displacement. The square CNFM was positive to begin with and, during its transformation to CN'F'M', it would only become larger, so it would not change sign. Both DNFG and BMFE, however, were zero when EN and GM coincided with AD and AB, respectively. Passing through point zero, the state of a-b should change, and so the signs of DNFG and BMFE. This meant that the above equation in the context of the new situation looked like this:

$$AE'F'G' = ABCD - CN'F'M' + DN'F'G' + BM'F'E'.$$

The last two terms in this equation are larger than CN'F'M'-ABCD, which guarantees that AE'F'G' is positive [Gelder 1815, pp. 117–120]. Another, rather similar proof, once again confirmed this conclusion and indicated that there was indeed perfect harmony between algebraic negative numbers and their geometrical interpretation.

D'Alembert's paradox was solved by stating that proportions were only relevant for absolute values ("hoegrootheid"). This reduced the meaning of 8:-6=-4:3 to $8\times 3=6\times 4$ (of course the signs had to match somehow). What remained, however, was the paradox of the ordering

 $^{^{17}}$ [Gelder 1815, p. 117]; literally: "[Wanneer men een] figuur, eens in beweging gesteld heeft, zoo moeten zij hare bewegingen blijven achtervolgen: hier mag men geene willekeurige sprongen, geene onnatuurlijke en gedwongene veranderingen aanneemen."

relation, which was solved by the axiomatic framework De Gelder had built. Having dealt with all paradoxes, and claiming to have a firm grasp on negative quantities, De Gelder closed his paper by rejecting Carnot's quantités directes and quantités inverses. These terms, he states explicitly, were simply equivalent to the ones they were supposed to replace with nothing more to gain [Gelder 1815, p. 242]. It seems that he felt no need for a geometrised mathematics, for he had just (again) shown that unity existed within the mathematical sciences.

De Gelder's geometrical 'proof' of the signs rule bears a remarkable resemblance to Carnot's geometric figures, continuously transformable into one another, and the related transformation of equalities by changing signs of terms depended on the "directes" or "inverses" nature of the lines in the figure. Whereas these continuously tranformable figures played a seminal role in [Carnot 1803] (where they offer a justification to algebraic manipulations), to De Gelder this proof simply showed that algebra and geometry are in complete accordance regarding results obtained from the manipulations of negative quantities.

De Gelder's views were influential in his days. He referred to his work on negative numbers many times later, in his algebra book for the gymnasia [Gelder 1826], when he had become a highly regarded influential Leyden professor. The algebra text book mentioned was most widely used in Dutch secondary education up to the 1840s [Smid 1997, p. 232]. As a distinguished member of the Dutch mathematical community De Gelder's views prevailed.

An alternative approach was offered by Utrecht professor of mathematics J.F.L. Schröder who defined a negative quantity in the 'German way', as a quantity opposed to a positive counterpart, which he linked to several physical examples. Multiplication with -1, he perceived as 'taking the opposite direction' [Schröder Notes, p. 16, 37]. Schröder's first textbook [Schröder 1831] was published much later, and he never had followers until after his death, when some of his former students started exporting his ideas by writing geometry textbooks (the most noteworthy of these being [Buys Ballot 1852]).

4. LACROIX'S TEXTBOOK

Lacroix's textbook was translated by I.R. Schmidt (1782–1826) and it was currently used at the Military Academy for a short period of time. Printed for the first time in 1821, it was used until it was replaced in 1838 by the textbook of J. Badon Ghyben (1798–1870) and H. Strootman (1799–1851) [Badon Ghyben and Strootman 1838]. This book was written in an entirely different didactical style: Strootman and Badon Ghyben wanted their students to acquire the mathematical prerequisites (mainly the ability to perform algebraic reckoning) for their future carreer. Their views of mathematics education were much more practical and less focussed on the exposition of a suitable mathematical theory than was Schmidt's. Both Schmidt's book, as well as the one that replaced it later, were used at Dutch secondary schools—sometimes because these schools wanted to offer their pupils a decent or more sophisticated preparation to what was to come at the Military Academy. So the two authors reached quite a large readership [Smid 1997, pp. 200–203].

According to its title page Schmidt's book was a free translation from the French original ("vrij gevolgd naar het Frans van . . ."). A quick glance at its table of contents reveals that the structure of Schmidt's book is tailored almost exactly on that of Lacroix's Élémens [Lacroix 1797] and Complément [Lacroix 1801]. The one exception is in Schmidt's added section on arithmetical and geometrical series – a subject which Lacroix treated in his arithmetic textbook [Lacroix 1808]. This may be considered a minor change, since the theory of series was not part of the arithmetics curriculum at the Military Academy, and neither was it at Dutch secondary schools. Most sections have been translated quite accurately. ¹⁸ The theory of negative quantities as it was introduced and applied by Lacroix, however, was largely transposed in Schmidt's own words.

¹⁸ Page references to Lacroix are always from the 11th edition of [Lacroix 1797], Paris: Courcier, 1815. For the Dutch translation page references are always to the second edition of [Lacroix 1821a]. The textbook by Lacroix has been revised several times, but on the points that are about to be discussed, from the 7th edition (1811) onwards no changes were made. The textbook by Schmidt was never revised. Schmidt died within a year after publication of the second edition. The statement about the similarity between Schmidt's and Lacroix's textbooks has been made on the basis of a sample of 20 sections from the editions indicated – excluding the ones discussed hereafter – which are almost literal translations.

Schmidt was a good friend of De Gelder. He was appointed to De Gelder's position after the latter's dismissal from the Military Academy. They nevertheless kept working together in harmony for the Dutch Mathematical Society. Schmidt published a whole series of textbooks for the Military Academy, which, in part, replaced De Gelder's. Most of these publications were translations of textbooks by Lacroix. ¹⁹ His original contributions consist of textbooks on calculus [Schmidt 1822a], analytic geometry [Schmidt 1822b], mechanics [Schmidt 1823, 1825], and one article about a property of plane triangles [Schmidt 1817] in the journal of the Dutch Mathematical Society. His posthumous treatises [Schmidt 1827], as well as the work he published during his lifetime, show that his main interest and concern were in the teaching of mathematics.

Schmidt agreed with Lacroix's view—which he replicated literally—that negative quantities are a difficult subject in algebra, which must be founded on a solid base. Lacroix notes that a-a=0 for all (positive) a and uses this fact to decompose a subtraction when the result is negative: for example, he wrote 3-5 as 3-3-2. Since this is always possible, he viewed a negative result as something impossible, but if (in the above example) 2 was added, the result would have been zero. In an abstract sense, a negative quantity was meaningless [Lacroix 1797, p. 92]. Translating Lacroix rather accurately at this point, Schmidt made however a subtle difference in his translation. Like De Gelder, Schmidt regarded a negative quantity as a positive one in another state which more or less naturally appeared in calculations: leaving out Lacroix's decomposition he wrote 3-5 as 5-3, but with a minus sign in front of it, to indicate that it is necessary to do the 'actual' subtraction in the reversed order [Lacroix 1821_a , pp. 92].

Though Schmidt translated Lacroix accurately, he departed from the French text in a subtle way, especially by introducing different notions. Lacroix brought in the sign rule by looking at the formula $-a \times (b-b)$, which can be written as $(-a) \times b + (-a) \times (-b)$; the first term is then -ab by the definition of the multiplication and, since the sum has to be equal to zero, the second term can only be equal to +ab, to 'compensate' for the first term. In his exposition, Lacroix carefully avoids using the

 $^{^{19}}$ Apart from the ones which are discussed here, he translated $\it G\acute{e}om\acute{e}trie~descriptive$ [Lacroix 1821b], $\it G\acute{e}om\acute{e}trie$ [Lacroix 1821c] and $\it Goniom\acute{e}trie$ [Lacroix 1822].

equality sign, replacing it by the words "doit donner" [Lacroix 1797, p. 93]. Lacroix, then, was not reckoning with negative quantities, instead he was establishing rules for what he called a "changement de forme": changing the form of an equation. Like Carnot, Lacroix encountered negative numbers while solving equations. Multiplying both sides of an equality by a negative number could, of course, be useful if one wanted to solve an equation. And that was really all that Lacroix wanted: a 'rule' for transforming an equation into an equivalent one. Multiplying by a negative quantity, however, was *not* an operation that could actually be performed.

In contrast Schmidt's proofs were about negative quantities. Whereas Lacroix carefully avoided using the equality sign when discussing the sign rule, Schmidt directly concludes that " $-a \times -b = +ab$ " [Lacroix 1821a, p. 93]. After proving the sign rule–for Schmidt the sentences he copied from Lacroix had the status of a proof–Schmidt adds some remarks that are nowhere found in Lacroix's original text. At this point he departs from the French original only to return to it after the exposé on negatives. Schmidt's remarks read as follows:

"The words positive and negative are general terms, that indicate the different states a quantity can be in, and that in special cases will have interpretations such as capital and debt, east and west, right and left, up and down, ascending and descending, winning and losing, etc. In each particular case it is up to us to choose which of the two states we wish to call positive, and thereby denote with the + sign, but once this is determined, we must consistently call the other state negative, and indicate it by the sign -." ²⁰

When an equation turned out to have a negative solution, Lacroix seized the opportunity to look carefully into the meaning of negative solutions, and find in what way they constitute a solution to the initial problem [Lacroix 1797, p. 94]. Schmidt, on the contrary, knew exactly

²⁰ [Lacroix 1821a, p. 94]; literally: "De woorden positief en negatief zijn dus algemeene namen, die de verschillende toestanden van eene grootheid aanwijzen, en die in bijzondere gevallen de namen kapitaal en schuld, oost en west, rechts en links, boven en beneden, klimmen en dalen, winnen en verliezen, enz. verkrijgen. Het staat nu bij elk onderzoek wel volmaakt vrij, welke van deze benamingen wij door het woord positief, en dus door het teeken + willen uitdrukken, maar dan moet ook, gedurende dit geheele onderzoek, de tegenovergestelde toestand als negatief worden aangemerkt."

what to do:

"If one looks closely at the definition of a negative quantity, as it has been given in the last section, it is *not* necessary to return to the original equation and look what changes would have to be made in the given problem in order to make these negative solutions comprehensible. Since the negative quantity -a can only be understood as the quantity which has to be added to +a in order to obtain zero as a result, this means that all one has to do to interpret the meaning of -a is to look what it takes to annihilate +a. 21 – my emphasis [DB]."

In his original text, Lacroix set a problem to illustrate his views on what to do when negative solutions occur. Consider two people who start running towards each other; 22 one starts from A and runs b kilometres per hour while the other starts from B and runs at a speed of c km/h; the distance between A and B is a. The question is at which point they will meet. Schmidt's translation is only slightly different: the question is nearly the same, but does not mention the fact that the runners are moving towards each other. 23 This, of course, is the prelude to two strikingly different solutions.

Lacroix gives a general solution to his problem, concluding that for all (naturally positive) values of a, b and c, the two persons will meet at a distance ab/(b+c) from A. Then he restates the problem for the case in which the two runners move in the same direction; the general solution is ab/(b-c). If b>c then all is well, but in the case that b<c the negative result means that there is no solution and that it is absurd to ask at which point A will overtake B if the former

 $^{^{21}}$ [Lacroix 1821a, p. 94]; literally: "Wanneer men de bepaling der negatieve grootheden, in de voorgaande § opgegeven, met aandacht nagaat, is het *niet* noodzakelijk, bij het verkrijgen van negatieve uitkomsten, tot de oorspronkelijke vergelijking terug te keeren, ten einde de wijzigingen op te speuren, welke men de opgaaf zou moeten doen ondergaan, om er deze uitkomsten op toepasselijk te maken. Daar namelijk door de negatieve grootheid -a zulk een uitdrukking moet worden verstaan, die met +a moet worden opgeteld, om nul te verkrijgen, zoo volgt hieruit, dat men in alle gevallen, ten einde de beteekenis van -a te verklaren, alleen zal behoeven te onderzoeken, wat men doen moet om +a te vernietigen."

 $^{^{22}}$ Although the choice of letters is a bit confusing, I opted for the same ones which Lacroix used for the respective variables.

²³ [Lacroix 1797, p. 94]; [Lacroix 1821a, pp. 94–95]; the Dutch translation mentions two "physical objects" in the most general sense (*ligchamen*) – as if it is an arbitrary mechanics problem. These objects are running (*lopen*), however!

is slower than the latter [Lacroix 1797, pp. 96–105]. Schmidt, on the contrary, immediately starts by indicating a positive and a negative direction.

Finding a similar solution to Lacroix's, Schmidt changes the signs of b and c and interprets the results in his framework of negative numbers. A velocity of -b is interpreted as a velocity of b in the opposite direction. In case the runners start off in the same direction, and the runner starting 'in front' is running faster, Schmidt interprets the resulting negative solution to be a virtual common starting point [Lacroix 1821a, pp. 95–105].

After discussing several cases Lacroix concluded:

"Ce qui précède fait voir bien clairement que les solutions algébriques, ou satisfont complètement à l'énoncé du problème, quand il est possible, ou indiquent une modification à faire dans l'énoncé, lorsque les données présentent des contradictions qui peuvent être levées, ou enfin font connaître une impossibilité absolue, lorsqu'il n'y a aucun moyen de résoudre avec les mêmes données, une question analogue dans un certain sens à la proposée" [Lacroix 1797, p. 105].

The corresponding sentences in Schmidt's "translation" read as follows:

"From all this we may conclude that the solutions we obtain always are exactly what was asked for, if we only keep in mind what was chosen to be positive when we solved the problem. For if we know this, then the sign [+ or -] of the solution will let us know all the circumstances that may take place." ²⁴

Here Lacroix returned to the problem of the two runners to illustrate what his remarks would mean in practice. Schmidt instead skipped these pages and picked up the text where Lacroix ended his exposé.

Schmidt's 'translation' of Lacroix's algebra textbook was thus in fact a fundamental rejection of Lacroix's and Carnot's views on negative quantities. While staying quite close to the original text, Schmidt changed crucial remarks. He opted for an approach that stood entirely in line with

²⁴ [Lacroix 1821a, p. 107]; literally: "Uit dit alles blijkt dan, dat de uitkomsten der stelkunstige oplossingen altijd volmaakt aan de opgaaf beantwoorden, indien men slechts met oplettendheid nagaat, wat men in de oorspronkelijke oplossing als positief heeft aangemerkt, daar men alsdan, door de teekens, welke de uitkomsten verkrijgen, in alle gevallen, op eene ongedwongene wijze, al de omstandigheden zal kunnen verklaren, die er bij de bijzondere gevallen, waarin het vraagstuk verkeeren kan, plaats kunnen grijpen."

the tradition of Dutch work on negative quantities.

5. CONCLUDING REMARKS

Present day mathematicians are trained to build on definitions and axioms in order to prove theorems. A proof is unacceptable if the theorem is about a concept whose definition is not used within the proof. For an early 19th century Dutch mathematician, this was not obvious. The definition of the negatives was related to physical quantities, since this is what mathematics was about. Understanding the concept itself was much more important to De Gelder and Schmidt than any logically sound proof. In that sense they had a blind spot for what d'Alembert, Carnot and Lacroix had striven to achieve.

Notions of proof and definition were shared by a majority of Dutch mathematicians as well as several mathematicians from other countries. It is tempting to attribute this to the working conditions in which these mathematicians worked. In the Netherlands (as in most other countries) no mathematical research was carried out as such and mathematical texts, such as De Gelder's, were largely produced for educational purposes. Most educators saw mathematics merely as a tool (though an indispensable one) for doing physical or technical research: according to De Gelder, Schmidt, and many of their contemporaries mathematical symbolism was not arbitrary, but linked to physical quantities. This did not conflict with their search for rigour; but rigour was subordinate to teaching. Dutch mathematicians did think that the foundations of mathematics were important, and wrote (or translated) textbooks in which they paid some attention to these foundations. The translation and partial mutilation of Lacroix's textbooks show how seriously these mathematicians held their own views on what negative quantities were about. This is why these books shed a light on the conceptions of rigour that prevailed in the early 19th century.

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