

**FROM ATTRACTION THEORY TO EXISTENCE PROOFS:
THE EVOLUTION OF POTENTIAL-THEORETIC METHODS
IN THE STUDY OF BOUNDARY-VALUE PROBLEMS, 1860–1890**

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ABSTRACT. — This paper examines developments in the study of boundary-value problems between about 1860 and 1890, in the context of the general evolution of this theory from the physical models in which the subject has its roots to a free-standing part of pure mathematics. The physically-motivated work of Carl Neumann and his method of the arithmetic mean appear as an initial phase in this development, one which employs physical models as an integral part of its reasoning and which concentrates on geometrical hypotheses concerning the regions under study. The alternating method of Hermann Amandus Schwarz, roughly contemporary to that of Neumann, exhibits more strongly the analytic influence of Weierstrass. Both methods form the essential background to Émile Picard's method of successive approximations, developed by him following a reading of both men's work. Picard's work, analytically rigorous and remote from physical argument, marks both a transition of the subject matter from applied to pure mathematics, and the full comprehension and mastery of Weierstrassian methods in the French context.

RÉSUMÉ. — DE LA THÉORIE DE L'ATTRACTION AUX THÉORÈMES D'EXISTENCE : L'ÉVOLUTION DES MÉTHODES DE LA THÉORIE DU POTENTIEL DANS L'ÉTUDE DES PROBLÈMES AUX LIMITES, 1860–1890. Cet article analyse les contributions à l'étude des problèmes aux limites, au cours des années 1860–1890, dans le contexte de l'évolution générale de la théorie qui, partant des modèles physiques où la question trouve ses racines, se constitue en domaine autonome relevant des mathématiques pures. Les travaux de Carl Neumann inspirés par la physique et sa méthode de la moyenne apparaissent comme la phase initiale de cette évolution, celle qui emploie des modèles physiques comme partie intégrante des raisonnements et qui se centre sur les hypothèses géométriques relatives aux régions considérées. Le procédé alterné dû à Hermann Amandus Schwarz, méthode à peu près contemporaine, porte nettement la marque de l'analyse weierstrassienne. Ces deux méthodes constituent pour l'essentiel le fonds où s'inscrira la méthode des approximations successives d'Émile Picard, que celui-ci a développée à la suite de la lecture des travaux des deux auteurs précédents. Les recherches de Picard, analytiquement rigoureuses et éloignées des argumentations

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physiques, marquent à la fois le passage du domaine des mathématiques appliquées à celui des mathématiques pures et l'avènement de la pleine compréhension et maîtrise des méthodes de Weierstrass en France.

1. INTRODUCTION

On July 6, 1937, Émile Picard was awarded the *Prix Mittag-Leffler* at the Institut de France. The prize was awarded by the Institut Mittag-Leffler for “*les découvertes qui constituent une source nouvelle et importante de progrès futurs pour les Sciences mathématiques*”, and consisted of a gold medal with the portrait of the winner, a diploma, and a personalized set of *Acta mathematica*. At the ceremony, Picard recounted the fame of Karl Weierstrass and his Swedish disciple Gösta Mittag-Leffler in the Paris of the mid-1880s:

“*Il arriva même que dans une de ces cérémonies, dites les Ombres, où les Polytechniciens font d’innocentes plaisanteries sur leurs professeurs, on annonça la découverte d’un nouveau verset de la Genèse, où il était écrit: ‘Dieu créa Weierstrass, puis, ne trouvant pas bon que Weierstrass fût seul, il créa Mittag-Leffler’*” [Picard 1938, pp. xxiii–xxiv].

The joke shows a widespread appreciation of the importance of Weierstrassian analysis in the French mathematics of the time, particularly those aspects of it most closely associated with the name of Mittag-Leffler: the theory of functions of a complex variable and its applications to other areas of analysis. Of course, France could lay claim to much of this theory thanks to the foundational work of Cauchy. Its later elaborations in Germany, due to Riemann and Weierstrass among others, had become known to the French mathematical community largely through the intermediary of Hermite, who lectured on these matters to Picard among others. In the next generation, Picard himself was instrumental in introducing these German techniques to French mathematicians and students, and so was an important figure in the development of an international style of mathematics from a congeries of distinct national schools. The requirements of Weierstrassian rigour, particularly in analysis, were instrumental in this transition. Originally conceived as a language of justification, Weierstrass’s analysis soon revealed itself to be a powerful tool for discovery as well; and this feature in part accounts for its success among his

students and adherents, as well as for its spread to mathematical communities outside Germany. In that context, it was natural that existence and uniqueness theory for partial differential equations should assume a front-line position.

Until the mid-nineteenth century, partial differential equations were not studied in a unified fashion, and there were few general results which could be considered to unify the theory. For the most part, individual equations were studied in the context where they arose; in the case of boundary-value problems, this meant that the Laplace-Poisson equation was studied in connection with the theory of gravitation, electrostatics, or steady-state heat conduction, while the wave equation arose in acoustics and optics, etc. The question of existence theorems for boundary-value problems was raised by the well-known critique by Weierstrass of Riemann's justification of the Dirichlet principle, which the latter had employed to show the existence of a solution to the Dirichlet problem for plane regions, given appropriate boundary conditions. The efforts to rehabilitate Riemann's proof were many. The first to succeed, beginning around 1870, were those of Carl Neumann — known as the method of the arithmetic mean, which established the existence of solutions for the Dirichlet problem by a method of approximate solutions; and those of Hermann Amandus Schwarz.

Both Neumann's work and that of Schwarz were seen by most readers as part of a specialty, called potential theory, which concerned itself not only with the theory of the Laplace-Poisson equation and associated boundary-value problems, but also with the associated special functions (spherical harmonics, etc.) and with applications especially in gravitation (attractions of ellipsoids, figures of planets) and electromagnetic theory (equilibrium electrostatic densities given an external force, forces given densities, etc.). However, the work of Neumann and Schwarz was generalized, in the hands of Emile Picard, to become the method of successive approximations, which Picard showed could be applied to a wide variety of boundary-value problems for second-order equations. At around the same time, Picard's Paris colleague Henri Poincaré began to systematically investigate the analogies between the various partial differential equations, mostly of second order, which are associated with physical problems.

These simultaneous efforts may be seen as part of the establishment of the subject of partial differential equations as a recognized research specialty, independent of its applications. At the same time, to an increasing degree, mathematical physics and pure mathematics were in the process of disciplinary separation. Hence fewer mathematicians undertook both kinds of research, and an increased specialization of institutions (such as journals, university departments and institutes) also occurred. This in turn led to a lessened emphasis on direct physical applications in potential theory, and to the subsuming of the latter into partial differential equations as a research specialty. We may see that this in a way completes the divorce of potential theory from physics, though of course certain problems were still of interest to physicists. These would however then be seen as applications of the theory, rather than as instances of it, and tended to be undertaken by different individuals from the pure mathematical problems.

It is the purpose of this paper to examine aspects of this transition. In particular, we shall concentrate on the background to the development of the method of successive approximations by Picard. As Lützen has discussed in detail, the method was used as early as 1830 by Liouville [Lützen 1990, pp. 447–448], though more as a solution method than as an existence proof, which is how Picard employs it. The pivotal position of the so-called Dirichlet problem in these developments makes it convenient to begin with a discussion of research related to this question.

2. CARL NEUMANN AND THE DIRICHLET PROBLEM

The Dirichlet problem is the following: given the values of a function on the boundary of a region in space or in the plane, find a function which is harmonic on the region and which takes on those boundary values. It is closely associated with the conformal mapping question; for if we can solve the problem for a particular region (*e.g.* a circular disc) we can extend the solution to other regions through composition with a harmonic function which provides a conformal representation of the region onto the disc. This idea was first worked out by Bernhard Riemann in his 1851 dissertation. There were well-known difficulties with Riemann's approach, however; his existence proof depended on the "Dirichlet principle", about which much has been written. In particular, Weierstrassian critiques called

into question the validity of the Riemann mapping theorem, one of the cornerstones of Riemann's function theory.

These critiques were addressed, from rather different standpoints, by Carl Neumann (1832–1925) and Hermann Amandus Schwarz (1843–1921), beginning in the 1860s and culminating in successful results about 1870. These works are indicative of the transitional state of affairs with regard to partial differential equations in Germany at the time, and were of particular importance to Picard.

Carl Neumann's earliest work on potential theory had revolved around the Dirichlet problem; his other interests in the period show that he was influenced by Riemann in this regard. In 1861, while at Halle, Neumann produced the first of his many papers on this question, The paper, "Ueber die Integration der partiellen Differentialgleichung $\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0$ ", treated the Dirichlet problem in the plane. It contains two principal results, both solving the problem explicitly for a limited class of regions. For Neumann, the work remained close to his physical investigations; he began by pointing out the analogy with the three-dimensional problem of steady state temperature distribution. The problem, specifically, is to find a function $F(x, y)$ which satisfies the Laplace equation inside a connected region R in the plane bounded by a curve of arbitrary form such that F and its first derivatives remain finite, single-valued, and continuous inside R , possessing given values on the boundary of R . In the three-dimensional case, Neumann points out, the use of the theory developed by Green and Gauss of the potential corresponding to the Newtonian attraction law is of great assistance with the problem, and further:

*"Likewise it is useful here in considering our planar problem to assume as an auxiliary a hypothetical matter or fluid which is distributed arbitrarily in the plane, for which the potential of two particles on one another is equal to the product of their masses multiplied by the logarithm of the distance between them."*¹

That one requires auxiliary fluids rather than auxiliary functions seems

¹ "Ebenso ist es hier bei Behandlung unseres Problems der Ebene zweckmässig, eine fingirte Materie oder ein fingirtes Fluidum zu Hülfe zu nehmen, welches auf beliebige Weise in der Ebene vertheilt wird, und für welches das Potential zweier Theilchen aufeinander gleich ist dem Product ihrer Massen multiplicirt mit dem Logarithmus ihrer Entfernung" [Neumann 1861, p. 336].

quite strange to modern eyes. It seems to insist on a close correspondence between the terms of an equation describing a physical system and the actual components of the system, even if the latter are hypothetical (*fingirte*). This is something that occurs later in the work of Neumann, and he continues to use this language throughout the 1861 paper. Not surprisingly, Neumann appears to have electrical problems in mind; when discussing the fact that a surface distribution can be replaced with a linear distribution on the bounding curve, he suggests the distribution of electrical fluid as a useful analogy to bear in mind. In this context, it is perhaps worth noting that the dominant physics in continental Europe in Neumann's day was one in which physical occurrences were governed by particulate interactions, occurring in obedience to (for the most part) central action-at-a-distance force laws between hypothetical point-masses (or point-charges).

Neumann then summarizes the main properties of the logarithmic potential which are of use for his problem. (I will not retain all the features of his prolix notation.) These are:

1) The logarithmic potential, V , resulting from a mass distribution lying outside a plane region R satisfies Laplace's equation inside R ; and if a function V is finite, continuous and single-valued, and satisfies Laplace's equation in a plane region, it may be considered as a potential resulting from a mass distribution outside R .

2) A potential function of the region R can always be represented as the potential of a certain distribution on the boundary of R .

He also discusses the Green's function of the region, $G(p, q)$, which he defines as the potential function of the given region which agrees with the logarithm of the distance (p, q) when the variable point p is located on the boundary. These results are unattributed in their specifics; he does mention the work of Gauss, and it seems likely that the other material is familiar to him from Riemann's work (though his specifically physical discussion of these functions points to an acquaintance with Green's 1828 *Essay*, which appeared in Crelle's *Journal* in the 1850s).

The first principal result of Neumann's paper states that, given a region R which is symmetric about the x -axis, if a function V is known which has the properties:

1) $\Delta V = 0$ inside the region;

2) V and its first partial derivatives are finite, single-valued and continuous everywhere in the region, with the possible exception of a segment of the x -axis;

3) V has a constant value on the boundary, and a (different) constant value on the segment of the x -axis,

then a method is available which allows us to solve the Dirichlet problem for the region. (Neumann overlooks the fact that in general $\Delta V = 0$ is not satisfied on the stated portion of the axis, which would imply a constant solution.)

The method is as follows. We introduce two variables θ and ω , where

$$\theta = 2\pi \frac{V - V_0}{\int \partial V / \partial n ds},$$

and ω is related to θ in such a way that $\theta + i\omega$ is a function of the complex variable $x + iy$. We note that θ and ω satisfy the equations

$$\frac{\partial \theta}{\partial x} = \frac{\partial \omega}{\partial y}, \quad \frac{\partial \theta}{\partial y} + \frac{\partial \omega}{\partial x} = 0$$

(the Cauchy-Riemann equations) as a result of the requirement of functionality. The integral in the denominator of the expression defining θ is taken around the bounding curve, n is the outward normal, and V_0 is the constant value of the potential on the exceptional segment.

Neumann further sets ω to 0 at one endpoint of the exceptional segment. The given definition of θ makes it constant on the boundary, which is therefore parametrized by ω alone. Referring parametrically to an arbitrary point a of the boundary by ω_a , Neumann then is able to write the solution of the Dirichlet problem in the form

$$\Phi(p) = \int_0^{2\pi} \Phi(a)\eta(a, p) d\omega_a,$$

where p is an arbitrary point of the interior, and η is a known function, for which Neumann gives both a series expansion and a closed form expression.

Neumann's studies in the field over the next few years concentrated on various special problems, and on finding specific solutions. In 1870, however, he addressed the challenge to Riemann's methods posed by Weierstrass by introducing a general technique for establishing the existence of solutions under specific circumstances. Neumann's solution was construc-

tive, but depended on special hypotheses about the region, most notably the convexity of the contours or surfaces bounding the mass. The principal features of the method may be outlined as follows.

Considering the case in the plane, suppose we have a region with a convex boundary σ which is smooth (in the sense that the radius of curvature is everywhere finite, for example). We are given a function U at all points of σ , and we want to extend U continuously into the interior of the region so that U is harmonic.

1) We begin by defining a function for any point x of the interior which is the so-called double-layer potential of the density U at the point x . This is of the form

$$W_x = \int \frac{\partial}{\partial \nu} \left(\log \frac{1}{r} \right) U \, d\sigma$$

which can be shown to be harmonic by differentiation under the integral sign. Neumann argues that the convexity and smoothness of σ guarantee the existence of the integral. (The integral in question is closely related to the mean value of U on the boundary, and for this reason the method is known as the method of the arithmetic mean.)

2) We note that if x is a point of the boundary, there will be a jump discontinuity in W as we pass from the boundary to the interior. The method is therefore to construct a convergent sequence of functions which smooth out this jump a little at a time, while preserving harmonicity and the boundary values.

It is noteworthy that Neumann here focuses entirely on the geometry of the region and the boundary. He does not examine the conditions that are necessary on U , simply assuming sufficient differentiability properties. This contrasts with Schwarz's approach, as we shall see.

Neumann's initial 1870 announcement

At the meeting of April 21, 1870 of the Royal Saxon Society of Sciences in Leipzig, Neumann presented his first communication on general potential theory. Neumann explained that he intended to present a series of communications on problems related to potential theory: conformal mapping, steady-state heat conduction, electrostatic and "electrokinetic" equilibrium. The core of these papers, said Neumann, was a collection of general methods, independent of choice of coordinate system, intended to replace the Dirichlet principle in such arguments, which he now saw as "rightly

held to be doubtful” (*mit Recht für bedenklich erklärten*) [Neumann 1870a, p. 49]. He nevertheless remained attached to the Dirichlet principle, later describing it as: “*the Dirichlet principle, so beautiful and once so much employed, but now probably forever sunk*”.² These papers would appear over the next two decades, many recalling Neumann’s title here: “*On the theory of the logarithmic and the Newtonian potential*”.

Of interest to us is the close connection in Neumann’s work between physical intuition and mathematical methods, in connection with the tight conception of rigour. He had gained the physical intuition by studying such problems in an “applied” context for over a decade, beginning with his 1858 treatment of the Faraday effect; his physical thought in this instance grew directly from the synthesis in German electromagnetic theory due to his father, the mathematical physicist Franz Neumann of Königsberg, and to Wilhelm Weber of Göttingen. His mathematical work had also begun in a vein strongly influenced by his father’s work, in the study of the solutions of Laplace’s equation (or spherical harmonics) as they appear in a variety of coordinate systems associated with certain boundary conditions, hence continuing the early nineteenth-century focus on what are usually called today “special functions”.

However, Neumann followed the nineteenth-century trend by at least partially shifting his concentration to functions as a class, and the view of differential equations less as objects to be solved and more as objects providing information about the functions which satisfy them. In this sense, his work represents a transitional phase in potential theory: bound to the physical models for the conceptual approach, yet increasingly distant from specific applications by virtue of the generality of the methods. For Neumann, this information was in the first instance geometric, and at first the methods he had adopted were closely allied to those of Riemann. Indeed, as an early interpreter and expositor of Riemann’s function theory, he desired to salvage that theory in the face of Weierstrass’ critique, which he came to appreciate keenly perhaps as a result of participating in the Weierstrass lectures in 1869, and again in 1870.

In the paper at hand, Neumann employs Riemann’s framework by treating the plane as a surface which is closed by a point at infinity (to

² “*das so schöne und dereinst so viel benutzte, jetzt aber wohl für immer dahingesunkene Dirichlet’sche Princip*” [Neumann 1887, p. 707].

form the Riemann sphere). A curve S then divides the plane into two parts P and Q , where Q contains the point at infinity. The conformal mapping problem (that is, the problem of mapping the region P or Q onto a given region conformally) is presented as a special case of the Dirichlet problem, exactly as in Riemann's work. Hence Neumann seeks a function f which satisfies Laplace's equation inside P , takes on given values $f(x, y)$ on S , and which (together with its first partial derivatives) is single-valued and continuous inside P . He also formulates a similar problem for Q , calling the function to be found y , and assuming that it has the same boundary values given by f on S . (In order to keep the coordinates in Q finite, he inverts the coordinates (x, y) in the unit circle, obtaining corresponding coordinates (x, h) .) The curve S is supposed to be convex and to be continuously curved (*stetig gekrümmt*), meaning that there are no cusps or corners.

Neumann then sets out to construct a convergent sequence of approximate solutions to his problem, the limit of which is a solution. Defining the moment of a differential element of S with respect to the point z as the angle swept out by the radius vector from z to the curve as it moves from one end of the element to the other, and assigning a sign to this angle in the usual way, Neumann observes that the total moment of the curve S with respect to z will be 2π , π , or 0 depending on whether the point is inside P , on P , or outside P respectively. He then generalizes this notion to assign a moment to the function f defined on S as the sum (*i.e.* integral) of the elementary moments each multiplied by the function value. This gives us three functions, denoted by Neumann as U_p , V_s and W_q , corresponding to the moment of a fixed continuous f along S with respect to a point in P , on S or in Q respectively. These satisfy the fundamental relations:

$$U_s = V_s + \pi f_s, \quad W_s = V_s - \pi f_s$$

where U_s and W_s are the limits of U and W as we approach a point s on the curve through P and Q respectively [Neumann 1870a, p. 52]. Notice that in defining these moments the hypothesis of convexity guarantees that each portion of the curve will be included only once in the sum.³

³ For interior points this is clear; for exterior points, Neumann achieves this by treating the exterior points as the complement of the interior points on the Riemann sphere and transforming coordinates to their reciprocals in the manner usual in complex analysis.

Neumann prefers to work with a slightly different function, however. He defines the function u_p as $U_p/2\pi$, calling it the arithmetic mean of the boundary values f with respect to the interior point p . He likewise defines $v_s = V_s/\pi$ and $w_q = W_q/2\pi$ as the arithmetic means of f with respect to a boundary point s and an exterior point q . Convexity is a necessary hypothesis in defining these means, since otherwise we would not know what to divide by to take the mean. These means turn out to be a critical step in his method of constructing a solution, as we shall see presently.

Hence from f we get a mean v_s for any point on the curve. These too will vary continuously with s , and by repeating the same procedure with v_s in place of f we get new means u' , v' and w' . (I'm omitting the subscripts for the moment.) Continuing this procedure leads to a sequence of functions f, v, v', v'', \dots . The sequence of v 's must converge, says Neumann, by the following argument. Suppose that M and m are the maximum and minimum values of the modulus of f on the curve S . Let a be a fixed point on the curve, s a variable point. For any number n of iterations of the above procedure, occurs the inequality

$$|v_s^{(n)} - v_a^{(n)}| \leq (M - m)\kappa^{n+1}$$

where κ is a positive constant, depending only on the curve, which is less than 1. This in turn implies convergence of the sequence of means.

Neumann gives no proof of this convergence in this paper. Indeed none appeared until his 1877 monograph on Newtonian and logarithmic potential. The constant κ was referred to by Neumann as the configuration constant of the curve [Neumann 1877, p. 164]. This omission is curious, and it seems likely that Neumann had only worked out the situation for some special cases at this point. Even in the 1877 work the proof is given only for a restricted class of convex curves.

The fact that the limits of the interior and exterior means u_s and w_s can be expressed in terms of the v 's allowed Neumann to use the same boundedness argument in constructing a solution of the Dirichlet problem given. I will limit myself to the problem of finding f satisfying the required conditions in P . Define a function

$$\varphi_p = v_a^{(2n+1)} + 2[u_p - u_p^{(1)} + u_p^{(2)} - \dots - u_p^{(2n+1)}]$$

where the symbols are as above, a being a fixed point on S . Because the u 's and v 's are constructed as arithmetic means, each of them satisfies

the Laplace equation in P , and continuity follows by hypothesis and construction. Hence the only thing that remains is to show that the boundary values are satisfied. But by the relation given above between u_s and v_s , we have

$$2u_s = v_s + f_s,$$

and similar relations link $u^{(n)}$ to $v^{(n)}$ and $v^{(n-1)}$. This leads to a telescoping sum in the expression for f , yielding

$$\varphi(s) = f(s) + v_a^{2n+1} - v_s^{2n+1}.$$

But the difference between the last two terms has limit zero as n approaches infinity, by the same argument involving the configuration constant. Hence f approaches the boundary values as p approaches s , and the problem is solved.

This method, Neumann's method of the arithmetic mean, gives a constructive solution of the Dirichlet problem. Since f satisfies the Laplace equation, it is of course a potential, a fact which Neumann makes explicit by finding the density on S yielding this (logarithmic) potential in the plane.

3. SCHWARZ, CONFORMAL MAPPING, AND THE DIRICHLET PROBLEM

Almost simultaneously, another approach to the existence question for the Dirichlet problem was being developed by Hermann Amandus Schwarz. This has recently been studied by Tazzioli [1994], so the following treatment is quite abbreviated. Schwarz began his mathematical studies in Berlin around 1860 or 1861. He was an early participant in the mathematical seminar organized by Kummer and Weierstrass; he was also a founding member of the *Mathematischer Verein*, intended to promote mathematical interest among the students [Biermann 1973, pp. 96–97]. He obtained a doctorate in 1864 for a geometrical study of developable surfaces apparently suggested by Kummer. He was recognized early as quite brilliant; his dissertation won the accolade "*eximia cum laude*". Schwarz remained in Berlin for several years, perhaps intending to habilitate; in 1866 he

completed the *staatliche Lehramtsprüfung* (state teacher's examination) and began his practice teaching, following the mathematical-pedagogical seminar of Schellbach at the same time. He received a call to be *extraordinarius* in Halle in 1867, and married Kummer's daughter the following year. The couple moved to Zürich in 1869, where Schwarz remained until 1875 as *ordinarius* at the Polytechnikum [Carathéodory 1927]. These remarks cover the period of Schwarz's activity with which we shall be concerned here.

In his final pre-doctoral year, Schwarz attended the lectures of Weierstrass on the theory of analytic functions (Wintersemester 1863-1864), where his attention was drawn to conformal mapping problems by his fellow-student Franz Mertens (1840–1927). According to Schwarz:

“[Mertens] happened to remark to me that it was rather odd that Riemann had proved the existence of a function which would for example map the surface of a plane triangle onto the surface of a disc conformally, but that no such function seemed so far to have been actually specified because the failures of smoothness at the corners seem still to lie beyond the powers of analysis at this time”.⁴

At the time of Mertens's remark, Schwarz later commented, he knew of no example at all of a specific conformal mapping of a simply-connected plane region onto the unit disc. He presented his first result in the area, an explicit conformal mapping of a square onto a disc, to the mathematics Seminar at Berlin in spring 1864, a work which he also presented at the time of his *Promotion* (graduation) in 1866. More than that, he was able to present an argument which provided such a mapping for any regular n -gon.

This paper also contained what is now frequently known as Schwarz's lemma, or Schwarz's reflection principle. This was, by the way, certainly known to Riemann, who used it without proof — indeed, without comment — in his paper on Nobili's rings [Riemann 1855] (see Archibald [1991]). The theorem, as Schwarz states it, is the following:

⁴ “[Mertens] machte gelegentlich mir gegenüber die Bemerkung, es sei doch eigenthümlich, dass Riemann von einer Function, welche z. B. die Fläche eines ebenen geradlinigen Dreiecks auf die Fläche eines Kreises conform abbildet, bereits die Existenz nachgewiesen habe, während die wirkliche Bestimmung einer solchen Function wegen der in den Ecken liegenden Unstetigkeiten der Begrenzungslinie die Kräfte der Analysis zur Zeit noch zu übersteigen scheine” [Schwarz 1869a, p. 65].

“If, in considering an analytic function, an interval of real values of the complex argument corresponds to an interval of real values of the function, then any pair of conjugate values of the argument corresponds to a pair of conjugate values of the function”.⁵

Schwarz’s proof is much like a contemporary textbook proof; he considers a region part of whose boundary includes a portion of the real line, and a function on that region. He then reflects the region in the real axis, extending the function by associating conjugates to conjugates, and uses Cauchy’s integral theorem to show that this continuation is analytic. This also shows that singularities are likewise mapped to singularities by reflection, something Schwarz mentions (and which Riemann also had used).

In the course of constructing the mappings which take n -gons to the disc, it is necessary to determine certain constants of integration. Schwarz was originally only able to prove that the constants could be determined for the case $n = 4$, that is, for the square. Weierstrass helped him to provide a general proof, as Schwarz acknowledged at the time [Schwarz 1869a, p. 77]. Schwarz observed later that the attempt to find an independent proof of the fact that the constants could be determined motivated him to write concerning the more general problem of finding an adequate basis for the Dirichlet principle [Schwarz 1890, p. 351]. This work appears to have been begun by Schwarz during his time in Halle. An early version of this paper, which attempted to solve the existence question for the Dirichlet problem in the plane in the case where the region in question is convex apparently was sent to Weierstrass in November of 1868 [Schwarz 1869a, p. 83].

After his move to Zürich, Schwarz continued this line of research, publishing a further example of a specific mapping (this time of a region bounded by an ellipse onto a disc) in the *Annali di matematica* [Schwarz 1869b]. It is perhaps worth remarking that the associated physical problem which may be solved by this mapping (in one form, the determination of the steady-state temperature distribution of an ellipse given a temperature distribution at the boundary) can also be solved using elliptical coordinates and Fourier analysis, a problem that had been posed by

⁵ “Entspricht bei einer analytischen Function einer stetigen Folge reeller Werthe des complexen Argumentes eine stetige Folge reeller Werthe der Function, so entsprechen je zwei conjugirten Werthen des Argumentes conjugirte Werthe der Function” [Schwarz 1869a, p. 66].

Dirichlet as an exercise for his students in the mathematical physics seminar at Göttingen. This was reported to Schwarz by Eduard Heine, who had been a student of Dirichlet in Berlin (well before the latter's move to Göttingen). Heine noted that the relation to conformal mapping went unremarked in the seminar, however [Schwarz 1890, pp. 355–356].

This work with specific mappings may have helped develop an intuition for the general Dirichlet problem. Whatever the case, Schwarz gave a proof of the existence of a solution for the Dirichlet problem in the convex case in a *Programmschrift*⁶ of the Zürich Polytechnikum in 1869, titled *Zur Theorie der Abbildung* [1869c]. The paper contains a proof, under fairly general hypotheses, of the Riemann mapping theorem, a proof which employed Weierstrassian notions and standards of rigour. While Schwarz employs a physical analogy to describe the underlying idea of his method, his discussion is resolutely mathematical both in its aims and in its methods: there are no more subtle fluids lurking as in Neumann's work.

With this in hand, Schwarz turned to the question of the Dirichlet problem on more general regions. He was definitely interested in constructive solutions, and hence investigated the problem of extending the solutions for which explicit conformal mappings onto the unit disc were known to regions which could be covered by regions for which mappings were known. In Picard's hands, the methods Schwarz constructed were to be transformed into a powerful tool for proving existence theorems for differential equations.

Schwarz described his "alternating method" (*alternirendes Verfahren*) in a report to the Zürich *Naturforschende Gesellschaft* in May 1870 [Schwarz 1870a]. However, we know that the method was known to him since the summer of 1869; in the fall of the same year, he sent a paper to Kronecker (on the solution of Laplace's equation) which employed this method, intending to publish it in the *Journal für die reine und angewandte Mathematik*, of which Kronecker was an editor [Schwarz 1890, pp. 356–357]. He also circulated copies to several other mathematicians, including Weierstrass. However, the paper was not published

⁶ It is difficult to translate the term *Programmschrift*. The term refers to papers published in connection with a sort of annual report (the *Programm*) of the activities of the institution. Sometimes, such papers were actually presented at a school or university ceremony; sometimes they were printed, either in connection with an annual volume or separately.

for some time, since Schwarz, on reading Neumann's 1870 preliminary announcement of his own work on the problem, concluded that Neumann had independently solved the problem using the same method. Accordingly, Schwarz appears to have withdrawn his paper. Neumann's detailed account, however [Neumann 1870b], acknowledged his debt to Schwarz's Zürich presentation for one of the methods he presented (the so-called *Methode der Kombinationen*). At this point, Schwarz decided the paper should be published, and sent an account of his results to Weierstrass. The latter read Schwarz's paper before the Berlin Academy in October of 1870, and the full version appeared in the Academy's *Monatsberichte* [Schwarz 1870b].

The general idea of Schwarz's method is the following. Suppose we have two overlapping regions T_1 and T_2 , bounded by analytic curves, and suppose that we have solutions to the Dirichlet problem for each of them individually (with prescribed boundary conditions of appropriate continuity and differentiability). Let T^* be the intersection of the two regions. We then see that except on the two portions of boundary constituting the boundary of T^* , the old solutions are harmonic. The question is, can they be modified to find a solution for $T_1 \cup T_2$?

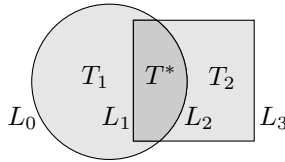


Figure 1. Schwarz's "alternating method" [1870a, p. 136].

Schwarz compared the functioning of his method to that of an air pump with two chambers, corresponding to $(T_1 - T^*)$ and $(T_2 - T^*)$. Let the wall of the first of these be L_0 , and that of the second be L_3 , while the remaining portions of the boundary (those in the interior of the combined region) are L_1 and L_2 respectively (see Figure 1). Let g be the maximum modulus of the original solution along the boundary of T_1 , let k be the minimum modulus, and let G be the difference $(g - k)$. We now assign the value k to the portion L_2 of the boundary, and obtain a new solution u_1 in T_1 to this reformulated problem. This corresponds to the first motion of the pump, which eliminates all of the original solution along L_2 above

a certain minimum k . We then fix the resulting values of u_1 along L_1 , and solve the new problem on T_2 in order to obtain a new solution u_2 . This completes the first cycle. By the construction, we obtain two harmonic functions such that

$$|u_2 - u_1| < |u_2 - k| < G$$

along L_2 . Continuing this process we obtain a sequence of functions, harmonic on the interiors of the regions, which according to Schwarz converged to a common function on T^* , harmonic along the overlapped boundaries. However, the proof contains many statements such as the following:

*“It is now not difficult to prove that the functions with odd index and those with even index [an essential technical point in the proof] approach limit functions . . . without bound.”*⁷

or again

*“By repeated application and appropriate modification . . . one can also demonstrate the existence of a function which besides the boundary conditions also satisfies prescribed discontinuity conditions, or (as with Abelian integrals) has discontinuity conditions alone.”*⁸

Schwarz’s arguments are difficult to reconstruct, however, and it is easy to see why Picard might have been thought the paper a worthy project for a student. Such a project presupposes a solid knowledge of Riemannian results, but also the Weierstrassian proof methods and the language of function elements, as well as notions of analytic continuation. Such concepts were available only with difficulty *via* the lecture notes of Weierstrass’s students. Obtaining a useful copy would therefore depend on having contact with a student of Weierstrass — moreover, one of sufficiently high calibre to make the notes comprehensible and correct. Even so, Weierstrass reworked the material repeatedly, so that there was

⁷ *“Es ist nun nicht schwer, nachzuweisen, dass die Functionen mit ungradem und diejenigen mit gradem Index sich mit wachsendem Index bestimmten Grenzfunctionen u' und u'' unbegrenzt nähern”* [Schwarz 1870a, p. 138].

⁸ *“Durch wiederholte Anwendung und geeignete Modification des erwähnten Grenzüberganges durch alternirendes Verfahren kann die Existenz einer Function u für ein gegebenes Gebiet auch dann, wenn ausser den Grenzbedingungen noch Unstetigkeitsbedingungen, oder wie bei den Abelschen Integralen Unstetigkeitsbedingungen allein vorgeschrieben sind, in den Fällen dargethan werden”* [Schwarz 1870a, p. 139].

no definitive version. This situation was described by Gösta Mittag-Leffler, a student of Weierstrass, in a letter to Charles Hermite written in 1879 (thus 23 years after Weierstrass began lecturing in Berlin):

“Les Allemands eux-mêmes ne sont pas en général assez au courant des idées de Monsieur Weierstrass pour pouvoir saisir sans difficulté une exposition qui soit faite strictement d’après le modèle classique qu’a donné le grand géomètre. Regardez par exemple Monsieur Fuchs. . . il regarde la méthode de [Weierstrass] comme bien supérieure à la méthode de Riemann. Et pourtant il écrit toujours dans le genre de Riemann. Tout le mal vient de ce que M. Weierstrass n’a pas publié ses cours. C’est vrai que la méthode de Weierstrass est enseignée maintenant dans plusieurs universités allemandes, mais tout le monde n’est pas pourtant l’élève de Weierstrass ou l’élève de quelqu’un de ses élèves” [Dugac 1973, Appendix XI, p. 154].

In part as a result of this situation, when Picard wished to understand it — probably around 1885, when his attention was drawn to another paper of Schwarz’s [1885] — he asked his student, Jules Riemann, to work out the details.

4. PICARD, THE DIRICHLET PROBLEM, AND THE METHOD OF SUCCESSIVE APPROXIMATIONS

The thesis of Jules Riemann: Schwarz’s work comes to France

Jules Riemann defended a thesis, *Sur le problème de Dirichlet*, at the Faculté des Sciences of the Paris University in November 1888. I have little information about J. Riemann; like Picard a *normalien*, he was a member of the graduating class of 1883, the year when Picard began to teach there, and their association may date from then. Like many other possessors of mathematics doctorates in Paris, he became a secondary school teacher in mathematics, first at the Lycée Condorcet, subsequently at the more prestigious Lycée Louis le Grand (a standard preparatory school for the *normaliens* of the time, and for many years to follow) [Poggendorff IV, p. 1250]. The thesis is dedicated to Picard. J. Riemann’s aim in the thesis was the clarification of Schwarz’s 1870 *Monatsberichte* paper. J. Riemann cites Schwarz’s main theorem in the following way:

“Soient S_1 et S_2 deux aires se recouvrant en partie, S l’aire constituée par leur ensemble. Si l’on sait résoudre le problème généralisé pour S_1 et pour S_2 , on sait aussi le résoudre pour S , à condition que S_1 et S_2 soient limitées par des lignes analytiques” [J. Riemann 1888, p. 2].

This theorem holds under the condition that the unknown function doesn’t become infinite near the singular point.

Much of the thesis consists of a careful exposition of the background to Schwarz’s work, using the Weierstrassian language of power series and function elements, and concentrating on results relevant to conformal mapping questions. While the proofs in the main follow Schwarz, there are some departures, and some efforts to generalize Schwarz’s results. Interestingly, though this does not touch on our subject immediately, the thesis is directly critical of Schwarz’s proof of the conformal representation theorem [Schwarz 1869c], claiming Schwarz has assumed without proof that if a function u is harmonic in a region and continuous on its boundary, then its harmonic conjugate v must also be continuous on the boundary [J. Riemann 1888, p. 3]. Schwarz himself rejected this criticism out of hand, stating simply that it could not be acknowledged to be well-founded [Schwarz 1890, p. 359].

J. Riemann’s effort at providing an exposition of recent German work was not an isolated one in the Paris of the day. As Gispert has pointed out, such expository theses were a response to the fact that in many fields students were obliged to master very difficult foreign work in order to embark on mathematical research. Frequently, such work was not presented with exemplary clarity, and the task of the student was not only to grasp the work but to render it accessible to a French audience. Gispert gives several examples, including theses by J. Tannery on work of Fuchs in 1874, by G. Floquet on recent German developments in the theory of linear differential equations in 1879, and by A. Niewenglowski on work of B. Riemann [Gispert 1991, pp. 81–83 and 325–340]. Of direct relevance to our study, in 1882 Georges Simart presented an expository thesis on (Bernhard) Riemann’s work, *Commentaire sur deux mémoires de Riemann relatifs à la théorie générale des fonctions et au principe de Dirichlet*. The thesis was defended on May 1, 1882, with Hermite as president of the thesis jury and Darboux and Bouquet as examiners. The work contains no detailed hints as to the origin of the project; but we may suspect Darboux, who

had a long-standing interest in Riemann's work, of initiating the study. (There is a relevant review by Darboux in an early issue of his *Bulletin des sciences mathématiques*.)

Since J. Riemann's work contributed little new, I shall pass to Picard's work with the remark that both the language of Riemann's work (neighbourhoods, function elements) and its content reflect the importance of Weierstrass for Parisian mathematics of the mid-1880s — as the student joke mentioned at the beginning of this paper indicated in less mathematical terms.

Picard and the method of successive approximations

Émile Picard was born on July 24, 1856, in Paris. He studied at the *École normale supérieure* and the Paris *Faculté des Sciences* from 1874–1877, receiving a *doctorat ès sciences* in 1877. He then obtained a position as *maître de conférences* at the Paris *Faculté des Sciences* for 1878–1879. There followed a brief exile as *professeur* at the Toulouse *Faculté des Sciences* (1879–1881). He then returned to Paris, serving as *chargé de cours* at the *Faculté des sciences* from 1881 to 1886, and as *maître de conférences* at the *École normale supérieure* from 1883 to 1886. He finally obtained a professorial appointment in 1886, becoming professor of differential and integral calculus at the *Faculté*. In 1898 the title altered to professor of higher analysis and higher algebra. Though I have omitted various other appointments, this covers the portion of his career with which we have to deal here.

Let us recall the essential features of the Picard method by considering an example. Consider the first-order equation

$$\frac{dx}{dt} = g(x, t)$$

with initial condition $x = 0$ when $t = 0$. We choose an initial function as an approximation to the solution, usually $x_0(t) = 0$, and define a sequence recursively by the formula

$$x_{n+1}(t) = \int_0^t g(\tau, x_n(\tau)) d\tau.$$

This sequence may converge; if so it converges to a (local) solution, which has been found by the method of successive approximations.

If for example we have $\frac{dx}{dt} = x + t$, with $x = 0$ when $t = 0$, choosing $x_0 = 0$ yields $x_1(t) = \int_0^t \tau d\tau = \frac{1}{2}t^2$, where the constant of integration vanishes because of the initial condition. Successive integrations yield

$$x_n(t) = \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^{n+1}}{(n+1)!}$$

so that, in the limit, we have

$$x(t) = e^t - t - 1,$$

which is easily seen to be a solution.

Picard's interest in partial differential equations dates from early in his career; beginning in 1880, he published a number of papers — one jointly with Paul Appell — on linear partial differential equations of first and second order. Existence theory for such equations had depended, until that point, largely on Cauchy's method of majorants. The idea for the method of successive approximations seems to have occurred to Picard in late 1888, following the completion of J. Riemann's thesis; his first reports on the matter were in the *Comptes rendus* of December 1888 and September 1889, with the details appearing for the first time in 1890 [Picard 1888, 1889b, 1890a,b]. In what follows, I shall first examine his treatment of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(u, u_x, u_y, x, y).$$

In this case Picard was able to show that, if we construct a sequence of approximate solutions in a fashion similar to that in the example, and if we assume boundary values for each element u_n along a closed contour C , then (subject to the analyticity of C) the sequence u_n will converge to a solution locally. In the event that F is linear in u and its first partial derivatives, the solution is unique; this is the case he treats in greatest detail.

The application of potential-theoretic methods occurs in the transition from local to global. Picard found conditions when the local solution could be made global, for example in the case when F is independent of the derivatives of u , by using Schwarz's theorem. With the additional

assumption that F is an increasing function of u , Picard was able to establish the existence of an extension of a local solution:

“De ce cas particulier, nous montrons ensuite qu’on peut passer à un contour quelconque. En effet, le problème étant traité pour deux contours ayant une partie commune pourra être résolu pour le contour limitant extérieurement l’ensemble des deux aires. Le procédé alterné, dont ont fait usage M. Schwarz et M. Neumann dans leurs mémorables travaux sur l’équation de Laplace $\Delta u = 0$, peut, avec des modifications d’ailleurs assez sensibles, s’étendre à notre équation générale, et, par suite, se trouve complètement effectué la recherche de l’intégrale, d’ailleurs unique, de l’équation $\Delta u = F(u, x, y)$ prenant une succession continue donnée de valeurs sur un contour fermé quelconque” [Picard 1890b, p. 388].

Thus Picard explicitly acknowledges his debt to Schwarz and Neumann; there seems little doubt that the German writers provided him with direct inspiration. Recent German work is cited elsewhere in the paper as well; namely, Otto Hölder’s condition for the solution of the Dirichlet problem using the Green’s function method, as outlined in Hölder’s Stuttgart dissertation [Hölder 1882]. Picard knew of the Hölder condition from the very recent treatise of Axel Harnack on logarithmic potentials, which he cites as *Sur le potentiel logarithmique* [Harnack 1887].

Extending the method to deal with linear equations, Picard employed known methods for dealing with the solution of the Laplace-Poisson equation on a closed contour C (assumed analytic, though this is implicit since the geometry of the boundary is not the main focus of Picard’s study), $\Delta u = f(x, y)$, where f is continuous and u is assumed to vanish on C , being continuous inside C . The standard solution to such a problem is given by the expression

$$u(x, y) = -\frac{1}{2\pi} \iint f(\xi, \eta) G(\xi, \eta, x, y) d\xi d\eta$$

where G is the Green’s function for the region bounded by C . In treating the more general linear problem, establishing the fact that the solution u is twice differentiable revolves around finding bounds for u and its derivatives inside C . For the restricted problem, these bounds are established by examining a situation where the Green’s function is known — namely, the case of a circular contour — and then generalizing to regions bounded by

arbitrary (sufficiently small) closed analytic contours C by means of conformal mapping. (This doesn't perturb the necessary boundedness properties of u and its derivatives.)

Turning to the linear equation

$$\Delta u = a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu$$

where a , b , and c are functions of x and y , the method of successive approximations is used to derive a sequence of possible solutions u_n , so that one has a sequence of equations

$$\Delta u_n = a \frac{\partial u_{n-1}}{\partial x} + b \frac{\partial u_{n-1}}{\partial y} + cu_{n-1}.$$

Each of the functions u_n is assumed to satisfy the same boundary conditions on C . This enables Picard to invoke the special case discussed earlier by defining a new sequence of functions $v_n = u_n - u_{n-1}$, each of which satisfies an equation of the same form because of linearity, but which vanish on the boundary. This permits an argument establishing the boundedness of each v and its derivatives inside the contour; indeed, in a sufficiently small contour, each v_n is bounded by a number of the form k^{n-3} , where k is a positive constant less than unity depending on the coefficients a , b , and c . Hence as n tends to infinity, v_n approaches 0. An argument involving Green's functions shows that the sequence u_n does indeed converge to a limit u .

By its construction, u clearly satisfies the boundary conditions. It remains to show that it satisfies the equation, and indeed that it is twice differentiable. Here I omit all details, but remark that this comes from a careful manipulation of inequalities in a fashion strongly reminiscent of the Weierstrass school.

5. CONCLUDING REMARKS

I claim, then, that Picard's work provides evidence of the acceptance of the Weierstrassian "arithmetisation of analysis" in France, not only in the general character of the arguments, but also in the choice of subject matter and in the precise terminology employed. This and related work by Picard and by his students found a wide audience *via* the published lectures and

textbooks of Picard (his *Traité d'analyse*, for example, especially in the later editions [Picard *Traité*]); an account of it also appeared in the Darboux's popular and influential textbook, *La théorie générale des surfaces* [Darboux 1896, pp. 353–367]. Picard's methods, and the textbooks which describe them, show a full comprehension of the recent German trends in analysis; in fact, Picard's work on partial differential equations rather outstripped German work of the same period. Corresponding developments elsewhere in Europe, especially in Britain, were stimulated by the work of Picard, which is thus both an example of the development of international standards in analysis, and an agent in this development.

APPENDIX: LATER FORMULATIONS

It is perhaps of some interest to note that Picard's method was subsequently generalized by a number of individuals, or employed for particular applications. In particular, for systems of linear equations, a functional-analytic generalization of Picard's method was obtained by L. Kantorovich of Leningrad [Kantorovich 1939]. Kantorovich uses the language of Banach, formulating the result as a fixed-point theorem for a linear operator on a Banach space. (The formulation as a fixed-point theorem was also doubtless influenced by the work of Schauder, though the latter was not specifically acknowledged.) The main result of Kantorovich's paper is the following. Let Z be a real normed space, complete, with a partial order (which need not be induced by the norm). We let V be a functional $V: Z \rightarrow Z$ on such a space, and let z' be a fixed element of Z which we suppose, without loss of generality, to be greater than zero. We suppose that V is defined for all z between 0 and z' , that V is monotone (so that larger elements have larger images) on $[0, z']$, and that V is a contraction at z' . Kantorovich further assumes that V takes convergent sequences of elements to convergent sequences in such a way that V commutes with passage to the limit (a requirement which is satisfied automatically, by Harnack's theorem, for harmonic operators, as was the case in Picard's examination of the Dirichlet problem). Then the fixed point expression $V(z) = z$ has a solution z^* between 0 and z' , and the solution can be found by the method of successive approximations.

The proof is straightforward, and shows dramatically the advantage of the functional-analytic standpoint. As our first approximation we

let $z_0 = 0$; we then define recursively $z_n = V(z_{n-1})$, which gives us an increasing sequence in the interval between $[0, z']$. This sequence is bounded above and monotone by hypothesis, hence converges to some limit in the interval, z^* . This solves the fixed-point expression by hypothesis. This formulation highlights clearly the important properties that the operator must possess, though it provides little insight into the sources of Picard's formulation.

A more geometric interpretation was provided by Theodor Zech of Darmstadt [Zech 1938]. He interprets each iteration as providing a proposed integral curve of a given tangent vector field; when the appropriate hypotheses are satisfied the field vectors will progress to a state closer to tangency with successive iterations. Zech attributes the basic idea of his method to Vietoris [Zech 1938, p. 209]. It also seems that the hypotheses he employs (like those of Kantorovich) were first formulated by Bendixson [1897].

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