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# Form factors, KdV and Deformed Hyperelliptic Curves. <sup>9</sup>

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À la mémoire de Claude ITZYKSON.

We review and summarize recent works on the relation between form factors in integrable quantum field theory and deformation of geometrical data associated to hyper-elliptic curves. This relation, which is based on a deformation of the Riemann bilinear identity, in particular leads to the notion of null vectors in integrable field theory and to a new description of the KdV hierarchy.

## 1 Form factor formula.

Let us first recall what form factors are. We shall consider the Sine-Gordon theory. The Sine-Gordon equation follows from the action:

$$S = \frac{\pi}{\gamma} \int \mathcal{L} d^2x, \quad \text{with} \quad \mathcal{L} = (\partial_\mu \varphi)^2 + m^2(\cos(2\varphi) - 1)$$

where  $\gamma$  is the coupling constant,  $0 < \gamma < \pi$ . The free fermion point is at  $\gamma = \frac{\pi}{2}$ . In the quantum theory, the relevant coupling constant is:

$$\xi = \frac{\pi\gamma}{\pi - \gamma}.$$

We shall always use the constant  $\xi$ , which plays the role of the Planck constant.

We shall actually not consider the Sine-Gordon model but a restriction of it. The Sine-Gordon theory contains two subalgebras of local operators which, as operator algebras are generated by  $\exp(i\varphi)$  and  $\exp(-i\varphi)$  respectively. Let us concentrate on one of them, say the one generated by  $\exp(i\varphi)$ . It is known that this subalgebra can be considered independently of the rest of the operators as the operator algebra of the theory with a modified energy-momentum tensor. This modification changes the trace of the stress tensor, and therefore changes the ultraviolet limit of the correlation functions. This modification

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corresponds to the restricted Sine-Gordon theory (RSG). For rational  $\frac{\xi}{\pi}$  it describes the  $\Phi_{1,3}$ -perturbations of the minimal models of CFT, but it can be considered for generic values of  $\xi$  as well.

The asymptotic states of the Sine-Gordon theory is made of solitons, anti-solitons and their bound states. We will denote  $n$  solitons,  $n$  anti-solitons states by:

$$|\beta_1, \beta_2, \dots, \beta_{2n}\rangle\rangle_{\epsilon_1, \epsilon_2, \dots, \epsilon_{2n}}$$

We shall consider the case  $\xi = \frac{\pi}{\nu}$  for  $\nu = 1, 2, \dots$ , when the reflection of solitons and anti-solitons is absent. The S-matrix is then diagonal and given by

$$S(\beta) = \prod_{j=1}^{\nu-1} \left( \frac{Bq^j - 1}{B - q^j} \right), \quad \text{with } q = e^{i\frac{\pi}{\nu}}$$

We shall use the following notations:  $B = \exp(\beta)$  and  $b = \exp(\frac{2\pi}{\xi}\beta) = \exp(2\nu\beta)$

The form factors are the matrix elements of local fields between two asymptotic states. By crossing symmetry they can be computed from the form factors between the vacuum and any  $n$  solitons,  $n$  anti-solitons states :

$$f_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{\epsilon_1, \epsilon_2, \dots, \epsilon_{2n}} = \langle\langle 0 | \mathcal{O}(0) | \beta_1, \beta_2, \dots, \beta_{2n} \rangle\rangle_{\epsilon_1, \epsilon_2, \dots, \epsilon_{2n}}$$

where  $\mathcal{O}(x)$  denotes any local operator. The next section is devoted to a brief description of integral formula for these form factors.

### 1.1 Form factors at the reflectionless points.

At the reflectionless points ( $\xi = \frac{\pi}{\nu}$ ,  $\nu = 1, 2, \dots$ ) there is a wide class of local operators  $\mathcal{O}$  for which the form factors in the (restricted) Sine-Gordon model corresponding to a state with  $n$ -solitons and  $n$ -anti-solitons are given by

$$\begin{aligned} & f_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{- \dots - + \dots +} = \\ & = c^n e^{-\frac{1}{2}(\nu(n-1)-n)\sum_j \beta_j} \prod_{i < j} \zeta(\beta_i - \beta_j) \prod_{i=1}^n \prod_{j=n+1}^{2n} \frac{1}{\sinh \nu(\beta_j - \beta_i - \pi i)} \widehat{f}_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{- \dots - + \dots +} \end{aligned} \quad (1)$$

The function  $\zeta(\beta)$ , without poles in the strip  $0 < \text{Im } \beta < 2\pi$ , satisfies  $\zeta(-\beta) = S(\beta)\zeta(\beta)$  and  $\zeta(\beta - 2\pi i) = \zeta(-\beta)$ . The S-matrix  $S(\beta)$  is defined above. The constant  $c$  is given by  $c = 2\nu(\zeta(-i\pi))^{-1}$ . The most essential part of the form factor is given by [1]:

$$\begin{aligned} & \widehat{f}_{\mathcal{O}}(\beta_1, \beta_2, \dots, \beta_{2n})_{- \dots - + \dots +} = \\ & = \frac{1}{(2\pi i)^n} \int dA_1 \dots \int dA_n \prod_{i=1}^n \prod_{j=1}^{2n} \psi(A_i, B_j) \prod_{i < j} (A_i^2 - A_j^2) L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) \prod_{i=1}^n a_i^{-i} \end{aligned} \quad (2)$$

where  $B_j = e^{\beta_j}$  and

$$\psi(A, B) = \prod_{j=1}^{\nu-1} (B - Aq^{-j}), \quad \text{with } q = e^{i\pi/\nu} \quad (3)$$

As usual we define  $a = A^{2\nu}$ . Here and later if the range of integration is not specified the integral is taken around 0. Notice that the operator dependence of the form factors (1) only enters in  $\widehat{f}_{\mathcal{O}}$ .

Different local operators  $\mathcal{O}$  are defined by different functions  $L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n})$ . These functions are symmetric polynomials of  $A_1, \dots, A_n$ . For the primary operators  $\Phi_{2k} = \exp(2ki\varphi)$  and their Virasoro descendants,  $L_{\mathcal{O}}$  are symmetric Laurent polynomials of  $B_1, \dots, B_{2n}$ . For the primary operators  $\Phi_{2k+1} = \exp((2k+1)i\varphi)$ , they are symmetric Laurent polynomials of  $B_1, \dots, B_{2n}$  multiplied by  $\prod B_j^{\frac{1}{2}}$ . Our definition of the fields  $\Phi_m$  is related to the notations coming from CFT as follows:  $\Phi_m$  corresponds to  $\Phi_{[1, m+1]}$ . The requirement of locality for the operator  $\mathcal{O}$  is guaranteed by the following simple recurrent relation for the polynomials  $L_{\mathcal{O}}^{(n)}$ :

$$\begin{aligned} L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) \Big|_{B_{2n} = -B_1, A_n = \pm B_1} &= \\ &= -\epsilon^{\pm} L_{\mathcal{O}}^{(n-1)}(A_1, \dots, A_{n-1} | B_2, \dots, B_{2n-1}) \end{aligned} \quad (4)$$

where  $\epsilon = +$  or  $-$  respectively for the operators  $\Phi_{2k}$  and their descendants, or for  $\Phi_{2k+1}$  and their descendants. In addition to the simple formula (2) we have to add the requirement

$$res_{A_n = \infty} \left( \prod_{i=1}^n \prod_{j=1}^{2n} \psi(A_i, B_j) \prod_{i < j} (A_i^2 - A_j^2) L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) a_n^{-k} \right) = 0, \quad k \geq n+1 \quad (5)$$

This is true in particular if  $\deg_{A_n}(L_{\mathcal{O}}) < 2\nu$ , and therefore the restriction (5) disappears only in the classical limit  $\nu \rightarrow \infty$ . This class of local operators is not complete for the reason that the ansatz (1) is too restrictive. We obtain the complete set of operators only in the classical limit. However there is a possibility to define the form factors of local operators which correspond to polynomials satisfying the relation (4) without any restriction of the kind (5). To do that for the reflectionless points one has to consider the coupling constant in generic position (in which case the formulae for the form factors are much more complicated [1]) and to perform carefully the limit  $\xi = \frac{\pi}{\nu} + \epsilon$ ,  $\epsilon \rightarrow 0$ . An example of such calculation for  $\xi = \pi$  is given in [2]. We would like to emphasize that the local operator can be defined for any polynomial satisfying (4) but its form factors are not necessarily given by the ansatz (1). Physically the existence of local operators for the reflectionless case whose form factors are not given by the ansatz (1) is related to the existence of additional local conserved quantities which constitute the algebra  $sl(2)$ . In spite of the fact that the form factors of the form (1) do not define all the operators they provide a good example for explaining the properties valid in generic case.

The explicit form of the polynomials  $L_{\mathcal{O}}$  for the primary operators  $\Phi_m = e^{im\varphi}$  is as follows

$$L_{\Phi_m}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) = \prod_{i=1}^n A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}$$

We shall consider the Virasoro descendants of the primary fields. We shall restrict ourselves by considering only one chirality. Obviously, the locality relation (4) is not destroyed if we multiply the polynomial  $L_{\mathcal{O}}^{(n)}(A|B)$  either by  $I_{2k-1}(B)$  or by  $J_{2k}(A|B)$  with

$$I_{2k-1}(B) = \left( \frac{1+q^{2k-1}}{1-q^{2k-1}} \right) s_{2k-1}(B), \quad k = 1, 2, \dots \quad (6)$$

$$J_{2k}(A|B) = s_{2k}(A) - \frac{1}{2} s_{2k}(B), \quad k = 1, 2, \dots \quad (7)$$

Here we use the following definition:  $s_k(x_1, \dots, x_m) = \sum_{j=1}^m x_j^k$ .

The multiplication by  $I_{2k-1}$  corresponds to the application of the local integrals of motion. The normalization factor  $\left( \frac{1+q^{2k-1}}{1-q^{2k-1}} \right)$  is introduced for convenience. Since the boost operator acts by dilatation on  $A$  and  $B$ ,  $I_{2k-1}$  has spin  $(2k-1)$  and  $J_{2k}$  has spin  $2k$ .

The crucial assumption which we make is that the space of local fields descendents of the operator  $\Phi_m$  is generated by the operators obtained from the generating function

$$\mathcal{L}_m(t, y|A|B) = \exp\left(\sum_{k \geq 1} t_{2k-1} I_{2k-1}(B) + y_{2k} J_{2k}(A|B)\right) \left(\prod_{i=1}^n A_i^m \prod_{j=1}^{2n} B_j^{-\frac{m}{2}}\right) \quad (8)$$

This is our main starting point. As explained below, this assumption follows from the classical meaning of the variables  $A, B$  [3]. We will restrict ourselves to the descendents of the identity operator which correspond to  $m = 0$  in eq.(8).

## 1.2 Form factors and quantization of solitons.

We now describe how the integral formula can be understood as arising from a (special) quantization of the quantum mechanical problem describing the dynamic of a system of  $n$  solitons.

For each  $n$ -soliton solution we introduce pairs of conjugated variables  $A_i$  and  $P_i$  ( $i = 1 \dots, n$ ), which in the quantum case satisfy Weyl commutation relations. Every local operator  $\mathcal{O}$  can be considered as acting in this  $A$ -representation, and therefore can be identified with a certain operator  $\mathcal{O}(A, P)$ . The typical formula for the matrix element of  $\mathcal{O}$  between two  $n$ -soliton states can be presented as [3]:

$$\langle B' | \mathcal{O} | B \rangle = \int \Psi(A, B')^\dagger \mathcal{O}(A, P) \Psi(A, B) d\mu(A), \quad (9)$$

where  $\Psi(A, B)$  is the wave-function of the state of  $n$  solitons with momenta  $B_1, \dots, B_n$ . The  $P_j$  variables are related to the variables  $A_i$  and  $B_k$  by  $P_j = \prod_k \left(\frac{B_k - A_j}{B_k + A_j}\right)$ . The measure  $d\mu(A)$  include a specific weight admitting a natural interpretation in the  $n$ -soliton symplectic geometry. In formula (9), the variables  $A$  are complex. The integration domain specifies the configuration space of the quantum mechanical problem.

At the classical level, the conjugated variables  $A_i$  and  $P_j$  arise from a particular parametrization of the  $n$ -soliton solutions of the Sine-Gordon equation. They are naturally related to the zeroes and poles of the Jost solution of the associated linear problem. In particular, the Sine-Gordon field can be parametrized in terms of the  $A$  and  $B$  variables as :

$$e^{i\varphi} = \prod_{j=1}^n \left(\frac{A_j}{B_j}\right)$$

In comparing the formula (2) and (9), the soliton wave functions are more or less identified with the functions  $\prod_{i,j} \psi(A_i, B_j)$  with  $\psi(A, B)$  defined in eq.(3), and the integration measure  $d\mu(A)$  in eq.(9) is identified with the Vandermonde determinant  $\prod_{i < j} (A_i^2 - A_j^2)$  in eq.(2). Furthermore, as explained in [3], the factor  $\prod_i a_i^{-i}$  in eq.(2) is related to the positions of the soliton trajectories in the  $A$ -plane.

But the most important point is that the polynomials  $L_{\mathcal{O}}(A_1, \dots, A_n | B_1, \dots, B_n)$  in eq.(2) are identified with the representations of the operators  $\mathcal{O} = \mathcal{O}(A, P)$  in the  $A, B$  variables:

$$L_{\mathcal{O}}(A_1, \dots, A_n | B_1, \dots, B_n) \Longleftrightarrow \mathcal{O}(A, P(A, B))$$

This is particularly clear for the primary operators  $\Phi_m = e^{im\varphi}$ . This observation actually underlies the construction we describe in the following.

## 2 KdV equation and hyperelliptic curves.

The ultraviolet limit of the (restricted) Sine-Gordon model is a minimal conformal field theory. Its classical limit is therefore intimately related to the KdV equation. One may think of KdV as describing

one of the chiral sector of Sine-Gordon. In this section, we first present a new description of the space of local fields in KdV in terms of the local integrals of motion and their densities. We then describe various connexion between form factor formula and hyperelliptic curves and the associated finite zone solutions of KdV.

## 2.1 Local fields and null vectors in the KdV theory.

The KdV equation for a field  $u(t_1, t_3, \dots)$  is the following non linear equation:

$$\partial_3 u + \frac{3}{2} u u' - \frac{1}{4} u''' = 0 \quad (10)$$

We shall use both notations  $\partial_1$  and  $'$  for the derivatives with respect to  $x = t_1$ . As is well known this is one of a hierarchy of equations which can be written in a Lax form. Namely the field  $u$  depends on a set of time variables  $t_{2k-1}$ , and its evolution with respect to these times is encoded in the equations :

$$\frac{\partial L}{\partial t_{2k-1}} = \left[ \left( L^{\frac{2k-1}{2}} \right)_+, L \right] = \frac{1}{2^{2k-1}} u^{(2k-1)} + \dots \quad (11)$$

Here  $L$  is the Lax operator of KdV :

$$L = \partial_1^2 - u \quad (12)$$

We have used the pseudo-differential operator formalism of Gelfand and Dickey, cf. [6].

In the KdV theory, the local fields, which are the descendents of the identity operators, are simply polynomials in  $u(t)$  and its derivatives with respect to  $t_1$ :

$$\mathcal{O} = \mathcal{O}(u, u', u'', \dots) \quad (13)$$

Instead of the variables  $u, u', u'', \dots$ , we may replace the odd derivatives of  $u(x)$  by the higher time derivatives  $\partial_{2k-1} u$ , according to the equations of motion of the hierarchy (11). We may also replace the even derivatives of  $u(x)$  by the densities  $S_{2k}$  of the local integrals of motion,

$$S_{2k} = \text{res}_{\partial_1} L^{\frac{2k-1}{2}} = -\frac{1}{2^{2k-1}} u^{(2k-2)} + \dots,$$

In particular  $S_2 = -\frac{1}{2}u$ . For a reader who prefers the  $\tau$ -function language  $S_{2k} = \partial_1 \partial_{2k-1} \log \tau$ . They satisfy the conservation laws :  $\partial_{2l+1} S_{2k} = \partial_1 H_{2k+2l}$  for some local field  $H_{2k+2l}$ . Therefore, from analogy with the conformal case we suggested in [13] the following conjecture :

**Conjecture.** *We can write any local fields of the KdV theory as*

$$\mathcal{O}(u, u', u'', \dots) = F_{\mathcal{O},0}(S_2, S_4, \dots) + \sum_{\nu \geq 1} \partial^\nu F_{\mathcal{O},\nu}(S_2, S_4, \dots) \quad (14)$$

where  $\nu = (i_1, i_3, \dots)$  is a multi index,  $\partial^\nu = \partial_1^{i_1} \partial_3^{i_3} \dots$ ,  $|\nu| = i_1 + 3i_3 + \dots$ .

We checked this conjecture up to very high levels. To see that this conjecture is a non trivial one, let us compute the character  $\chi_1$  of the space of local fields eq.(13). Attributing the degree 2 to  $u$  and 1 to  $\partial_1$ , we find that :

$$\chi_1 = \prod_{j \geq 2} \frac{1}{1-p^j} = (1-p) \prod_{j \geq 1} \frac{1}{1-p^j} = 1 + p^2 + p^3 + 2p^4 + 2p^5 + \dots$$

On the other hand the character  $\chi_2$  of the elements in the right hand side of eq.(14) is :

$$\chi_2 = \prod_{j \geq 1} \frac{1}{1 - p^{2j-1}} \prod_{j \geq 1} \frac{1}{1 - p^{2j}} = \prod_{j \geq 1} \frac{1}{1 - p^j} = 1 + p + 2p^2 + 3p^3 + 5p^4 + 7p^5 + \dots$$

Hence  $\chi_1 < \chi_2$  and the two spaces in eq.(14) can be equal only if there are null-vectors among the elements in the right hand side of eq.(14). Let us give some examples of null-vectors :

$$\begin{aligned} \text{level 1} & : \partial_1 \cdot 1 = 0 \\ \text{level 2} & : \partial_1^2 \cdot 1 = 0 \\ \text{level 3} & : \partial_1^3 \cdot 1 = 0, \quad \partial_3 \cdot 1 = 0 \\ \text{level 4} & : \partial_1^4 \cdot 1 = 0, \quad \partial_1 \partial_3 \cdot 1 = 0, \quad (\partial_1^2 S_2 - 4S_4 + 6S_2^2) \cdot 1 = 0 \\ \text{level 5} & : \partial_1^5 \cdot 1 = 0, \quad \partial_1^2 \partial_3 \cdot 1 = 0, \quad \partial_5 \cdot 1 = 0, \\ & \quad \partial_1(\partial_1^2 S_2 - 4S_4 + 6S_2^2) \cdot 1 = 0, \quad (\partial_3 S_2 - \partial_1 S_4) \cdot 1 = 0 \end{aligned} \tag{15}$$

We wrote all the null-vectors explicitly to show that their numbers exactly match the character formulae. The non trivial null-vector at level 4 expresses  $S_4$  in terms of the original variable  $u$ :  $4S_4 = -\frac{1}{2}u'' + \frac{3}{2}u^2$ . With this identification the non-trivial null-vector at level 5,  $\partial_3 S_2 - \partial_1 S_4$ , gives the KdV equation itself.

In summary, null vectors code the hierarchy of equations of motion.

## 2.2 Hyperelliptic curves and Riemann bilinear identity.

Let us consider an hyperelliptic curve  $\Gamma$  of genus  $n$  described by the equation

$$\Gamma : Y^2 = X\mathcal{P}(X), \quad \text{with } \mathcal{P}(X) = \prod_{j=1}^{2n} (X - B_j^2), \tag{16}$$

We suppose that the coefficients  $B_i$  have been ordered :  $B_{2n} > \dots > B_2 > B_1 > 0$ . For historical reasons we prefer to work with the parameter  $A$  such that  $X = A^2$ . Thus the curve  $\Gamma$  is :

$$\Gamma : Y^2 = A^2 \mathcal{P}(A^2)$$

The surface is realized as the  $A$ -plane with cuts on the real axis over the intervals  $c_i = (B_{2i-1}, B_{2i})$  and  $\bar{c}_i = (-B_{2i}, -B_{2i-1})$ ,  $i = 1, \dots, n$ , the upper (lower) bank of  $c_i$  is identified with the upper (lower) bank of  $\bar{c}_i$ . The square root  $\sqrt{\mathcal{P}(A^2)}$  is chosen so that  $\sqrt{\mathcal{P}(A^2)} \rightarrow A^{2n}$  as  $A \rightarrow \infty$ . The canonical basis of cycles is chosen as follows: the cycle  $a_i$  starts from  $B_{2i-1}$  and goes in the upper half-plane to  $-B_{2i-1}$ , while the cycle  $b_i$  is an anti-clockwise cycle around the cut  $c_i$ .

Since  $\Gamma$  has genus  $n$ , there are  $n$  independent holomorphic differentials on it. A basis is given by  $d\sigma_k(A) = \frac{A^{2k-2}}{\sqrt{\mathcal{P}(A^2)}} dA$ , for  $k = 1, \dots, n$ . The normalized holomorphic differentials  $d\omega_i$  for  $i = 1, \dots, n$  are such that :

$$\int_{a_j} d\omega_i = \delta_{i,j},$$

They are linear combinations of the  $d\sigma_k(A)$  with coefficients depending on  $B_i$ . They can be written as  $n \times n$  determinants :

$$d\omega_k(A) = \frac{1}{\Delta} \det M(A) \quad \text{with} \quad \begin{cases} M(A)_{ij} = \int_{a_i} \frac{D^{2(j-1)}}{\sqrt{\mathcal{P}(D^2)}} dD, & \text{if } i \neq k, \\ M(A)_{kj} = \frac{A^{2j-2}}{\sqrt{\mathcal{P}(A^2)}} dA, & i, j = 1, \dots, n. \end{cases}$$

with

$$\Delta = \det \left( \int_{a_i} \frac{D^{2(j-1)}}{\sqrt{\mathcal{P}(D^2)}} dD \right)_{i,j=1,\dots,n} \quad (17)$$

A particular role is also played by the differentials of the second kind with singularities at infinity. These are meromorphic differentials whose only singularities are poles of order bigger or equal to two at infinity. Such differentials are linear combinations of differentials of the form  $\frac{A^{2n+2p}}{\sqrt{\mathcal{P}(A^2)}} dA$  with  $p \geq 0$ . The normalized second kind differentials with singularity at infinity  $d\tilde{\omega}_{2i-1}$ ,  $i \geq 1$  are defined by :

$$\int_{a_j} d\tilde{\omega}_{2i-1} = 0, \quad \text{and} \quad d\tilde{\omega}_{2i-1}(A) = d(A^{2i-1}) + \mathcal{O}(A^{-2})dA \quad \text{for} \quad A \sim \infty$$

On Riemann surfaces there is a natural symplectic pairing between meromorphic differentials. Namely, let  $d\Omega_1$  and  $d\Omega_2$  be two meromorphic differentials on  $\Gamma$ . The pairing  $(d\Omega_1 \bullet d\Omega_2)$  is then defined by integrating them along the canonical cycles as follows :

$$(d\Omega_1 \bullet d\Omega_2) = \sum_{i=1}^n \left( \int_{a_j} d\Omega_1 \int_{b_j} d\Omega_2 - \int_{a_j} d\Omega_2 \int_{b_j} d\Omega_1 \right)$$

The Riemann bilinear identity expresses this quantity in terms of sum over residues :

$$(d\Omega_1 \bullet d\Omega_2) = \frac{1}{2i\pi} \sum_{poles} res(\Omega_1 d\Omega_2) \quad (18)$$

In particular, the pairing between the normalized holomorphic differentials is trivial :  $(d\omega_i \bullet d\omega_j) = 0$  for  $i, j = 1, \dots, n$ .

As formulated in the previous equations, the Riemann bilinear identity gives an expression for the pairing between one-forms. We now want to formulate it in a dual form, ie. in a form which gives an expression for the pairing between one-cycles. More precisely, let  $C_1$  and  $C_2$  be two cycles, the pairing is simply the intersection number :

$$(C_1 \circ C_2) = \sum_{j=1}^n (n_j^1 m_j^2 - m_j^1 n_j^2) \quad (19)$$

if  $C_1 = \sum_{j=1}^n (n_j^1 a_j + m_j^1 b_j)$ , and similarly for  $C_2$ . The dual form of the bilinear Riemann identity is:

**Proposition.** *Let  $d\omega_j$  be the normalized holomorphic differentials. Let  $d\xi_j$ , for  $j = 1, \dots, n$ , be differentials of the second kind dual to the holomorphic differentials, ie. such that*

$$(d\omega_i \bullet d\xi_j) = \delta_{ij}, \quad \text{and} \quad (d\xi_i \bullet d\xi_j) = (d\omega_i \bullet d\omega_j) = 0$$

*Then the intersection number between two cycles  $C_1$  and  $C_2$  can be written as :*

$$(C_1 \circ C_2) = \sum_{j=1}^n \left( \int_{C_1} d\omega_j \int_{C_2} d\xi_j - \int_{C_2} d\omega_j \int_{C_1} d\xi_j \right) \quad (20)$$



Alternatively, the intersection number is given by :

$$(C_1 \circ C_2) = \frac{1}{2i\pi} \int_{C_1} \frac{dA_1}{\sqrt{\mathcal{P}(A_1^2)}} \int_{C_2} \frac{dA_2}{\sqrt{\mathcal{P}(A_2^2)}} C_{cl}(A_1, A_2) \quad (21)$$

where the anti-symmetric polynomial  $C_{cl}(A_1, A_2)$  is given by

$$C_{cl}(A_1, A_2) = \sqrt{\mathcal{P}(A_1^2)} \frac{d}{dA_1} \left( \sqrt{\mathcal{P}(A_1^2)} \frac{A_1}{A_1^2 - A_2^2} \right) - (A_1 \longleftrightarrow A_2) \quad (22)$$

Proof.

See, for example, [5] for a relevant discussion. The normalization condition for the differentials  $d\omega_j$  and  $d\xi_j$  means that the matrix  $P$  defined by,

$$P_{ij} = \begin{pmatrix} \int_{a_j} d\omega_i & \int_{b_j} d\omega_i \\ \int_{a_j} d\xi_i & \int_{b_j} d\xi_i \end{pmatrix}$$

is a symplectic matrix. ie:

$$P J {}^t P = J \quad \text{with} \quad J = \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix} \quad (23)$$

where  ${}^t P$  denotes the transposed matrix. Notice that since  $J^2 = -id$ , eq.(23) means that the right inverse of  $P$  is  $-J {}^t P J$ . Using the fact the right and left inverse are identical, eq.(23) is therefore equivalent to  ${}^t P J P = J$ .

Now let  $C_1$  and  $C_2$  be our two cycles. By definition the intersection number is  $(C_1 \circ C_2) = \langle C_1 | J | C_2 \rangle$ , where  $\langle C_1 | = (n_j^1, m_j^1)$  and similarly for  $|C_2\rangle$ . Using the relation  ${}^t P J P = J$ , we can rewrite the intersection number as :

$$(C_1 \circ C_2) = \langle C_1 | {}^t P J P | C_2 \rangle$$

This is equivalent to the relation (20) since the vector  $\langle C_1 | {}^t P$  is the vector of the periods of the forms  $d\omega_j$  and  $d\xi_j$  along the cycles  $C_1$  :

$$\langle C_1 | {}^t P = \left( \int_{C_1} d\omega_j, \int_{C_1} d\xi_j \right)$$

and similarly for  $P | C_2 \rangle$ .

The second formulation (21) can be proved in two ways. Either one verifies directly that the integral (21) gives the intersection numbers (the integral is localized on the intersection of the cycles), and then by expanding  $C_{cl}(A_1, A_2)$  this gives a formula for the differentials  $d\xi_j$ . Indeed, the explicit expression of  $C_{cl}(A_1, A_2)$  is :

$$C_{cl}(A_1, A_2) = \frac{1}{A_1^2 - A_2^2} \left( A_1^2 \mathcal{P}'(A_1^2) + A_2^2 \mathcal{P}'(A_2^2) - \frac{A_1^2 + A_2^2}{A_1^2 - A_2^2} (\mathcal{P}(A_1^2) - \mathcal{P}(A_2^2)) \right)$$

It is an anti-symmetric polynomials of degree at most  $4n - 2$ . It can be expanded as:

$$C_{cl}(A_1, A_2) = \sum_{k=1}^n (A_1^{2k-2} Q_k(A_2^2) - A_2^{2k-2} Q_k(A_1^2))$$

where  $Q_k(A^2)$  are polynomials of degree  $(4n - 2k)$  given by,

$$Q_k(A^2) = \sum_{p=2k}^{2n} (-)^{p+1} (p+1-2k) s_{4n-2p}(B) A^{2p-2k}$$

These polynomials define the differentials of the second kind dual to the holomorphic forms  $d\sigma_k(A)$ . Alternatively, one may determine directly the differentials  $d\xi_j$  by solving their normalization conditions, and then by resumming  $\sum_j d\omega_j \wedge d\xi_j$  this gives the formula for  $C_{cl}(A_1, A_2)$ . ■

It is the dual formulation of the Riemann bilinear identities which admits a simple quantum deformation in the form factor problem [5].

### 2.3 Baker-Akhiezer functions and finite zone solutions.

As is well known, to any hyperelliptic curve we can associate a solution of the KdV equation. We first need certain informations about the Baker-Akhiezer function. The Baker-Akhiezer function  $w(t, A)$  is a eigenfunction of the Shroedinger equation defined by  $L$  with eigenvalue  $A^2$ ,

$$L w(t, A) = A^2 w(t, A) \quad (24)$$

which admits an asymptotic expansion at  $A = \infty$  of the form

$$w(t, A) = e^{\zeta(t, A)} \left( 1 + O\left(\frac{1}{A}\right) \right); \quad \text{with} \quad \zeta(t, A) = \sum_{k \geq 1} t_{2k-1} A^{2k-1}$$

In these formulae, higher times are considered as parameters. The second solution of equation (24), denoted by  $w^*(t, A)$ , has the asymptotics

$$w^*(t, A) = e^{-\zeta(t, A)} \left( 1 + O\left(\frac{1}{A}\right) \right)$$

These definitions do not fix completely the Baker-Akhiezer functions since we can still multiply them by constant asymptotic series of the form  $(1 + O(1/A))$ . Since normalizations will be important, let us give a more precise definition. We first introduce the dressing operator  $\Phi$ , which is an element of the algebra of pseudo-differential operators, by :

$$L = \Phi \partial_1^2 \Phi^{-1}; \quad \text{with} \quad \Phi = 1 + \sum_{i > 1} \Phi_i \partial_1^{-i}$$

We then define the Baker-Akhiezer functions by :

$$w(t, A) = \Phi e^{\zeta(t, A)}, \quad \text{and} \quad w^*(t, A) = (\Phi^*)^{-1} e^{-\zeta(t, A)}$$

where  $\Phi^* = 1 + \sum_{i > 1} (-\partial_1)^{-i} \Phi_i$  is the formal adjoint of  $\Phi$ . Clearly,  $w(t, A)$  is a solution of the Shroedinger equation with eigenvalue  $A^2$ . Moreover [13],

**Proposition.** *With the above definitions, one has*

- 1) *The wronskian  $W(A) = w(t, A)' w^*(t, A) - w^*(t, A)' w(t, A)$  takes the value  $W(A) = 2A$ .*
- 2) *The generating function of the local densities  $S(A) = 1 + \sum_{k > 0} S_{2k} A^{-2k}$  is related to the Baker-Akhiezer function by*

$$S(A) = w(t, A) w^*(t, A) \quad (25)$$

The solutions of KdV associated to hyperelliptic curves are the so-called finite-zone solutions. The Baker-Akhiezer function is then an analytical function on the spectral curve. Let us recall briefly the construction [7, 8]. Consider the hyperelliptic curve (16) which we have introduced in the previous section. Let us consider in addition a divisor of order  $n$  on the surface  $\Gamma$ :

$$\mathcal{D} = (P_1, \dots, P_n)$$

With these data we construct the Baker-Akhiezer function which is the unique function with the following analytical properties:

- It has an essential singularity at infinity:  $w(t, A) = e^{\zeta(t, A)}(1 + O(1/A))$ .
- It has  $n$  simple poles outside infinity. The divisor of these poles is  $\mathcal{D}$ .

Considering the quantity  $-\partial_1^2 w + A^2 w$ , we see that it has the same analytical properties as  $w$  itself, apart for the first normalization condition. Hence, because  $w$  is unique, there exists a function  $u(t)$  such that

$$-\partial_1^2 w + u(t)w + A^2 w = 0 \quad (26)$$

We recognize eq.(24). One can give various explicit constructions of the Baker-Akhiezer function. Let us introduce the divisor  $\mathcal{Z}(t)$  of the zeroes of the Baker-Akhiezer function. It is of degree  $n$ :

$$\mathcal{Z}(t) = (A_1(t), \dots, A_n(t))$$

The equations of motion with respect to the first time for the divisor  $\mathcal{Z}(t)$  read [8]:

$$\partial_1 A_i(t) = -\frac{\sqrt{\mathcal{P}(A_i^2(t))}}{\prod_{j \neq i} (A_i^2(t) - A_j^2(t))} \quad (27)$$

The normalization of the Baker-Akhiezer function corresponds to a particular choice of the divisor of its poles  $\mathcal{D}$ . We shall specify the divisor which corresponds to the normalization of the Baker-Akhiezer function which was required above. We have the following proposition [13].

**Proposition.** *For the Baker-Akhiezer functions  $w(t, A)$  and  $w^*(t, A)$  normalized such that their Wronskian is  $2A$ , ie.  $w(t, A)'w^*(t, A) - w^*(t, A)'w(t, A) = 2A$ . We have :*

$$S(A) = \frac{Q(A^2)}{\sqrt{\mathcal{P}(A^2)}} \quad (28)$$

where the polynomials  $Q(A^2)$  and  $\mathcal{P}(A^2)$  are :  $Q(A^2) = \prod_{i=1}^n (A^2 - A_i^2)$  and  $\mathcal{P}(A^2) = \prod_{i=1}^{2n} (A^2 - B_i^2)$ .

We now can make contact with the generating function (8) for the form factors of the descendent operators. Indeed, let us introduce a set of variables  $J_{2k}$  related to the generating function  $S(A)$  by :

$$S(A) \equiv \exp \left( - \sum_k \frac{1}{k} J_{2k} A^{-2k} \right) \quad (29)$$

The formula (28) gives :

$$J_{2k} = \sum_i A_i^{2k} - \frac{1}{2} \sum_i B_i^{2k}$$

They coincide with those appearing when defining form factors of descendent operators, cf. eq.(7). Clearly the quantities  $I_{2k} = \sum_i B_i^{2k}$  coincide with the value of the integral of motion for the finite zone solutions. In other words, the generating function of the descendent operators (8) is in correspondence with the integrals of motion and their densities for the finite zone solutions. Moreover, the variables  $A, B$  used in the integral representation of the form factors are in correspondence with the poles and zeroes of the Baker-Akhiezer functions.

## 2.4 The ultra-classical limit of the form factor formula.

There is a surprising relation between the form factor formula and the averaging formula occurring in the Whitham theory for KdV [9, 10, 11, 12]. The present section is devoted to the description of this relation.

Let us remind briefly what is the Whitham method about. Suppose we consider the solutions of KdV which are close to a given quasi-periodic solution. The latter is defined by the set of ends of zones  $B_1^2, \dots, B_{2n}^2$ . We know that for the finite-zone solution the dynamics is linearized by the Abel transformation to the Jacobi variety of the hyper-elliptic surface  $Y^2 = X\mathcal{P}(X)$  for  $\mathcal{P}(X) = \prod (X - B_j^2)$ . The idea of the Whitham method is to average over the fast motion over the Jacobi variety and to introduce "slow times"  $T_j$  which are related to the original KdV times as  $T_j = \epsilon t_j$  ( $\epsilon \ll 1$ ), assuming that the ends of zones  $B_j$  become functions of these "slow times" (recall that the ends of zones were the integrals of motion for the pure finite-zone solutions).

For the given finite-zone solution the observables can be written in terms of  $\theta$ -functions on the Jacobi variety, but this kind of formulae is inefficient for writing the averages. One has to undo the Abel transformation, and to write the observables in terms of the divisor  $\mathcal{Z} = (A_1, \dots, A_n)$ . The formulae for the observables are much more simple in these variables, and the averages can be written as abelian integrals, the Jacobian due to the Abel transformation is easy to calculate. The result of this calculation is as follows [10]. Every observable  $\mathcal{O}$  can be written as an even symmetric function  $L_{\mathcal{O}}(A_1, \dots, A_n)$  (depending on  $B$ 's as parameters). For the average we have

$$\langle\langle \mathcal{O} \rangle\rangle = \Delta^{-1} \int_{a_1} \frac{dA_1}{\sqrt{\mathcal{P}(A_1^2)}} \cdots \int_{a_n} \frac{dA_n}{\sqrt{\mathcal{P}(A_n^2)}} L_{\mathcal{O}}(A_1, \dots, A_n) \prod_{i < j} (A_i^2 - A_j^2) \quad (30)$$

where the normalization factor  $\Delta$  is defined as above in eq.(17).

The similarity of this formula with the formula for the form factors (1) is a surprising fact. We have the following dictionary:

For the local observables, we have

$$L_{\mathcal{O}} \Longleftrightarrow L_{\mathcal{O}}$$

For the weight of integration, we have

$$\frac{1}{\sqrt{\mathcal{P}(A^2)}} \Longleftrightarrow \prod_{j=1}^{2n} \psi(A, B_j)$$

For the integration cycles, we have

$$a_i - \text{cycles} \Longleftrightarrow \text{functions} \quad A_i^{2\nu} = a_i$$

The most striking feature is that the cycles of integration are replaced by functions of  $a_i = A_i^{2\nu}$ . The coincidence between the notations for  $a_i$ -variables and  $a_i$ -cycles is therefore not fortuitous. The explanation of the fact that the cycles are replaced by these functions is given in [3], where it was shown that the factor  $\prod_i a_i^{-i}$  selects the classical trajectory in the semi-classical approximation of eq.(2). So, the solution of a non trivial, full fledged, quantum field theory has provided us with a very subtle definition of a *quantum* Riemann surface.

For comparison with the quantum case, it is important to note that the average (30) can vanish for some observables  $L_{\mathcal{O}}(A_1, \dots, A_n)$ . More precisely, let  $M_{\mathcal{O}}(A_1, \dots, A_n)$  be the antisymmetric polynomials defined by :

$$M_{\mathcal{O}}(A_1, \dots, A_n) = \prod_{i < j} (A_i^2 - A_j^2) L_{\mathcal{O}}(A_1, \dots, A_n)$$

The degree of  $M_{\mathcal{O}}$  in any variable  $A_k$  is always greater than  $(2n - 2)$ . Since  $M_{\mathcal{O}}(A_1, \dots, A_n)$  are even antisymmetric polynomials, a basis of such functions is provided by  $n \times n$  determinants of  $n$  polynomials  $Q_{p_j}(A_k^2)$  of degree  $(2p_j - 2)$ . The average  $\langle\langle \mathcal{O}_{\{p_j\}} \rangle\rangle$  is then :

$$\langle\langle \mathcal{O}_{\{p_j\}} \rangle\rangle = \Delta^{-1} \det \left( \int_{a_i} \frac{dA}{\sqrt{\mathcal{P}(A^2)}} Q_{p_j}(A^2) \right)_{i,j=1,\dots,n}$$

For certain observable  $\mathcal{O}$  the average  $\langle\langle \mathcal{O} \rangle\rangle$  vanishes. There are two origins for this vanishing :

**1. Exact forms.** The integral (30) vanishes if  $M_{\mathcal{O}}(A_1, \dots, A_n)$  is an exact form, ie.:

$$M_{\mathcal{O}}(A_1, \dots, A_n) = \sum_k (-)^k \widehat{M}(A_1, \dots, \widehat{A}_k, \dots, A_n) \times Q(A_k^2)$$

such that the differential  $\frac{Q(A^2)}{\sqrt{\mathcal{P}(A^2)}} dA$  has vanishing integrals along the  $a_i$ -cycles, and for some antisymmetric polynomial  $M(A_1, \dots, A_{n-1})$ . Here  $\widehat{A}_k$  means that the variables  $A_k$  is omitted.

**2. Riemann bilinear identity.** Since we are integrating on  $n$  non-intersecting cycles, the integral (30) vanishes if:

$$M_{\mathcal{O}}(A_1, \dots, A_n) = \sum_{i < j} \widehat{M}(A_1, \dots, \widehat{A}_i, \dots, \widehat{A}_j, \dots, A_n) C_{cl}(A_i, A_j)$$

where  $C_{cl}(A_1, A_2)$  is defined in (22), and  $\widehat{M}(A_1, \dots, A_{n-2})$  is an anti-symmetric polynomial. This fact is a direct consequence of the dual form of the Riemann bilinear identities.

The null-vectors of the quantum theory originate in the quantum deformation of these two properties.

### 3 The deformed Riemann bilinear identity and null-vectors.

We now describe how a quantum deformation of the geometrical structure we just recalled leads to the notion of null vectors in integrable field theory. The existence of these null vectors yields differential equations for the correlation functions or for the form factors, which reduce to the KdV hierarchy in the classical limit.

#### 3.1 Null vectors in integrable field theory.

By definition, null-vectors correspond to operators with all the form factors vanishing. Therefore, consider the fundamental integrals  $\widehat{f}_{\mathcal{O}}$  of the form factor formula :

$$\frac{1}{(2\pi i)^n} \int dA_1 \dots \int dA_n \prod_{i=1}^n \prod_{j=1}^{2n} \psi(A_i, B_j) \prod_{i < j} (A_i^2 - A_j^2) L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) \prod_{i=1}^n a_i^{-i} \quad (31)$$

Instead of  $L_{\mathcal{O}}^{(n)}$ , it is more convenient to use the anti-symmetric polynomials  $M_{\mathcal{O}}^{(n)}$ :

$$M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n}) = \prod_{i < j} (A_i^2 - A_j^2) L_{\mathcal{O}}^{(n)}(A_1, \dots, A_n | B_1, \dots, B_{2n})$$

The dependence on  $B_1, \dots, B_{2n}$  in the polynomials  $M_{\mathcal{O}}^{(n)}$  will often be omitted.

There are several reasons why this integral can vanish. Some of them depend on a particular value of the coupling constant or on a particular number of solitons. We should not consider these occasional

situations. In parallel to the classical case discussed above, there are three general reasons for the vanishing of the integral, let us present them.

**1. Residue.** The integral (31) vanishes if vanishes the residue with respect to  $A_n$  at the point  $A_n = \infty$  of the expression

$$\prod_{j=1}^{2n} \psi(A_n, B_j) a_n^{-n} M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n)$$

Of course the distinction of the variable  $A_n$  is of no importance because  $M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n)$  is anti-symmetric.

**2. "Exact forms."** The integral (31) vanishes if  $M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n)$  happens to be an "exact form". Namely, if it can be written as:

$$M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n) = \sum_k (-1)^k M(A_1, \dots, \widehat{A}_k, \dots, A_n) (Q(A_k)P(A_k) - qQ(qA_k)P(-A_k)), \quad (32)$$

with

$$P(A) = \prod_{j=1}^{2n} (B_j + A)$$

for some anti-symmetric polynomial  $M(A_1, \dots, A_{n-1})$ . Here  $\widehat{A}_k$  means that  $A_k$  is omitted. The polynomial  $P(A)$  should not be confused with the polynomial  $\mathcal{P}(A^2)$ . They are related by:  $\mathcal{P}(A^2) = P(A)P(-A)$ . Eq.(32) is a direct consequence of the functional equation satisfied by  $\psi(A, B)$ :

$$\psi(qA, B) = \left( \frac{B - A}{B + qA} \right) \psi(A, B) \quad (33)$$

For  $Q(A)$  one can take in principle any Laurent polynomial, but since we want  $M_{\mathcal{O}}^{(n)}$  to be a polynomial the degree of  $Q(A)$  has to be greater or equal  $-1$ .

**3. Deformed Riemann bilinear relation.** The integral (31) vanishes if

$$M_{\mathcal{O}}^{(n)}(A_1, \dots, A_n) = \sum_{i < j} (-1)^{i+j} M(A_1, \dots, \widehat{A}_i, \dots, \widehat{A}_j, \dots, A_n) C(A_i, A_j)$$

where  $M(A_1, \dots, A_{n-2})$  is an anti-symmetric polynomial of  $n - 2$  variables, and  $C(A_1, A_2)$  is given by

$$C(A_1, A_2) = \frac{1}{A_1 A_2} \left\{ \frac{A_1 - A_2}{A_1 + A_2} (P(A_1)P(A_2) - P(-A_1)P(-A_2)) + (P(-A_1)P(A_2) - P(A_1)P(-A_2)) \right\} \quad (34)$$

This property needs some comments. For the case of generic coupling constant its proof is rather complicated. It is a consequence of the so called deformed Riemann bilinear identity [4]. The name is due to the fact that in the limit  $\xi \rightarrow \infty$  (which is the opposite of the classical limit which corresponds to  $\xi \rightarrow 0$ ) the deformed Riemann bilinear identity happens to be the same as the Riemann bilinear identity for hyper-elliptic integrals. The formula for  $C(A_1, A_2)$  given in [5] differs from (34) by simple "exact forms". Notice that the formula for  $C(A_1, A_2)$  does not depend on the coupling constant. For the reflectionless case a very simple proof is available.

**Proposition.** The function  $C(A_1, A_2)$  defined in eq.(34) satisfy :

$$\int dA_1 \int dA_2 \prod_{i=1}^2 \prod_{j=1}^{2n} \psi(A_i, B_j) C(A_1, A_2) a_1^k a_2^l = 0 \quad \forall k, l \quad (35)$$

Proof.

The reflectionless case is a rather degenerate one, so, the deformed Riemann bilinear identity [4] does not exist in complete form. However we only use the consequence of the deformed Riemann bilinear identity which allows a simple proof in the reflectionless case. Let us introduce the functions

$$F(A) = \prod_{j=1}^{2n} \psi(A, B_j) P(A), \quad G(A) = \prod_{j=1}^{2n} \psi(A, B_j) P(-A)$$

Recall that the function  $\psi(A, B)$  satisfies the difference equation (33) :  $\psi(Aq, B) = \left( \frac{B-A}{B+qA} \right) \psi(A, B)$ . It implies that

$$F(Aq) = G(A)$$

The integral (35) can therefore be rewritten as follows:

$$\int \frac{dA_1}{A_1} \int \frac{dA_2}{A_2} \left\{ \frac{A_1 - A_2}{A_1 + A_2} (F(A_1)F(A_2) - F(qA_1)F(qA_2)) + (F(qA_1)F(A_2) - F(A_1)F(qA_2)) \right\} a_1^k a_2^l$$

Changing variables  $A_i \rightarrow qA_i$  where needed one easily find that this integral equals zero. Recall that  $a_i = A_i^{2\nu}$  and  $q^{2\nu} = 1$ , so  $a_1^k a_2^l$  do not change under these changes of variables. ■

### 3.2 The deformed Riemann bilinear identity.

As understood in [4], the complete structure underlying the deformed bilinear identity only emerges when one consider the general case. Ie. one has to consider the Sine-Gordon model at generic coupling constant  $\xi = \pi/\nu$ . There is then two dual quantum parameters  $q$  and  $\tau$ :

$$q = e^{i\frac{\pi}{\nu}} \quad ; \quad \tau = e^{i\pi\nu}$$

The basic ingredient in the form factors at generic value of the coupling constant is a (special) pairing between polynomials  $L(A)$  and  $r(a)$  with  $a = e^\alpha$  and  $a = A^{2\nu} = e^{2\nu\alpha}$  for some  $\alpha$ . The pairing is defined as:

$$\langle L(A), r(a) \rangle = \left( \int_{\text{quant.}} \cdots \right) L(A) r(a) \quad (36)$$

where the package  $\left( \int_{\text{quant.}} \cdots \right)$  refers to some very complicated contour integrals [1] which reduce to those involved in eq.(2) in the reflexionless cases.

The main lesson from ref.[4] is that this pairing could be understood as the analogue of the pairing between one-forms and one-cycles simply defined by integrating the one-form, say  $d\omega$ , along the one-cycle, say  $C$  :

$$\langle L(A), r(a) \rangle \Longleftrightarrow \int_C d\omega$$

Under this analogy one has the following possible identification:

$$\begin{aligned} \text{forms} &\Longleftrightarrow L(A), & \text{around } \nu \simeq \infty \\ \text{cycles} &\Longleftrightarrow r(a) \end{aligned}$$

This identification is appropriate close to the semi-classical limit  $\nu \rightarrow \infty$ , as we hope to have convinced the reader. But in the opposite limit, this is the dual identification which is appropriate:

$$\begin{aligned} \text{cycles} &\iff L'(A), & \text{around } \nu \simeq 0 \\ \text{forms} &\iff r'(a) \end{aligned}$$

In other words, on quantum Riemann surfaces forms and cycles are on an equal footing.

This fact can be formulated in more mathematical terms:

**Proposition.** *For generic values of the coupling constant  $\xi$ , there exist two skew symmetric polynomials  $C(A_1, A_2)$  and  $C^*(a_1, a_2)$  such that if we decompose them as follows :*

$$\begin{aligned} C(A_1, A_2) &= \sum_k (L_k(A_1)M_k(A_2) - L_k(A_2)M_k(A_1)) \\ C^*(a_1, a_2) &= \sum_j (s_j(a_1)r_j(a_2) - s_j(a_2)r_j(a_1)) \end{aligned}$$

then the “period matrix ”  $P$  defined by:

$$P_{ij} = \begin{pmatrix} \langle L_i, r_j \rangle & \langle L_i, s_j \rangle \\ \langle M_i, r_j \rangle & \langle M_i, s_j \rangle \end{pmatrix}_{ij} \quad (37)$$

is a symplectic matrix.

For the proof, see ref.[4]. A quick comparison with the previous sections shows that this is really the quantum analogue of the Riemann bilinear identity.

### 3.3 Null-vector equations.

As explained in [13], the occurrence of null vectors leads to a set of differential equations for the form factors, or for the correlation functions. As in the classical theory, they reflect the quantum equations of motion.

In ref.[13] these equations were written in a fermionic language. Here we will rewrite them in an alternative, but equivalent, bosonic language. We will just quote the results. Thus, let us introduce again the generating function of the descendent operators:

$$\mathcal{L}(t, y) = \exp \left( \sum_{k \geq 1} t_{2k-1} I_{2k-1} + y_{2k} J_{2k} \right) \cdot \mathbf{1} \quad (38)$$

The functions  $\mathcal{L}(t, y)$  may be understood as the generating function of the expectation values of the descendents of the identities between any states of the theory. Choosing these states to be the  $n$ -soliton states allows us to identify  $\mathcal{L}(t, y)$  with the generating function of the form factors. But choosing these states to be those created by auxiliary operators allows us to interpret  $\mathcal{L}(t, y)$  as the generating function of the correlation functions.

**Proposition.** *The equations arising from the “exact forms” can be written as :*

$$\int dD e^{-\xi(D, y)} \mathcal{L}(t + [D]_o; y + [D]_e) = 0 \quad (39)$$

where  $\xi(D, y) = \sum_{k \geq 1} D^{2k} y_{2k}$  and  $[D]_o = (\dots, \left( \frac{1-q^{2k-1}}{1+q^{2k-1}} \right) \frac{D^{-2k+1}}{2k-1}, \dots)$  and  $[D]_e = (\dots, \frac{D^{-2k}}{k}, \dots)$ .

The equations arising from the “deformed Riemann bilinear identities” can be written as :

$$\int_{|D_2| > |D_1|} dD_1 dD_2 (D_1^2 - D_2^2) \tau \left( \frac{D_1}{D_2} \right) e^{-\xi(D_1, y) - \xi(D_2, y)} \mathcal{L}(t + [D_1]_o + [D_2]_o; y + [D_1]_e + [D_2]_e) = 0 \quad (40)$$



where the function  $\tau(x)$  is defined by:

$$\tau(x) = \sum_{k=1}^{\infty} \frac{1 - q^{2k-1}}{1 + q^{2k-1}} x^{2k-1} - \sum_{k=1}^{\infty} \frac{1 + q^{2k}}{1 - q^{2k}} x^{2k} \quad (41)$$

To make sense of the function  $\tau(x)$  we have to assume that the parameter  $q$  is not a root of unity. The function  $\xi(D, y)$  should not be confused with the function  $\zeta(A, t)$  which was introduced above.

Equations (39,40) are linear equations for the generating functions  $\mathcal{L}(t, y)$ . Thus, they give linear relations among the correlation functions of the descendent operators. They do seem to give non trivial information on these correlation functions until we find a way to close this hierarchy of equations.

In ref.[13], the null-vector equations (39) and (40) were used to show that the character of the space of local fields obtained by the bootstrap approach matches the character of the space of field of the ultraviolet conformal field theory.

## 4 A new description of the KdV hierarchy.

The classical limit, which corresponds to  $\nu \rightarrow \infty$ , of the quantum equations leads to a new formulation of the KdV hierarchy. In this description the fundamental variable is the generating function  $S(A)$  of the densities of the integrals of motion:

$$S(A) = 1 + \sum_{k \geq 1} A^{-2k} S_{2k} = \exp \left( - \sum_k \frac{A^{-2k}}{k} J_{2k} \right)$$

To write the equations we introduce the generating function  $\mathcal{L}^{cl}(t, y)$  defined by :

$$\mathcal{L}^{cl}(t, y) = \exp \left( \sum_{k \geq 1} y_{2k} J_{2k}(t) \right)$$

**Proposition.** *The KdV hierarchy then reduces in the set of two equations for  $\mathcal{L}^{cl}(t, y)$ :*

$$\int D e^{-\xi(D, y)} dI(D) \mathcal{L}^{cl}(t, y + [D]_e) = 0 \quad (42)$$

and

$$\begin{aligned} \int_{|D_2| > |D_1|} D_2 D_1 (D_1^2 - D_2^2) \log \left( 1 - \frac{D_1^2}{D_2^2} \right) e^{-\xi(D_1, y) - \xi(D_2, y)} dI(D_1) dI(D_2) \mathcal{L}^{cl}(t, y + [D_1]_e + [D_2]_e) + \\ + 8\pi i \int dD D^3 e^{-2\xi(D, y)} \mathcal{L}^{cl}(t, y + 2[D]_e) = 0 \end{aligned} \quad (43)$$

where  $[D]_e = (\dots, \frac{D^{-2k}}{k}, \dots)$  and  $\xi(D, y) = \sum_{k \geq 1} D^{2k} y_{2k}$  as before, and  $dI(D) = \sum_k D^{-2k} dD \partial_{2k-1}$ . The mixed operator  $dI(D)$  acts on  $\mathcal{L}^{cl}$  by differentiation with respect to the time variables, ie by  $\partial_{2k-1}$ .

Equations (42,43) provide a system of linear differential equations for the Taylor coefficients of  $\mathcal{L}^{cl}(t, y)$ . It becomes a system of non-linear differential equations for the  $J_{2k}$  only after the closure condition  $\mathcal{L}^{cl}(t, y) = \exp \left( \sum_{k \geq 1} y_{2k} J_{2k}(t) \right)$  has been imposed. Ie. we have to impose the following factorization relation :

$$\mathcal{L}^{cl}(t, y + [D]_e) = \frac{1}{S(D)} \mathcal{L}^{cl}(t, y) \quad (44)$$

One may think of this closure equation as a kind of Ward identity. With this closure condition, eqs.(42) and (43) are completely equivalent to those of the KdV hierarchy.

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