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Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1997, tome 48
« Conférences de M. Bauer, A. Beauville, O. Babelon, A. Bilal, R. Stora », , exp. nº 7,
p. 139-170
<http://www.numdam.org/item?id=RCP25_1997__48__139_0>

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ALGEBRAIC STRUCTURE
OF
COHOMOLOGICAL FIELD THEORY MODELS
AND
EQUIVARIANT COHOMOLOGY

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Abstract : The definition of observables within conventional gauge theories is settled by general consensus. Within cohomological theories considered as gauge theories of an exotic type, that question has a much less obvious answer. It is shown here that in most cases theses theories are best defined in terms of equivariant cohomologies both at the field level and at the level of observables.

Lectures at the 1st Caribbean Spring School of Mathematics and Theoretical Physics, Saint François, Guadeloupe, May 30 – June 13 1993

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Ce texte reprend une prépublication du :
Laboratoire de Physique Théorique ENSLAPP, ENSLAPP-A-481/94,
Institut de Physique Nucléaire d'Orsay, Division Physique Théorique, IPNO-TH-94/29.

I) Introduction.

In a classical article [W88], E. Witten proposed a Euclidean field theory scheme which should allow one to compute cohomology classes of orbit spaces using field theory methods. The example treated in that article is that of the Donaldson invariants [DK90]. Whereas the corresponding classical action was found thanks to $N=2$ supersymmetry arguments, it was progressively realized that equivariant cohomology could be thought of as the proper mathematical background at the root of such constructions [W88, BS88, B92, OSB89, BS91, K93].

While equivariant cohomology is more than twenty years old [C50, GHV73, AB84, MQ86, BGV91, K93] relatively little is known about the corresponding field theory models in which both ultraviolet and infrared problems arise. Here, we shall have in mind a perturbative local field theory approach which is probably suitable since it is conjectured that the semiclassical approximation is exact. This sheds no light on the infrared problem, and in particular, the question of integration over moduli. These notes will focus on algebraic aspects needed to constrain the above mentioned field theories. Two models will be studied to some extent : "topological" Yang-Mills in four dimensions (YM_4^{top}), pure topological gravity in two dimensions (Gr_2^{top}). These are the examples for which equivariant cohomology is needed. Topological σ-models [WBS88] barely need such refinements unless they are coupled to Gr_2^{top}. In all models, on the other hand, field theory is the ideal set up to perform "fiber" integration.

These notes will be divided into three parts :

Section II will be devoted to a description of equivariant cohomology with emphasis on the points needed in the following sections.

Section III is devoted to YM_4^{top}.

Section IV is devoted to Gr_2^{top}.

The point of view taken here will be as algebraic as possible since it is the first step to control the perturbative renormalization problems to be solved next.

II) Equivariant cohomology [K93].

Let $\mathcal{M}$ be a smooth manifold and $\Omega^*(\mathcal{M})$ the exterior algebra of differential forms on $\mathcal{M}$ endowed with the differential $d_{\mathcal{M}}$. A Lie group $\mathcal{G}$ is assumed to be acting on $\mathcal{M}$ and its Lie algebra will be denoted Lie$\mathcal{G}$. For any $\lambda \in \mathrm{Lie}\mathcal{G}$ there is a vector field $\lambda_{\mathcal{M}}$ representing the infinitesimal action of λ on $\mathcal{M}$. This vector field $\lambda_{\mathcal{M}}$ is usually called the fundamental vector field associated with λ. We shall denote $i_{\mathcal{M}}(\lambda) = i_{\mathcal{M}}(\lambda_{\mathcal{M}})$ and $l_{\mathcal{M}}(\lambda) = l_{\mathcal{M}}(\lambda_{\mathcal{M}}) = [i_{\mathcal{M}}(\lambda),\ d_{\mathcal{M}}]_+$ the contraction (or inner derivative) and Lie derivative acting on $\Omega^*(\mathcal{M})$. Let us recall that $i_{\mathcal{M}}(\lambda)$ takes n-forms into (n−1)-forms while $l_{\mathcal{M}}(\lambda)$ acts on forms without changing their degrees. Elements of $\Omega^*(\mathcal{M})$ which are annihilated by both $i_{\mathcal{M}}(\lambda)$ and $l_{\mathcal{M}}(\lambda)$, for any $\lambda \in \mathrm{Lie}\mathcal{G}$, are the so-called ***basic*** elements of $\Omega^*(\mathcal{M})$ for the action of $\mathcal{G}$. The ***basic cohomology*** of $\mathcal{M}$, for the action of $\mathcal{G}$, is accordingly defined [C50].

We now consider the Weil algebra $\mathcal{W}$ of Lie$\mathcal{G}$. It is generated by the "connection" ω and its curvature Ω :

$$\Omega = d_{\mathcal{W}}\omega + \frac{1}{2}[\omega,\omega] \tag{2.1}$$

where $d_{\mathcal{W}}$ is the differential of $\mathcal{W}$. Of course, one has the Bianchi identity :

$$d_{\mathcal{W}}\Omega + [\omega,\Omega] = 0 \tag{2.2}$$

There is an action $i_{\mathcal{W}}(\lambda)$, $l_{\mathcal{W}}(\lambda)$ for $\lambda \in$ Lie$\mathcal{G}$:

$$i_{\mathcal{W}}(\lambda)\omega = \lambda \quad , \quad l_{\mathcal{W}}(\lambda)\omega = -[\lambda , \omega] \tag{2.3a}$$

$$i_{\mathcal{W}}(\lambda)\Omega = 0 \quad , \quad l_{\mathcal{W}}(\lambda)\Omega = -[\lambda , \Omega] \tag{2.3b}$$

For instance, ω may be a connection on a principal $\mathcal{G}$-bundle Π and Ω its curvature. In that case $i_{\mathcal{W}}(\lambda)$ and $l_{\mathcal{W}}(\lambda)$ are generated by the action of $\mathcal{G}$ on Π, and $\mathcal{W}$ will be referred to as $\mathcal{W}_{\Pi}$.

We now consider the graded algebra $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ equipped with the differential $d_{\mathcal{M}} + d_{\mathcal{W}}$ so that $(\Omega^*(\mathcal{M})\otimes\mathcal{W},\ d_{\mathcal{M}} + d_{\mathcal{W}})$ turns into a graded differential algebra. Finally, the operations $i_{\mathcal{M}} + i_{\mathcal{W}}$ and $l_{\mathcal{M}} + l_{\mathcal{W}}$ are defined on $(\Omega^*(\mathcal{M})\otimes\mathcal{W},\ d_{\mathcal{M}} + d_{\mathcal{W}})$. The so-called ***equivariant cochains*** are the elements of $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ that are annihilated by $(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)$ and $(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)$ for any $\lambda \in$ Lie$\mathcal{G}$, and the ***equivariant cohomology***, for the action of $\mathcal{G}$, is accordingly defined. This is what is called the ***Weil scheme*** for equivariant cohomology.

Equivariant cohomology can be alternatively described in the so-called ***intermediate scheme***, which was introduced in [K93] and which will be repeatedly used in the sequel. It is obtained from the Weil scheme via of the following algebra isomorphism :

$$x \to \exp\{-i_{\mathcal{M}}(\omega)\}x \tag{2.4}$$

for any $x \in \Omega^*(\mathcal{M})\otimes\mathcal{W}$. This isomorphism changes the original differential and operations on $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ by conjugation[3] :

$$d_{\mathcal{M}} + d_{\mathcal{W}} \quad \to \quad D = d_{\mathcal{W}} + d_{\mathcal{M}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega) \tag{2.5a}$$

$$(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda) \quad \to \quad i_{\mathcal{W}}(\lambda) = e^{-i_{\mathcal{M}}(\omega)}(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\omega)} \tag{2.5b}$$

$$(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) \quad \to \quad (l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) = e^{-i_{\mathcal{M}}(\omega)}(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\omega)} \tag{2.5c}$$

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[3] These equations can be easily obtained by introducing the family of isomorphisms $x \to \exp\{-t \cdot i_{\mathcal{M}}(\omega)\}x$, $0 \le t \le 1$, and solving the differential equations for the transformed differential and operations, recalling that $i_{\mathcal{W}}(\lambda)\omega = \lambda$.

Finally, the so-called ***Cartan model*** is obtained from the intermediate scheme by putting $\omega = 0$ so that $D^2|_{\omega=0}$ vanishes when restricted to invariant cochains. This is the most popular model, although many calculations are better automatized in the intermediate scheme.

Another item which will be repeatedly used is "Cartan's theorem 3" [C50] : let us assume that $(\Omega^*(\mathcal{M}), d_{\mathcal{M}}, i_{\mathcal{M}}, l_{\mathcal{M}})$ admits a $\mathcal{G}$-connection, that is to say a Lie$\mathcal{G}$-valued 1-form θ on $\mathcal{M}$ such that $i_{\mathcal{M}}(\lambda)\theta = \lambda$ and $l_{\mathcal{M}}(\lambda)\theta = -[\lambda, \theta]$ for any $\lambda \in$ Lie$\mathcal{G}$, with curvature Θ. Then any equivariant cohomology class of $\Omega^*(\mathcal{M}) \otimes \mathcal{W}$ with representative $P(\omega,\Omega)$ gives rise canonically to a basic cohomology class of $\Omega^*(\mathcal{M})$ with representative $P(\theta,\Theta)$. This can be easily proven by using the homotopy which allows to prove the triviality of the cohomology of the Weil algebra [MSZ85]. It follows from the construction that the cohomology class of $P(\theta,\Theta)$ does not depend on θ (see Appendix B).

One convenient way to produce equivariant cohomology classes is as follows [BGV91] : we consider an H-bundle $\mathcal{P}(\mathcal{M},H)$ on which there exists an action of $\mathcal{G}$ which lifts the action of $\mathcal{G}$ on $\mathcal{M}$. In general, the Lie group H has nothing to do with the Lie group $\mathcal{G}$. As before, $\mathcal{P}(\mathcal{M},H)$ is endowed with a differential $d_{\mathcal{P}}$, a contraction $i_{\mathcal{P}}$ and a Lie derivative $l_{\mathcal{P}}$.

Next, let Γ be a $\mathcal{G}$-invariant H-connection on $\mathcal{P}(\mathcal{M},H)$:

$$l_{\mathcal{P}}(\lambda)\Gamma = 0 \qquad \text{for any } \lambda \in \text{Lie}\mathcal{G} \tag{2.6}$$

The pull-back $\hat{\Gamma}$ of Γ on $\Omega^*(\mathcal{M}) \otimes \mathcal{W} \otimes \text{LieH}$ is a 1-form on $\mathcal{P}(\mathcal{M},H)$ and a 0-form in $\mathcal{W}$. It follows that[4] :

$$i_{\mathcal{W}}(\lambda)\hat{\Gamma} = 0 \tag{2.7}$$

for any $\lambda \in$ Lie$\mathcal{G}$.

In $\Omega^*(\mathcal{P}(\mathcal{M},H)) \otimes \mathcal{W}$, the equivariant curvature of $\hat{\Gamma}$ is defined by :

$$R^{eq}_{int}(\hat{\Gamma},\omega,\Omega) = D\hat{\Gamma} + \frac{1}{2}[\hat{\Gamma},\hat{\Gamma}] \tag{2.8}$$

where $D = d_{\mathcal{W}} + d_{\mathcal{P}} + l_{\mathcal{P}}(\omega) - i_{\mathcal{P}}(\Omega)$. Then, if I_H is a symmetric invariant polynomial on LieH, we consider the H-characteristic class $I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = I_H\left(R^{eq}_{int}(\hat{\Gamma},\omega,\Omega)\right)$. It is defined on $\mathcal{M}$ and fulfills :

$$(d_{\mathcal{W}} + d_{\mathcal{M}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega)) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9a}$$

$$i_{\mathcal{W}}(\lambda) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9b}$$

$$(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9c}$$

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[4]This construction may be extended by choosing for $\mathcal{W}$ a $\mathcal{W}_\Pi$ for some Π as above, and have Γ depending parametrically on points of Π. Equation (2.6) has then to be replaced by : $(l_\Pi + l_{\mathcal{P}})(\lambda)\Gamma = 0$ whereas (2.7) still holds.

In the Weil scheme, the equivariant curvature is defined by :

$$\mathrm{R}^{\mathrm{eq}}_{\mathrm{W}}(\hat{\Gamma},\omega,\Omega)=\left(\mathrm{d}_{\mathcal{P}}+\mathrm{d}_{\mathcal{W}}\right)\left(\hat{\Gamma}+i_{\mathcal{P}}(\omega)\hat{\Gamma}\right)+\frac{1}{2}\left[\left(\hat{\Gamma}+i_{\mathcal{P}}(\omega)\hat{\Gamma}\right),\left(\hat{\Gamma}+i_{\mathcal{P}}(\omega)\hat{\Gamma}\right)\right]_{\mathrm{H}} \quad (2.10)$$

We may similarly consider $I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\omega,\Omega\right)=I_{\mathrm{H}}\left(\mathrm{R}^{\mathrm{eq}}_{\mathrm{W}}(\hat{\Gamma},\omega,\Omega)\right)=\exp\left\{-i_{\mathcal{P}}(\omega)\right\}I_{\mathrm{H}}\left(\mathrm{R}^{\mathrm{eq}}_{\mathrm{int}}(\hat{\Gamma},\omega,\Omega)\right)$ which fulfills :

$$(\mathrm{d}_{\mathcal{M}}+\mathrm{d}_{\mathcal{W}})I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11a)$$

$$(i_{\mathcal{M}}+i_{\mathcal{W}})(\lambda)I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11b)$$

$$(l_{\mathcal{M}}+l_{\mathcal{W}})(\lambda)I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11c)$$

Finally, if $\mathcal{M}$ admits a $\mathcal{G}$-connection θ with curvature Θ, we can apply "Cartan's theorem 3", and substitute θ and Θ instead of ω and Ω in $I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\omega,\Omega\right)$[5], so that :

$$\mathrm{d}_{\mathcal{M}}I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12a)$$

$$i_{\mathcal{M}}(\lambda)I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12b)$$

$$l_{\mathcal{M}}(\lambda)I^{\mathrm{eq}}_{\mathrm{H,W}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12c)$$

By standard arguments, these cohomology classes do not depend either on $\hat{\Gamma}$ or on θ. In the following $\mathcal{P}(\mathcal{M},\mathrm{H})$ will be a family of H-bundles over a finite dimensional manifold Σ and $\mathcal{M}$ will be itself an infinite dimensional fibered manifold with fiber Σ and base, a space of fields defined on Σ. In this set up, the generators of the Weil algebra can be also realized as fields on Σ.

As we shall see in the sequel, this rather modest equipment proves quite useful to understand many features of the cohomological theories. The interesting aspects lie in the interconnection between various equivariant cohomologies, schematically, one attached to fields and one attached to observables as just described.

One final remark is in order : the above constructions only involve Lie algebras. In practice, this may not be enough and global group properties may have to be checked.

III) Topological Yang-Mills (YM_4^{top}) [W88, BS88].

At the geometric level as well as at the field theory level, one has to distinguish the fields and the observables.

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[5] One may wonder why one does not use such a connection right from the beginning. The reader may convince himself that doing so would spoil the main algebraic properties of the whole construction, e.g. $\mathrm{D}^2 = 0$, with D the differential of equation (2.5a).

In YM_4^{top}, the idea is to produce cohomology classes of $\mathcal{A}/\mathcal{G}$ where $\mathcal{A}$ is a suitably defined space of connections a on some principal G-bundle P(Σ,G) over a four-dimensional space-time manifold Σ and $\mathcal{G}$ is a suitably defined gauge group (group of vertical automorphisms of P(Σ,G)) [DK90]. The differential and operations are respectively denoted by d_Σ, i_Σ and l_Σ for Σ, d_P, i_P and l_P for P(Σ,G) and δ, $\mathcal{I}$ and $\mathcal{L}$ for $\mathcal{A}$.

To produce the structure equations of the model, we follow section II. Here, $\mathcal{M}=\mathcal{A}$ and $\mathcal{W}$ is realized by a $\mathcal{G}$-connection $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ on another copy $\tilde{\mathcal{A}}$ of $\mathcal{A}$. The differential and operations on $\tilde{\mathcal{A}}$ are denoted by $\tilde{\delta}$, $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{L}}$. The fields will be chosen as a, δa, $\tilde{\omega}$ and $\tilde{\Omega}$.

The structure equations then read :

$$s^{top}a=\Psi+\mathcal{L}(\tilde{\omega})a=\Psi+\mathcal{L}^{top}(\tilde{\omega})\,a\equiv\Psi-D_a\tilde{\omega} \tag{3.1a}$$

$$s^{top}\Psi=-\mathcal{L}^{top}(\tilde{\Omega})a+\mathcal{L}^{top}(\tilde{\omega})\Psi\equiv-D_a\tilde{\Omega}+[\Psi,\tilde{\omega}] \tag{3.1b}$$

$$s^{top}\tilde{\omega}=\tilde{\Omega}-\frac{1}{2}[\tilde{\omega},\tilde{\omega}] \tag{3.1c}$$

$$s^{top}\tilde{\Omega}=[\tilde{\Omega},\tilde{\omega}] \tag{3.1d}$$

where :

$$s^{top}=\tilde{\delta}+\delta+\mathcal{L}(\tilde{\omega})-\mathcal{I}(\tilde{\Omega}) \quad , \quad \Psi=\delta a\equiv\Psi_{int} \tag{3.2}$$

in the intermediate scheme, whereas :

$$s^{top}=\tilde{\delta}+\delta \quad , \quad \Psi=\delta a-\mathcal{L}(\tilde{\omega})a=\delta a-\ \mathcal{L}^{top}(\tilde{\omega})\,a\equiv\Psi_W \tag{3.3}$$

in the Weil scheme, and $\mathcal{L}^{top}=\tilde{\mathcal{L}}+\mathcal{L}$ in both schemes[6]. One can check that :

$$\Psi_{int}=\exp\{-\mathcal{I}(\tilde{\omega})\}\Psi_W \tag{3.4}$$

$$\mathcal{I}^{top}(\lambda)\tilde{\omega}=\lambda \qquad , \qquad \mathcal{I}^{top}(\lambda)(\text{other})=0 \tag{3.5}$$

for any $\lambda\in\text{Lie}\mathcal{G}$, with $\mathcal{I}^{top}(\lambda)=\tilde{\mathcal{I}}(\lambda)$ in the intermediate scheme and $\mathcal{I}^{top}(\lambda)=\tilde{\mathcal{I}}(\lambda)+\mathcal{I}(\lambda)$ in the Weil scheme.

Now choose $\mathcal{M}=\mathcal{A}\times\Sigma$, $\mathcal{P}(\mathcal{M},H)=\mathcal{P}(\mathcal{A}\times\Sigma,G)=\mathcal{A}\times P(\Sigma,G)$ and $\hat{\Gamma}=\hat{a}$: for any point a of $\mathcal{A}$ we consider the principal bundle P(Σ,G) equipped with the connection a. This is a family $\hat{a}$ of G-connections such that $\hat{a}(a,p)=a(p)$ for any $(a,p)\in\mathcal{A}\times P(\Sigma,G)$, which defines a G-connection on $\mathcal{P}(\mathcal{A}\times\Sigma,G)$. We extend $\hat{a}$ to $\tilde{\mathcal{A}}\times\mathcal{P}(\mathcal{A}\times\Sigma,G)$. As a zero-form on $\mathcal{A}$ and a LieG-valued 1-form on P(Σ,G), $\hat{a}$ is a LieG-valued 1-form on $\mathcal{P}(\mathcal{A}\times\Sigma,G)$.

[6] To get equation (3.1b) in the Weil scheme, one can either use (3.4) together with (2.5c) or directly compute it by using : $\tilde{\mathcal{L}}(\tilde{\omega})\mathcal{L}(\tilde{\omega})a=\mathcal{L}\big(\tilde{\mathcal{L}}(\tilde{\omega})\tilde{\omega}\big)a=-\mathcal{L}\big([\tilde{\omega},\tilde{\omega}]\big)a=2\mathcal{L}^{top}(\tilde{\omega})\mathcal{L}(\tilde{\omega})a$.

From Appendix A, the fundamental vector field $\underline{\lambda}$ associated with the action of $\lambda \in \mathrm{Lie}\mathcal{G}$ on $\mathcal{P}(\mathfrak{A}\times\Sigma,G)$ takes the following expression at $(a,p)\in\mathcal{P}(\mathfrak{A}\times\Sigma,G)$:

$$\underline{\lambda} = l_P(\lambda_P)a_\mu \frac{\delta}{\delta a_\mu} - \lambda_P^\alpha e_\alpha \tag{3.6a}$$

where λ_P is the fundamental vector field on $P(\Sigma,G)$ associated with λ (for the natural left-action of $\mathcal{G}$ on $P(\Sigma,G)$) and e_α the fundamental vector field associated with a basis of LieG indexed by α. Noting that $\hat{a}$ does not really depend on $\widetilde{\mathfrak{A}}$, the actions of $\mathrm{Lie}\mathcal{G}$ on $\hat{a}$ reads :

$$\begin{aligned}
&\left(\widetilde{\mathcal{I}} + \mathcal{I} + i_P\right)(\lambda)\hat{a} \equiv \mathcal{I}(\lambda)\hat{a} - i_P(\lambda)\hat{a} = -\hat{\lambda} \\
&\left(\widetilde{\mathcal{L}} + \mathcal{L} + l_P\right)(\lambda)\hat{a} \equiv \mathcal{L}(l_P(\lambda_P)a_\mu \frac{\delta}{\delta a_\mu})\hat{a} - l_P(\lambda_P)\hat{a} \\
&\qquad = l_P(\lambda_P)\hat{a} - l_P(\lambda_P)\hat{a} = 0
\end{aligned} \tag{3.6b}$$

where $\hat{\lambda}$ is a LieG-valued function on $\mathfrak{A}\times P(\Sigma,G)$ defined by : $\hat{\lambda}(a,p) = \lambda(p)$ for any element (a,p) of $\mathfrak{A}\times P(\Sigma,G)$. From equations (3.6b), one sees that $\hat{a}$ is $\mathrm{Lie}\mathcal{G}$-invariant.

In the intermediate scheme, the equivariant curvature of $\hat{a}$ is :

$$F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right) = D\hat{a} + \frac{1}{2}[\hat{a},\hat{a}] \tag{3.7}$$

with :

$$D = \tilde{\delta} + \left(\delta + d_P\right) + \left(\mathcal{L} + l_P\right)(\tilde{\omega}) - \left(\mathcal{I} + i_P\right)(\tilde{\Omega}) \tag{3.8}$$

Taking into account the $\mathrm{Lie}\mathcal{G}$-invariance of $\hat{a}$, we get :

$$F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right) = \hat{F}(\hat{a}) + \delta\hat{a} + i_P(\tilde{\Omega})\hat{a} = \hat{F}(a) + \hat{\Psi}_{\mathrm{int}} + \hat{\tilde{\Omega}} \tag{3.9}$$

where $\hat{F}(\hat{a}) = d_P\hat{a} + \frac{1}{2}[\hat{a},\hat{a}]$. Notice the similarity of equation (3.9) with equation (3.2) up to the symbol ^. Moreover, using the $\mathcal{G}$-invariance of $\hat{a}$, one can verify the $\mathcal{G}$-basicity condition :

$$\widetilde{\mathcal{I}}(\lambda)F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right) = 0 \quad , \quad \left(\widetilde{\mathcal{L}} + \mathcal{L} + l_P\right)(\lambda)F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right) = 0 \tag{3.10}$$

holding for any $\lambda \in \mathrm{Lie}\mathcal{G}$.

In order to go to the Weil scheme, we transform $\hat{a}$ as follows :

$$\hat{a} \to e^{(\mathcal{I}+i_P)(\tilde{\omega})}\hat{a} = \hat{a} + \left(\mathcal{I} + i_P\right)(\tilde{\omega})\hat{a} = \hat{a} - i_P(\tilde{\omega})\hat{a} \equiv \hat{a} + \hat{\tilde{\omega}} \tag{3.11}$$

The corresponding equivariant curvature is :

$$F_W^{\mathrm{eq}}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right) = \left(\tilde{\delta} + \delta + d_P\right)\left(\hat{a} - i_P(\tilde{\omega})\hat{a}\right) + \frac{1}{2}\left[\left(\hat{a} - i_P(\tilde{\omega})\hat{a}\right),\left(\hat{a} - i_P(\tilde{\omega})\hat{a}\right)\right] \tag{3.12}$$

or equivalently :

$$F_W^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = \hat{F}(\hat{a}) + (\delta\hat{a} + D_a\hat{\tilde{\omega}}) + \hat{\tilde{\Omega}} = \hat{F}(\hat{a}) + \hat{\tilde{\Psi}}_W + \hat{\tilde{\Omega}} \tag{3.13}$$

By construction : $F_W^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = \exp\{(\mathcal{I} + i_P)(\tilde{\omega})\}F_{int}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega})$ and consequently :

$$\begin{aligned}&(\tilde{\mathcal{I}} + \mathcal{I} + i_P)(\lambda)F_W^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = 0\\ &(\tilde{\mathcal{L}} + \mathcal{L} + l_P)(\lambda)F_W^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = 0\end{aligned} \tag{3.14}$$

Now, for any symmetric invariant polynomial I_G on LieG, $I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = I_G(F_W^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}))$ fulfills :

$$(\tilde{\delta} + \delta + d_P)I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = 0 \tag{3.15a}$$

$$(\tilde{\mathcal{I}} + \mathcal{I} + i_P)(\lambda)I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = 0 \tag{3.15b}$$

$$(\tilde{\mathcal{L}} + \mathcal{L} + l_P)(\lambda)I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega}) = 0 \tag{3.15c}$$

for any $\lambda \in \mathrm{Lie}\mathcal{G}$.

Last but not least, we apply "Cartan's theorem 3" to $I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega})$. Let ω be a $\mathcal{G}$-connection on $\mathcal{A}$ and Ω its curvature. It does define a $\mathcal{G}$-connection on $\mathcal{M} = \mathcal{A} \times \Sigma$. Accordingly, we just replace $\tilde{\omega}$ and $\tilde{\Omega}$ respectively by ω and Ω in $I_{G,W}^{eq}(\hat{a},\tilde{\omega},\tilde{\Omega})$. Then :

$$(\delta + d_P)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = (\delta + d_\Sigma)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = 0 \tag{3.16a}$$

$$(\mathcal{I} + i_P)(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = (\mathcal{I} + i_\Sigma)(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = 0 \tag{3.16b}$$

$$(\mathcal{L} + l_P)(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = (\mathcal{L} + l_\Sigma)(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = 0 \tag{3.16c}$$

for any $\lambda \in \mathrm{Lie}\mathcal{G}$. Recall that :

$$F_W^{eq}(\hat{a},\omega,\Omega) = \hat{F}(a) + (\delta\hat{a} + D_a\hat{\omega}) + \hat{\Omega} = \hat{F}(a) + \hat{\Psi}_W + \hat{\Omega} \tag{3.17}$$

with : $\hat{\omega} \equiv -i_P(\omega)\hat{a}$ and $\hat{\varphi} \equiv i_P(\Omega)\hat{a}$.

In fact, $I_{G,W}^{eq}(\hat{a},\omega,\Omega)$ fulfills a horizontality property stronger than (3.16b), namely :

$$\mathcal{I}(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = 0 = i_P(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) = i_\Sigma(\lambda)I_{G,W}^{eq}(\hat{a},\omega,\Omega) \tag{3.18}$$

and is defined on $\mathcal{A} \times \Sigma$. Now, let us decompose $I_{G,W}^{eq}(\hat{a},\omega,\Omega)$ according to :

$$I_{G,W}^{eq}(\hat{a},\omega,\Omega) = \sum_{k=0}^{2n} I_{2n-k}^{k} \tag{3.19}$$

where I_{2n-k}^{k} is a form of degree $2n - k$ on Σ and of degree k on $\mathcal{A}$ such that :

$$
\begin{aligned}
& d_\Sigma I^{k}_{2n-k} + \delta I^{k-1}_{2n-k+1} = 0 \\
& \mathcal{I}(\lambda) I^{k+1}_{2n-k-1} - i_\Sigma(\lambda) I^{k}_{2n-k} = 0 \qquad (3.20) \\
& \mathcal{L}(\lambda) I^{k}_{2n-k} - l_\Sigma(\lambda) I^{k}_{2n-k} = 0
\end{aligned}
$$

Then, the integration over a cycle γ_{2n-k} on Σ yields a k-form on $\mathcal{A}$:

$$
\mathcal{O}^k = \int_{\gamma_{2n-k}} I^{k}_{2n-k} \qquad (3.21)
$$

From the descent equations (3.20), we deduce :

$$
\delta\mathcal{O}^k = -\int_{\gamma_{2n-k}} d_\Sigma I^{k+1}_{2n-k-1} = 0 \qquad (3.22)
$$

and because of the detailed horizontality condition expressed in equation (3.18) :

$$
\mathcal{I}(\lambda)\mathcal{O}^k = \int_{\gamma_{2n-k}} \mathcal{I}(\lambda) I^{k}_{2n-k} = 0 \qquad (3.23)
$$

Finally :

$$
\begin{aligned}
\mathcal{L}(\lambda)\mathcal{O}^k &= \int_{\gamma_{2n-k}} l_\Sigma(\lambda) I^{k}_{2n-k} = \int_{\gamma_{2n-k}} i_\Sigma(\lambda) d_\Sigma I^{k}_{2n-k} = \\
&= \int_{\gamma_{2n-k}} i_\Sigma(\lambda)\delta I^{k-1}_{2n-k+1} = \delta \int_{\gamma_{2n-k}} i_\Sigma(\lambda) I^{k-1}_{2n-k+1} = 0
\end{aligned} \qquad (3.24)
$$

Hence, the k-form $\mathcal{O}^k$ defines a basic cohomology class[7]. This class does not depend on $\hat{a}$, $\hat{\omega}$, $\hat{\Omega}$ and $\hat{\Psi}_W$ provided that they are related by equation (3.17), so that one may average it out over these fields variables, which is the formal reason why the topological YM_4^{top} field theory should be a tool able to construct such cohomology classes. Of course, this is so provided the field theory treatment (e.g. renormalized perturbation theory which in the present case ought to be exact) retains enough properties of the averaging out process, which, in turn will be insured by the fulfillment of the proper Ward identities entailed by the requirement of s^{top}, $\mathcal{I}(\lambda)$ and $\mathcal{L}(\lambda)$ invariance. Note that the equivalence between the structure equations (3.1-3) and those leading to the construction of the observables (equations (3.17)) is insured by "Cartan's theorem 3", which, at the cohomology level allows one to replace $\tilde{\omega}$ and $\tilde{\Omega}$ by $\hat{\omega}$ and $\hat{\Omega}$.

For a review of the field theory context, we refer to [OSB89] supplemented with the proof, provided in [K93], that the basic cohomology proposed there is isomorphic with that proposed in [W88] in view of the equivalence between the intermediate model and the Cartan model. Of course, these (ultraviolet) considerations do not touch the problem of the integration of the relevant cohomology classes over orbit space.

---

[7] An alternative much faster construction is given in Appendix C. This one is identical to that used for 2d topological gravity. That of Appendix C takes advantage of the product structure $\mathcal{P}(\mathcal{M},G) = \mathcal{A} \times P(\Sigma,G)$.

IV) Topological 2d gravity (Gr_2^{top}).

Let Σ be a compact Riemann surface without boundary, of genus larger than one. We recall that the space of complex structures on Σ can be canonically identify with the space $\mathcal{B}(\Sigma)$ of Beltrami differentials on Σ. The origin in $\mathcal{B}(\Sigma)$ is nothing but the complex analytic structure defining Σ itself. Let us introduce more notations : $\mathcal{M}(\Sigma)$ is the space of metrics on Σ; $\mathcal{W}(\Sigma)$ is the group of Weyl transformations acting on $\mathcal{M}(\Sigma)$ by local scaling of the metrics; the space $\mathcal{M}(\Sigma) / \mathcal{W}(\Sigma)$ of conformal classes of metrics on Σ is denoted by $\mathcal{CM}(\Sigma)$ and is naturally isomorphic to $\mathcal{B}(\Sigma)$; finally, $\mathcal{D}_0(\Sigma)$ is the component of the group $\mathcal{D}(\Sigma)$ of diffeomorphisms of Σ connected to the identity. We recall that the Lie algebra of $\mathcal{D}_0(\Sigma)$ is the opposite of the Lie algebra $\mathcal{V}(\Sigma)$ of vector fields on Σ [Mi].

Let $\{(U_\alpha,(z_\alpha,\bar{z}_\alpha))\}$ be an atlas defining the complex analytic structure of Σ, and let g be a metric on Σ. With respect to this atlas, the metric element takes the form :

$$ds^2 = \rho_{z_\alpha \bar{z}_\alpha} \left| dz_\alpha + \mu^{z_\alpha}{}_{\bar{z}_\alpha} d\bar{z}_\alpha \right|^2 \tag{4.1}$$

where $\mu^{z_\alpha}{}_{\bar{z}_\alpha}$ is the component in $(z_\alpha,\bar{z}_\alpha)$ of the Beltrami differential $\mu = \mu^{z_\alpha}{}_{\bar{z}_\alpha} \partial_{z_\alpha} \otimes d\bar{z}_\alpha$ parametrizing the conformal class of the metric g. Note that equation (4.1) produces an isomorphism between $\mathcal{CM}(\Sigma)$ and $\mathcal{B}(\Sigma)$.

In topological (Euclidean) 2d gravity, one first wishes to study the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ and later go over to the moduli space (as already explain in section II, we do not consider the global group properties and hence do not look at the whole group of diffeomorphisms). There are two ways to define $\mathcal{T}(\Sigma)$. In the first one, that we shall refer to as the "Riemannian route", one considers $\mathcal{M}(\Sigma)$ as the parameter space together with the action of $\mathcal{W}(\Sigma) \mathbin{\tilde{\times}} \mathcal{D}_0(\Sigma)$[8] on it :

$$\mathcal{T}(\Sigma) = \frac{\mathcal{M}(\Sigma)}{\mathcal{W}(\Sigma) \mathbin{\tilde{\times}} \mathcal{D}_0(\Sigma)} \tag{4.2a}$$

In the second approach, the space of parameters is $\mathcal{B}(\Sigma)$ and the "gauge group" $\mathcal{D}_0(\Sigma)$ so that :

$$\mathcal{T}(\Sigma) = \frac{\mathcal{B}(\Sigma)}{\mathcal{D}_0(\Sigma)} \tag{4.2b}$$

This will be referred to as the "Conformal route". The equivalence between the Conformal and Riemannian routes comes from the canonical identification of $\mathcal{CM}(\Sigma)$ with $\mathcal{B}(\Sigma)$. The former is natural from the mathematical point of view but presumably less amenable to a field theory treatment by virtue of the non-linearities involved. The latter, closer to field theory [BCI94], will be exhibited as an alternative.

[8] Where $\tilde{\times}$ denotes the semi-direct product.

In the Conformal route, the Weyl transformations are eliminated from the start by fixing the factor ρ of equation (4.1), as a function of μ and $\bar{\mu}$, through the $\mathcal{D}(\Sigma)$ invariant constraint :

$$R(\rho,\mu,\bar{\mu}) = -1 \tag{4.3}$$

where R is the scalar curvature of the metric (4.1). (Recall this is possible because the genus of Σ was assumed to be larger than 1).

We take a $\mathcal{D}_0(\Sigma)$-connection $\tilde{\omega}$ on another copy $\tilde{\mathcal{B}}(\Sigma)$ of $\mathcal{B}(\Sigma)$. So, $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ are vector field on Σ :

$$\begin{aligned} \tilde{\omega} &= \tilde{\omega}^z\partial_z + \tilde{\omega}^{\bar{z}}\partial_{\bar{z}} \\ \tilde{\Omega} &= \tilde{\Omega}^z\partial_z + \tilde{\Omega}^{\bar{z}}\partial_{\bar{z}} \end{aligned} \tag{4.4}$$

If we denote δ, $\mathcal{I}$ and $\mathcal{L}$ the differential and operations on $\mathcal{B}(\Sigma)$ and $\tilde{\delta}$, $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{L}}$ those on $\tilde{\mathcal{B}}(\Sigma)$, the action of $\lambda = \lambda^z\partial_z + \lambda^{\bar{z}}\partial_{\bar{z}} \in \mathcal{V}(\Sigma)$ on $\mathcal{B}(\Sigma)$ is :

$$\begin{aligned} \mathcal{L}(\lambda)\mu &= \left[\left(\partial_{\bar{z}} - \mu^z{}_{\bar{z}}\partial_z + \left(\partial_z\mu^z{}_{\bar{z}}\right)\right)\Lambda^z_\mu\right] d\bar{z}\otimes\partial_z = \left(\mathcal{L}(\lambda)\mu^z{}_{\bar{z}}\right) d\bar{z}\otimes\partial_z \\ &= \left(D_{\mu,\bar{z}}\Lambda^z_\mu\right) d\bar{z}\otimes\partial_z = \bar{D}_\mu\Lambda_\mu \end{aligned} \tag{4.5}$$

where we have introduced the type (1,0) vector field :

$$\Lambda_\mu = \Lambda^z_\mu\,\partial_z = \left(\lambda^z + \mu^z_{\bar{z}}\lambda^{\bar{z}}\right)\partial_z \tag{4.6}$$

and to emphasize the similarities with YM_4^{top} we have defined the operator $\bar{D}_\mu$:

$$\bar{D}_\mu = D_{\mu,\bar{z}}\, d\bar{z} = \left(\partial_{\bar{z}} - \mu^z{}_{\bar{z}}\partial_z + \left(\partial_z\mu^z{}_{\bar{z}}\right)\right) d\bar{z} = \bar{\partial} - \{\mu,\,\} \tag{4.7}$$

acting on type (1,0) vector fields. In equation (4.7), $\bar{\partial}$ is the usual Dolbeault operator, μ is considered as a $\mathcal{V}(\Sigma)$-valued one-form on Σ and $\{\,,\}$ is the natural Lie bracket that turns $\mathcal{V}(\Sigma)$ into a Lie algebra[9]. Finally, noting that $\tilde{\omega}$,$\bar{D}_\mu$ and μ are odd while $\tilde{\Omega}$ is even, we get the structure equations :

$$s^{top}\mu = \nu + \mathcal{L}(\tilde{\omega})\mu = \nu + \mathcal{L}^{top}(\tilde{\omega})\mu = \nu - \bar{D}_\mu\tilde{\omega}_\mu \tag{4.8a}$$

$$s^{top}\nu = -\mathcal{L}^{top}(\tilde{\Omega})\mu + \mathcal{L}^{top}(\tilde{\omega})\nu = -\bar{D}_\mu\tilde{\Omega}_\mu - \left\{\nu,\tilde{\omega}_\mu\right\} \tag{4.8b}$$

$$s^{top}\tilde{\omega}_\mu = \tilde{\Omega}_\mu + \frac{1}{2}\left\{\tilde{\omega}_\mu,\tilde{\omega}_\mu\right\} \tag{4.8c}$$

$$s^{top}\tilde{\Omega}_\mu = -\left\{\tilde{\Omega}_\mu,\tilde{\omega}_\mu\right\} \tag{4.8d}$$

where we have introduced the μ-dependent basis :

---

[9] Since $\mathcal{V}(\Sigma)$ is the opposite of Lie$\mathcal{D}(\Sigma$: $\{\underline{\lambda}_1,\underline{\lambda}_2\} = -[\underline{\lambda}_1,\underline{\lambda}_2]$, where $[\,,\,]$ is the Lie bracket of Lie$\mathcal{D}(\Sigma)$ and we have denoted by the same symbol an element of Lie$\mathcal{D}(\Sigma)$ and its image in $\mathcal{V}(\Sigma)$ by the canonical isomorphism between these two spaces [Mi]. Now, compare equation (4.9) with equation (3.1).

$$\tilde{\omega}_\mu = \tilde{\omega}^z_\mu \partial_z = \left(\tilde{\omega}^z + \mu^z{}_{\bar z}\tilde{\omega}^{\bar z}\right)\partial_z$$
$$\tilde{\Omega}_\mu = \tilde{\Omega}^z_\mu \partial_z = \left(\tilde{\Omega}^z + \mu^z{}_{\bar z}\tilde{\Omega}^{\bar z} + v^z{}_{\bar z}\tilde{\omega}^{\bar z}\right)\partial_z \tag{4.9}$$

with :

$$s^{top} = \tilde{\delta} + \delta + \mathcal{L}(\tilde{\omega}) - \mathcal{I}(\tilde{\Omega}) \quad , \quad v = \delta\mu = \left(\delta\mu^z{}_{\bar z}\right) d\bar z \otimes \partial_z \tag{4.10}$$

in the intermediate scheme, whereas :

$$s^{top} = \tilde{\delta} + \delta \quad , \quad v = \delta\mu + \overline{D}_\mu \tilde{\omega}_\mu = \left(\delta\mu^z{}_{\bar z} - D_{\mu,\bar z}\tilde{\omega}^z_\mu\right) d\bar z \otimes \partial_z \tag{4.11}$$

in the Weil scheme, and $\mathcal{L}^{top}(\lambda) = \tilde{\mathcal{L}}(\lambda) + \mathcal{L}(\lambda)$ in both schemes.

The action of $\mathcal{V}(\Sigma)$ is given by :

$$\mathcal{I}^{top}(\lambda)\tilde{\omega}_\mu = \Lambda_\mu$$
$$\mathcal{I}^{top}(\lambda)\tilde{\Omega}_\mu = -\left(v^z{}_{\bar z}\lambda^{\bar z}\right)\partial_z \tag{4.12}$$
$$\mathcal{I}^{top}(\lambda)\text{other} = 0$$

with $\mathcal{I}^{top}(\lambda) = \tilde{\mathcal{I}}(\lambda)$ in the intermediate scheme and $\mathcal{I}^{top}(\lambda) = \tilde{\mathcal{I}}(\lambda) + \mathcal{I}(\lambda)$ in the Weil scheme. The formulae for $\mathcal{L}(\lambda)$ follow from equations (4.8) and (4.12). Had we stuck to the initial basis, there would be complete similarity with the gauge case YM_4^{top}. The μ dependent basis, more appropriate to discuss holomorphic factorization [KLS91] introduces however an inconvenience : the second of equations (4.12).

Now, choose $\mathcal{M} = \mathcal{B}(\Sigma) \times \Sigma$ equipped with the complex structure defined by the complex variables μ, Z_μ, where Z_μ are complex coordinates on Σ which fulfill (locally) the Beltrami equation :

$$\left(\partial_{\bar z} - \mu^z{}_{\bar z}\partial_z\right)Z_\mu = 0 \tag{4.13}$$

allowing to construct from Σ and μ a Riemann surface denoted by Σ_μ.

For each $\mu \in \mathcal{B}(\Sigma)$ we consider the holomorphic tangent bundle of Σ_μ. This generate the family $T^{(1,0)}_{\{\mu\}}(\Sigma)$ of holomorphic tangent bundles of Σ, and the associated $GL(1,\mathbb{C})$ principal bundle is denoted $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ ($\mathcal{P}(\mathcal{M}$,H) of section II). A set of holomorphic coordinates on $T^{(1,0)}_{\{\mu\}}(\Sigma)$ is given locally by μ, Z_μ, V^{Z_μ}, or $E^{Z_\mu} \in GL(1,\mathbb{C})$ on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$[10]. $\mathcal{D}(\Sigma)$ acts holomorphically on these coordinates so that along an orbit of $\mathcal{D}(\Sigma)$ one gets :

[10] The fiber of $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ over (μ,x) is the set of all frames (i.e. bases) of $T^{(1,0)}_x\Sigma_\mu$. Hence, with respect to the chart (U,Z_μ) at $x \in \Sigma_\mu$, the coordinates E^{Z_μ} of a frame E_x are the entries of the $GL(1,\mathbb{C})$ matrix transforming the natural frame ∂_{Z_μ} of $T^{(1,0)}_x\Sigma_\mu$ into E_x : $E_x = E^{Z_\mu}\,\partial_{Z_\mu}$.

$$
\begin{aligned}
\mu &\longrightarrow \mu^{\varphi} \\
Z_\mu(z,\bar z) &\longrightarrow Z_{\mu^\varphi}(\varphi^{-1}(z,\bar z),\bar\varphi^{-1}(z,\bar z)) \\
E^{Z_\mu}(z,\bar z) &\longrightarrow E^{Z_{\mu^\varphi}}(\varphi^{-1}(z,\bar z),\bar\varphi^{-1}(z,\bar z))
\end{aligned}
\tag{4.14}
$$

The fundamental vector field on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ representing $\lambda\in\mathcal{V}(\Sigma)$ is given in Appendix D. The Dolbeault operators on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ are accordingly given by :

$$
\mathcal{D} = \int_\Sigma \delta\mu \left.\frac{\delta}{\delta\mu}\right|_{Z_\mu,E^{Z_\mu}} d^2z + dZ_\mu \frac{\partial}{\partial Z_\mu} + dE^{Z_\mu}\frac{\partial}{\partial E^{Z_\mu}}
\tag{4.15}
$$

and its complex conjugate, and the total differential is :

$$
D = \mathcal{D} + \overline{\mathcal{D}}
\tag{4.16}
$$

By construction, the contraction I and the Lie derivative L on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ split up :

$$
\begin{aligned}
I(\lambda) &= I(\underline{\lambda}^h) + I(\underline{\lambda}^{\bar h}) \equiv I^h(\lambda) + I^{\bar h}(\lambda) \\
L(\lambda) &= L(\underline{\lambda}^h) + L(\underline{\lambda}^{\bar h}) \equiv L^h(\lambda) + L^{\bar h}(\lambda)
\end{aligned}
\tag{4.17}
$$

with :

$$
\begin{aligned}
L^h(\lambda) &= \left[I^h(\lambda),\mathcal{D}\right]_+ \\
L^{\bar h}(\lambda) &= \left[I^{\bar h}(\lambda),\overline{\mathcal{D}}\right]_+
\end{aligned}
\tag{4.18}
$$

so that :

$$
\left[L^h(\lambda),\mathcal{D}\right] = \left[L^{\bar h}(\lambda),\overline{\mathcal{D}}\right] = 0
\tag{4.19}
$$

and the operators carrying the label h, together with $\mathcal{D}$ commute (in the graded sense) with those carrying the label $\bar h$ together with $\overline{\mathcal{D}}$.

Now, for each $\mu\in\mathcal{B}(\Sigma)$ choose the metric $ds^2_\mu = \rho_{Z_\mu\bar Z_\mu} dZ_\mu d\bar Z_\mu$ solution of the constant curvature equation :

$$
\partial_{\bar Z_\mu}\partial_{Z_\mu} \ln \rho_{Z_\mu\bar Z_\mu} = \rho_{Z_\mu\bar Z_\mu}
\tag{4.20}
$$

equivalent to equation (4.3). In local coordinates, the canonical GL(1,$\mathbb{C}$)-connection on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ associated with $\rho_{Z_\mu\bar Z_\mu}$ is :

$$
\Gamma = \mathcal{D}\ln\rho_{Z_\mu\bar Z_\mu} + D\ln E^{Z_\mu}
\tag{4.21}
$$

where $D\ln E^{Z_\mu}$ denotes the Maurer-Cartan form of GL(1,$\mathbb{C}$).

Using the uniqueness of the solution of equation (4.20), one deduces that :

$$
L(\lambda)\Gamma = 0
\tag{4.22}
$$

Now, given a $\mathcal{D}(\Sigma)$-connection $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ on another copy of $\mathcal{B}(\Sigma)$, the connection becomes in the Weil scheme :

$$\hat{\Gamma} = \mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + D \ln E^{Z_\mu} + I(\tilde{\omega}) \mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \tag{4.23}$$

and its equivariant curvature is given by :

$$\begin{aligned} R_W^{eq}(\hat{\Gamma}) = \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + I(\tilde{\omega}) \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + \frac{1}{2} I(\tilde{\omega}) I(\tilde{\omega}) \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \\ - I(\tilde{\Omega}) \left(\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + D \ln E^{Z_\mu} \right) \end{aligned} \tag{4.24}$$

(details are given in Appendix E).

There remains to replace $\tilde{\omega}$ and $\tilde{\Omega}$ by a $\mathcal{D}(\Sigma)$-connection θ on $\mathcal{B}(\Sigma) \times \Sigma$ and its curvature Θ (Cartan's theorem 3). This is obtained by pulling back on $\mathcal{B}(\Sigma) \times \Sigma$ a $\mathcal{D}(\Sigma)$-connection on $\mathcal{B}(\Sigma)$. All in all, we may write :

$$R \equiv R_W^{eq}(\hat{\Gamma}, \theta, \Theta) = R_2^0 + R_1^1 + R_0^2 \tag{4.25}$$

where the lower index labels the form degree on Σ and the upper index labels the form degree on $\mathcal{B}(\Sigma)$, and :

$$(\delta + d_\Sigma) R = (\mathcal{I} + i_\Sigma)(\lambda) R = (\mathcal{L} + l_\Sigma)(\lambda) R = 0 \tag{4.26}$$

Observables are extracted from[11] :

$$\begin{aligned} R^n &= \left(R_0^2\right)^n + n\left(R_0^2\right)^{n-1} R_1^1 + \left(n\left(R_0^2\right)^{n-1} R_2^0 + \frac{n(n-1)}{2} \left(R_0^2\right)^{n-1} \left(R_1^1\right)^2 \right) \\ &= \mathcal{O}_0^2 + \mathcal{O}_1^{2n-1} + \mathcal{O}_2^{2n-2} \end{aligned} \tag{4.27}$$

with :

$$\begin{cases} \delta \mathcal{O}_0^{2n} = 0 \\ d_\Sigma \mathcal{O}_0^{2n} + \delta \mathcal{O}_1^{2n-1} = 0 \\ d_\Sigma \mathcal{O}_1^{2n-1} + \delta \mathcal{O}_2^{2n-2} = 0 \end{cases} \tag{4.28a}$$

$$\begin{cases} (\mathcal{L} + l_\Sigma)(\lambda) \mathcal{O}_0^{2n} = (\mathcal{L}(\lambda) - l_\Sigma(\lambda)) \mathcal{O}_0^{2n} = 0 \\ (\mathcal{L} + l_\Sigma)(\lambda) \mathcal{O}_1^{2n-1} = (\mathcal{L}(\lambda) - l_\Sigma(\lambda)) \mathcal{O}_1^{2n-1} = 0 \\ (\mathcal{L} + l_\Sigma)(\lambda) \mathcal{O}_2^{2n-2} = (\mathcal{L}(\lambda) - l_\Sigma(\lambda)) \mathcal{O}_2^{2n-2} = 0 \end{cases} \tag{4.28b}$$

[11] There is no need of making $Gl(1,\mathbb{C})$ invariant polynomial, because R is already invariant.

$$\begin{cases} \mathcal{G}(\lambda)\mathcal{O}_2^{2n-2} = 0 \\ \mathcal{G}(\lambda)\mathcal{O}_1^{2n-1} - i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} = 0 \\ \mathcal{G}(\lambda)\mathcal{O}_0^{2n} - i_\Sigma(\lambda)\mathcal{O}_1^{2n-1} = 0 \end{cases} \tag{4.28c}$$

Let us introduce :

$$\begin{aligned} \mathcal{O}^{2n-2} &= \int_\Sigma \mathcal{O}_2^{2n-2} \\ \mathcal{O}_{(\gamma)}^{2n-1} &= \int_\gamma \mathcal{O}_1^{2n-1} \\ \mathcal{O}_{(x)}^{2n-1} &= \mathcal{O}_0^{2n}(x) \end{aligned} \tag{4.29}$$

where γ (resp. x) is a one cycle (resp. 0 cycle) in Σ. One verifies that $\mathcal{O}^{2n-2}$ represents a basic cohomology class on $\mathcal{B}(\Sigma)$ since :

$$\delta\mathcal{O}^{2n-2} = \mathcal{G}(\lambda)\mathcal{O}^{2n-2} = \mathcal{L}(\lambda)\mathcal{O}^{2n-2} = 0 \tag{4.30}$$

However :

$$\delta\mathcal{O}_{(\gamma)}^{2n-1} = 0 \tag{4.31a}$$

but :

$$\mathcal{G}(\underline{\lambda})\mathcal{O}_{(\gamma)}^{2n-1} = \int_\gamma i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} \neq 0 \tag{4.31b}$$

$$\begin{aligned} \mathcal{L}(\lambda)\mathcal{O}_{(\gamma)}^{2n-1} &= \int_\gamma l_\Sigma(\lambda)\mathcal{O}_1^{2n-1} = \int_\gamma i_\Sigma(\lambda)\mathrm{d}_\Sigma\mathcal{O}_1^{2n-1} \\ &= \delta\int_\gamma i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} \neq 0 \end{aligned} \tag{4.31c}$$

Hence, $\mathcal{O}_{(\gamma)}^{2n-1}$ does not represent a basic cohomology class. Similarly :

$$\delta\mathcal{O}_{(x)}^{2n} = 0 \quad \text{while} \quad \mathcal{G}(\lambda)\mathcal{O}_{(x)}^{2n} = i_\Sigma(\lambda)\mathcal{O}_1^{2n-1} \neq 0 \tag{4.32}$$

This is different from what happened in the YM_4^{top} case and is essentially due to the fact that $\mathcal{D}(\Sigma)$ moves points on Σ.

One should realize at this point that one should make sure that whatever cohomology classes have been constructed are non trivial. It is known that modular invariance plays a crucial role in that respect [Mu, BCI94].

On the other hand, the choice of the metric ρ fulfilling the constant negative curvature condition (4.20) is immaterial provided it behaves properly under diffeomorphisms, i.e. a change in ρ produces a coboundary.

Whereas the holomorphic fibration of $\mathcal{B}(\Sigma) \times \Sigma$ over $\mathcal{T}(\Sigma)$ is essential (the smooth fibration being trivial), a real approach is possible whereby the GL(1,$\mathbb{C}$) bundle is reduced to U(1) and the canonical connection Γ is replaced by the unitary connection :

$$\Gamma^{\text{unit}} = \frac{\mathcal{D} - \overline{\mathcal{D}}}{2i} \ln \rho_{Z\overline{Z}} \tag{4.33}$$

This make the bridge with the Riemannian route chosen in [BCI94], as follows.

The structure equations for the action of $\mathcal{D}(\Sigma)$ on $\mathcal{M}(\Sigma)$ read :

$$s^{\text{top}} g = \gamma + \mathcal{L}(\tilde{\omega}) g = \gamma + \mathcal{L}^{\text{top}}(\tilde{\omega}) g \tag{4.34a}$$

$$s^{\text{top}} \gamma = -\mathcal{L}^{\text{top}}(\tilde{\Omega}) g + \mathcal{L}^{\text{top}}(\tilde{\omega}) \gamma \tag{4.34b}$$

$$s^{\text{top}} \tilde{\omega} = \tilde{\Omega} - \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \tag{4.34c}$$

$$s^{\text{top}} \tilde{\Omega} = [\tilde{\Omega}, \tilde{\omega}] \tag{4.34d}$$

where $g \in \mathcal{M}(\Sigma)$, $\tilde{\omega}$ a $\mathcal{D}_0(\Sigma)$-connection on another copy $\tilde{\mathcal{M}}(\Sigma)$ of $\mathcal{M}(\Sigma)$ and $\tilde{\Omega}$ its curvature, $\gamma = \delta g$ in the intermediate scheme, $\gamma = \delta g - \mathcal{L}^{\text{top}}(\tilde{\omega}) g$ in the Weil scheme, and $\mathcal{L}^{\text{top}} = \tilde{\mathcal{L}} + \mathcal{L}$ in both schemes. Of course, δ, $\mathcal{I}$ and $\mathcal{L}$ are the differential and operations on $\mathcal{M}(\Sigma)$ while $\tilde{\delta}$, $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{L}}$ are those on $\tilde{\mathcal{M}}(\Sigma)$.

One may wonder why one does not write down the structure equations for the action of $\mathcal{W}(\Sigma) \tilde{\times} \mathcal{D}(\Sigma)$. The main reason is that there is no known Weyl invariant connection on a bundle over $\mathcal{M}(\Sigma) \times \Sigma$ to provide non trivial cohomology classes.

We now consider the family $\mathcal{P}F_{\{g\}}(\Sigma) = \mathcal{M}(\Sigma) \times P(\Sigma, Gl(2,\mathbb{R}))$ of frame bundles over Σ indexed by $g \in \mathcal{M}(\Sigma)$[12]. As usual, we wish to provide $\mathcal{P}F_{\{g\}}(\Sigma)$ with a $\mathcal{D}(\Sigma)$-invariant Gl(2,$\mathbb{R}$)-connection. Accordingly, we look for a Gl(2,$\mathbb{R}$)-connection Γ that leaves g invariant. In terms of local coordinates :

$$(\delta + d_\Sigma) g_{\mu\nu} - \Gamma^\lambda{}_\mu g_{\lambda\nu} - \Gamma^\lambda{}_\nu g_{\lambda\mu} = 0 \tag{4.35}$$

A solution of this compatibility equation is given by :

$$\Gamma^\lambda{}_\mu = {}^{LC}\Gamma^\lambda{}_\mu + \frac{1}{2} g^{\lambda\nu} \delta g_{\nu\mu} \tag{4.36}$$

where ${}^{LC}\Gamma$ is the Levi-Civita connection :

$${}^{LC}\Gamma^\lambda{}_\mu = \frac{1}{2} g^{\lambda\nu} \left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}\right) dx^\rho \tag{4.37}$$

while the general solution is obtained according to :

---

[12] It is a Gl(2,$\mathbb{R}$) principal bundle over $\mathcal{M}(\Sigma) \times \Sigma$ whose fiber $F_{(g,x)}\Sigma$ over (g,x) is made of all the frames of $T_x\Sigma$. With respect to coordinates (x^k) of x, the coordinates of a frame $E_{(g,x)} = (E_k)$ of $F_{(g,x)}\Sigma$ will be the entries $A^j{}_k$ of the Gl(2,$\mathbb{R}$) matrix A changing the natural frame (∂_k) associated with (x^k) into $E_{(g,x)}$: $E_j = A^k{}_j \partial_k$.

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} + a_{[\mu\nu]} \tag{4.38}$$

$a_{[\mu\nu]}$ being antisymmetric[13]. In equation (4.36) we have chosen $a_{[\mu\nu]} = 0$.

One lifts Γ to get a $gl(2,\mathbb{R})$-valued one form $\hat{\Gamma}$ on $\mathcal{P}F_{\{g\}}(\Sigma)$:

$$\hat{\Gamma}^{\sigma}{}_{\tau} = (A^{-1})^{\sigma}{}_{\sigma'}\Gamma^{\sigma'}{}_{\tau'}A^{\tau'}{}_{\tau} + (A^{-1})^{\sigma}{}_{\sigma'}d_{\mathcal{P}}A^{\sigma'}{}_{\tau} \tag{4.39a}$$

or in a more compact notation :

$$\hat{\Gamma} = A^{-1}\Gamma A + A^{-1}d_{\mathcal{P}}A \tag{4.39b}$$

As explained in Appendix F, the action of $\mathcal{D}(\Sigma)$ extends to $\mathcal{P}F_{\{g\}}(\Sigma)$, and the fundamental vector field associated with $\lambda \in \mathcal{V}(\Sigma)$ reads :

$$\underline{\lambda} = \left((l_{\Sigma}(\lambda)g_{\mu\nu})\frac{\delta}{\delta g_{\mu\nu}} - \lambda^{\alpha}\partial_{\alpha} - (\partial_{\rho}\lambda^{\sigma})A^{\rho}{}_{\tau}\frac{\delta}{\delta A^{\sigma}{}_{\tau}} \right) \tag{4.40}$$

It follows that :

$$\begin{aligned}(i_{\mathcal{P}}(\lambda)\hat{\Gamma})^{\sigma}{}_{\tau} &\equiv (i_{\mathcal{P}}(\underline{\lambda})\hat{\Gamma})^{\sigma}{}_{\tau} = (A^{-1})^{\sigma}{}_{\sigma'}\left(\frac{1}{2}g^{\sigma'\rho}\left({}^{LC}D_{\rho}\bar{\lambda}_{\tau'} - {}^{LC}D_{\tau'}\bar{\lambda}_{\rho}\right)\right)A^{\tau'}{}_{\tau} \\ &\underset{\text{Def}}{=} \left[A^{-1}\left(\frac{1}{2}{}^{LC}D \wedge \bar{\lambda}\right)A\right]^{\sigma}_{\tau}\end{aligned} \tag{4.41}$$

and :

$$l_{\mathcal{P}}(\lambda)\hat{\Gamma} \equiv l_{\mathcal{P}}(\underline{\lambda})\hat{\Gamma} = 0 \tag{4.42}$$

where $\bar{\lambda}_{\rho} = g_{\rho\sigma}\lambda^{\sigma}$ and ${}^{LC}D$ is the covariant derivative associated with ${}^{LC}\Gamma$. Hence, $\hat{\Gamma}$ is the $\mathcal{D}(\Sigma)$-invariant $Gl(2,\mathbb{R})$-connection on $\mathcal{P}F_{\{g\}}(\Sigma)$ we are looking for, and its curvature reads :

$$\begin{aligned}\hat{R}(\hat{\Gamma}) &= d_{\mathcal{P}}\hat{\Gamma} + \frac{1}{2}[\hat{\Gamma},\hat{\Gamma}] = A^{-1}\left((\delta + d_{\Sigma})\Gamma + \frac{1}{2}[\Gamma,\Gamma]\right)A \\ &= A^{-1}R(\Gamma)A\end{aligned} \tag{4.43}$$

with :

$$\begin{aligned}R(\Gamma) &= (\delta + d_{\Sigma})\left({}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right) + \frac{1}{2}\left[{}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g, {}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right] \\ &= {}^{LC}R + {}^{LC}D\left(\frac{1}{2}g^{-1}\delta g\right) + \delta\,{}^{LC}\Gamma - \left(\frac{1}{2}g^{-1}\delta g\right)^{2}\end{aligned} \tag{4.44}$$

LC referring to the Levi-Civita part of the connection.

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[13] We wish to thank M. Dubois Violette for communicating the above construction of $\hat{\Gamma}$.

Let $\tilde{\omega}$ be a $\mathcal{V}(\Sigma)$-valued connection on another copy $\widetilde{\mathcal{M}}(\Sigma)$ of $\mathcal{M}(\Sigma)$, and $\tilde{\Omega}$ its curvature. In the intermediate scheme, the equivariant curvature of $\hat{\hat{\Gamma}}$ (the pull-back of $\hat{\Gamma}$ on $\widetilde{\mathcal{M}}(\Sigma) \times \mathcal{G}F_{\{g\}}(\Sigma)$) is given by :

$$\hat{\hat{R}}^{eq}_{int}(\hat{\Gamma},\tilde{\omega},\tilde{\Omega}) = \left(\tilde{\delta} + d_{\mathcal{G}} + l_{\mathcal{G}}(\tilde{\omega}) - i_{\mathcal{G}}(\tilde{\Omega})\right)\hat{\hat{\Gamma}} + \frac{1}{2}\left[\hat{\hat{\Gamma}},\hat{\hat{\Gamma}}\right]$$
$$= \hat{\hat{R}}(\hat{\Gamma}) - i_{\mathcal{G}}(\tilde{\Omega})\hat{\hat{\Gamma}} = A^{-1}\left(\hat{R}(\hat{\Gamma}) - \frac{1}{2}{}^{LC}D \wedge \bar{\tilde{\Omega}}\right)A \tag{4.45}$$
$$\equiv A^{-1}\hat{R}^{eq}_{int}A$$

with $\hat{R}(\hat{\Gamma})$ the pull-back of $R(\Gamma)$ on $\widetilde{\mathcal{M}}(\Sigma) \times \mathcal{G}F_{\{g\}}(\Sigma)$. In view of the particular form taken by $\hat{\hat{R}}^{eq}_{int}$, and since the invariants we are looking for are constructed in terms of curvatures, one can forget about the Gl(2,$\mathbb{R}$) fibration (represented by the A^{-1} and A terms) since one deals with forms globally defined on Σ, such as $\hat{R}(\hat{\Gamma})$ and $\hat{R}^{eq}_{int}$.

As before, $\mathcal{I}$, $\mathcal{L}$ and i_Σ, l_Σ refer to the action of $\mathcal{D}(\Sigma)$ on $\mathcal{M}(\Sigma)$ and Σ respectively. In the Weil scheme, the equivariant curvature is given by :

$$R^{eq}_{W}(\hat{\Gamma},\tilde{\omega},\tilde{\Omega}) = \exp\{(\mathcal{I} + i_\Sigma)(\tilde{\omega})\}R^{eq}_{int}(\hat{\Gamma},\tilde{\omega},\tilde{\Omega})$$
$$= \hat{R}(\hat{\Gamma}) + (\mathcal{I} + i_\Sigma)(\tilde{\omega})\hat{R}(\hat{\Gamma}) + \frac{(\mathcal{I} + i_\Sigma)(\tilde{\omega})(\mathcal{I} + i_\Sigma)(\tilde{\omega})}{2}\hat{R}(\hat{\Gamma}) \tag{4.46}$$
$$-\frac{1}{2}{}^{LC}D \wedge \bar{\tilde{\Omega}}$$

The equivariant Euler class, which plays the role of the invariant polynomial $I^{eq}_{H,W}$ of section II, is defined by :

$$\mathcal{E}^{eq}_{W} = \frac{\varepsilon^{\mu\rho}}{\sqrt{g}} g_{\rho\nu} \left(R^{eq}_{W}(\hat{\Gamma})\right)^{\nu}_{\mu} \tag{4.47}$$

Once equation (4.46) has been made explicit (using equations (G.8) and (G.11) of Appendix G), $\mathcal{E}^{eq}_{W}$ can be written as :

$$\mathcal{E}^{eq}_{W} = \frac{\varepsilon^{\mu\rho}}{\sqrt{g}} g_{\rho\nu}\left({}^{LC}R - i_\Sigma(\tilde{\omega}){}^{LC}R + \frac{i_\Sigma(\tilde{\omega})i_\Sigma(\tilde{\omega})}{2}{}^{LC}R\right.$$
$$\left. + \frac{1}{2}{}^{LC}D \wedge \bar{\tilde{\gamma}} - \frac{1}{2}i_\Sigma(\tilde{\omega}){}^{LC}D \wedge \bar{\tilde{\gamma}} - \frac{1}{4}\tilde{\psi}\tilde{\psi}\right)^{\nu}_{\mu} \tag{4.48}$$
$$+ \frac{1}{2}\frac{\varepsilon^{\mu\nu}}{\sqrt{g}}\left({}^{LC}D_\mu \bar{\tilde{\Omega}}_\nu - {}^{LC}D_\nu \bar{\tilde{\Omega}}_\mu\right)$$

with :

$$\bar{\tilde{\gamma}}_\mu = \tilde{\gamma}_{\lambda\mu}dx^\lambda = \left(\delta g_{\lambda\mu} - l_\Sigma(\tilde{\omega})g_{\lambda\mu}\right)dx^\lambda = \left(\delta \bar{g}_\mu - l_\Sigma(\tilde{\omega})\bar{g}_\mu\right)$$
$$\tilde{\psi}^{\nu}_{\mu} = g^{\nu\lambda}\left(\delta g_{\lambda\mu} - l_\Sigma(\tilde{\omega})g_{\lambda\mu}\right) = g^{\nu\lambda}\tilde{\gamma}_{\lambda\mu} = \left(g^{-1}\tilde{\gamma}\right)^{\nu}_{\mu} \tag{4.49}$$

and ${}^{LC}D \wedge \bar{\bar{\gamma}}$ as defined in Appendix G.

Using the basicity property :

$$\left(\tilde{\mathcal{I}} + \mathcal{I} + i_\Sigma\right)(\lambda)\bar{\bar{\gamma}} = 0 \tag{4.50}$$

one easily checks :

$$\left(\tilde{\mathcal{I}} + \mathcal{I} + i_\Sigma\right)(\lambda)\mathcal{E}_W^{eq} = 0 \tag{4.51}$$

Now, because $\frac{\varepsilon^{\mu\rho}}{\sqrt{g}} g_{\rho\nu}$ is covariant constant (see Appendix G) and because of the Bianchi identity for $\hat{R}_W^{eq}$:

$$\left(\tilde{\delta} + \delta + d_\Sigma\right)\mathcal{E}_W^{eq} = 0 \tag{4.52}$$

And consequently :

$$\left(\tilde{\mathcal{L}} + \mathcal{L} + l_\Sigma\right)(\lambda)\mathcal{E}_W^{eq} = 0 \tag{4.53}$$

(Compare with [BCI94]).

The last step is to apply Cartan's theorem 3 : one has to replace $\tilde{\omega}$ by a $\mathcal{D}(\Sigma)$-connection on $\mathcal{M}(\Sigma) \times \Sigma$ [BGV91]. An obvious solution is given by a $\mathcal{D}(\Sigma)$-connection ω on $\mathcal{M}(\Sigma)$, so that the form of equation (4.46) is maintained with the replacement :

$$\begin{cases} \tilde{\omega} \longrightarrow \omega \\ \tilde{\Omega} \longrightarrow \Omega = \delta\omega + \frac{1}{2}[\omega, \omega] \\ \tilde{\gamma} \longrightarrow \gamma = \delta g - \mathcal{L}(\omega)g \\ \tilde{\psi} \longrightarrow \psi = g^{-1}\gamma = g^{-1}(\delta g - \mathcal{L}(\omega)g) \end{cases} \tag{4.54}$$

Furthermore, since $\mathcal{M}(\Sigma)$ is a principal bundle over $\mathcal{CM}(\Sigma)$ with structure group the Weyl group, one may choose ω a $\mathcal{D}(\Sigma)$-connection on $\mathcal{CM}(\Sigma)$. It is very likely that the conformal picture is recovered by choosing the section provided by the (negative) constant scalar curvature condition (4.20), but, at the time of writing this has not been explicitely checked.

V) Concluding Remarks.

Cohomological field theories are gauge theories of an exotic type. The question of the definition of the observables is crucial. The definition has to be such that "physics" - e.g. correlation functions of observables - be gauge independent, i.e. be independent of the parameters or external fields needed to define a perturbatively computable Lagrangian, namely, a Lagrangian whose quadratic part is non degenerate. The fact that the equivariant cohomology classes defined in the previous section do not depend on the various connections used to define them suggests that they be computed by "averaging out" over these connections. This is formally realizable in terms of functional integrals. The well known difficulties in defining those result in ambiguities which are well understood at the perturbative level provided that they are properly constrained algebraically. The equivariant cohomology framework exhibited here both at the level of fields and at the level of observables is a compelling ingredient whose necessity has often not been fully appreciated.

The construction reviewed here may not give all observables. Note that those which have been constructed here emerge as integrated local expressions in the fields. Whereas these are basic cohomology classes [OSB89], it is not clear a priori which cohomology classes the local densities represent. Another delicacy in the definition of observables has to do with global aspects which are known to be crucial [DK90, Mu83].

The corresponding mathematics should in each case guarantee that one is not describing a trivial cohomology class via complicated formulae. A final remark is in order since it provides a bridge with the origin of cohomological theories. The introduction of the connection ω -the Faddeev-Popov ghost introduced by L. Baulieu and I.M. Singer-, rather natural from the geometrical point of view may however look somewhat redundant, since only curvatures are involved in the final formulae. A similar impression may prevail from the field point of view. It has however several advantages, one being the necessity to introduce the operations $\tilde{\mathcal{I}}$, $\mathcal{I}$ and i. The devoted reader will easily establish the bridge between equivariant cohomology and twisted $N = 2$ supersymmetry.

ACKNOWLEDGMENTS

We wish to thank M. Dubois Violette for communicating the construction of the connection Γ used in the Riemannian route. We also wish to thank J. Kalkman for interesting exchanges of ideas.

Our thanks also go to the organizers of the 1st Caribbean Spring School of Mathematics and Theoretical Physics for giving us the opportunity to present these notes.

APPENDIX A

Some basic facts and conventions about associated bundles.

Let $\Pi(B,\mathcal{G})$ be a smooth principal fiber bundle with a right action of the Lie group $\mathcal{G}$. The right-transformed by $\gamma\in\mathcal{G}$ of a point $\pi\in\Pi$ is written π^{γ}. An infinitesimal transformation, represented by $\lambda\in\mathrm{Lie}\mathcal{G}$, gives rise to a so-called fundamental vector field λ_{Π} on Π. The operations on Π are denoted by :

$$i_{\Pi}(\lambda) \underset{\mathrm{Def}}{=} i_{\Pi}(\lambda_{\Pi}) \quad \text{and} \quad l_{\Pi}(\lambda) \underset{\mathrm{Def}}{=} l_{\Pi}(\lambda_{\Pi}) \tag{A.1}$$

while the differential is d_{Π}.

Now, let us consider a smooth manifold $\mathcal{F}$ with a left-action of $\mathcal{G}$ on it. The left-transformed by $\gamma\in\mathcal{G}$ of a point $f\in\mathcal{F}$ is written $\gamma(f)$. Here again, to any $\lambda\in\mathrm{Lie}\mathcal{G}$ there corresponds a fundamental vector field $\lambda_{\mathcal{F}}$ on $\mathcal{F}$, and the operations on $\mathcal{F}$ are also written :

$$i_{\mathcal{F}}(\lambda) \underset{\mathrm{Def}}{=} i_{\mathcal{F}}(\lambda_{\mathcal{F}}) \quad \text{and} \quad l_{\mathcal{F}}(\lambda) \underset{\mathrm{Def}}{=} l_{\mathcal{F}}(\lambda_{\mathcal{F}}) \tag{A.2}$$

Finally, we consider the right-action on the smooth product manifold $\Pi\times\mathcal{F}$ defined by :

$$(\pi,f)^{\gamma} = (\pi^{\gamma},\gamma^{-1}(f)) \tag{A.3}$$

for $\gamma\in\mathcal{G}$. Hence, for an infinitesimal transformation $\lambda\in\mathrm{Lie}\mathcal{G}$, the corresponding fundamental vector field on $\Pi\times\mathcal{F}$ will be :

$$\lambda_{\Pi\times\mathcal{F}} = \lambda_{\Pi} - \lambda_{\mathcal{F}} \tag{A.4}$$

where λ_{Π} and $\lambda_{\mathcal{F}}$ are the fundamental vector fields associated to the original right and left actions of λ on Π and $\mathcal{F}$ respectively.

One can show that $\Pi\times\mathcal{F}$ with this right-action of $\mathcal{G}$ can be made into a smooth principal bundle with structure group $\mathcal{G}$, whose base space, denoted by $\Pi\times_{\mathcal{G}}\mathcal{F}$, is itself a smooth fiber bundle (not principal) over B, with typical fiber $\mathcal{F}$ [GHV73].

Whereas the differentials on $\Pi\times\mathcal{F}$ are $d_{\Pi\times\mathcal{F}} = d_{\Pi} + d_{\mathcal{F}}$, the operations become :

$$\begin{aligned} i_{\Pi\times\mathcal{F}}(\lambda) &\underset{\mathrm{Def}}{=} i_{\Pi\times\mathcal{F}}(\lambda_{\Pi\times\mathcal{F}}) = (i_{\Pi}+i_{\mathcal{F}})(\lambda_{\Pi\times\mathcal{F}}) \\ &= i_{\Pi}(\lambda_{\Pi}) - i_{\mathcal{F}}(\lambda_{\mathcal{F}}) = i_{\Pi}(\lambda) - i_{\mathcal{F}}(\lambda) \end{aligned} \tag{A.5}$$

and the same for $l_{\Pi\times\mathcal{F}}$. Note that $\lambda_{\mathcal{F}}$ is the fundamental vector field associated with the original action of $\mathcal{G}$ on $\mathcal{F}$ (here a left action) which explains the relative sign in the last term of (A.5).

APPENDIX B

Cartan's Theorem 3 [C50].

Let $P(\omega,\Omega)$ represent an equivariant cohomology class of $\Omega^*(\mathcal{M})\otimes\mathcal{W}_\Pi$. Let θ be a connection in $\Omega^*(\mathcal{M})$, i.e. a Lie$\mathcal{G}$ valued one form on $\mathcal{M}$ such that :

$$i_{\mathcal{M}}(\lambda)\omega=\lambda \qquad \text{and} \qquad l_{\mathcal{M}}(\lambda)\omega=-[\lambda\,,\omega] \tag{B.1}$$

for any $\lambda\in$ Lie$\mathcal{G}$, and Θ its curvature.

Let :

$$\begin{aligned}\omega_t &= t\omega+(1-t)\theta\\ \Omega_t &= \left(d_\Pi+d_{\mathcal{M}}\right)\Omega+\frac{1}{2}[\omega_t,\omega_t]\end{aligned}\qquad 0\le t\le 1 \tag{B.2}$$

It is easy to check that :

$$\begin{aligned}\left(i_\Pi(\lambda)+i_{\mathcal{M}}(\lambda)\right)\omega_t &= \lambda\\ \left(l_\Pi(\lambda)+l_{\mathcal{M}}(\lambda)\right)\omega_t &= -[\lambda,\omega_t]\end{aligned} \tag{B.3}$$

For all polynomials P in ω and Ω, define [MSZ85] the derivation k by :

$$(kP)(\omega,\Omega)=\int_{[0,1]} k_t P(\omega_t,\Omega_t) \tag{B.4}$$

with :

$$\begin{aligned}k_t\omega_t &= 0\\ k_t\Omega_t &= d_t\omega_t\end{aligned} \tag{B.5}$$

such that :

$$k_t\left(d_\Pi+d_{\mathcal{M}}\right)-\left(d_\Pi+d_{\mathcal{M}}\right)k_t=d_t \tag{B.6}$$

and :

$$\left[i_\Pi(\lambda)+i_{\mathcal{M}}(\lambda),k_t\right]=\left[l_\Pi(\lambda)+l_{\mathcal{M}}(\lambda),k_t\right]=0 \tag{B.7}$$

Thus :

$$\left[i_\Pi(\lambda)+i_{\mathcal{M}}(\lambda),k\right]=\left[l_\Pi(\lambda)+l_{\mathcal{M}}(\lambda),k\right]=0 \tag{B.8}$$

and :

$$\begin{aligned}P(\omega,\Omega)-P(\theta,\Theta) &= \int_{[0,1]} d_t P(\omega_t,\Omega_t)\\ &= \int_{[0,1]} k_t\left[\left(d_\Pi+d_{\mathcal{M}}\right)P\right](\omega_t,\Omega_t)-\int_{[0,1]}\left(d_\Pi+d_{\mathcal{M}}\right)k_t P(\omega_t,\Omega_t)\\ &= k\left(d_\Pi+d_{\mathcal{M}}\right)P-\left(d_\Pi+d_{\mathcal{M}}\right)kP\end{aligned} \tag{B.9}$$

Now, since P represents an equivariant cohomology class :

$$\left(d_\Pi + d_{\mathcal{M}}\right)P = \left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)P = \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)P = 0 \tag{B.10}$$

Then :

$$P(\omega,\Omega) = P(\theta,\Theta) - \left(d_\Pi + d_{\mathcal{M}}\right)kP \tag{B.11}$$

Because of the commutativity of k with $i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)$ and $l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)$,

$$\left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)kP = \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)kP = 0 \tag{B.12}$$

It follows that :

$$\begin{gathered}
\left(d_\Pi + d_{\mathcal{M}}\right)P(\theta,\Theta) = d_{\mathcal{M}}P(\theta,\Theta) = 0 \\
\left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)P(\theta,\Theta) = i_{\mathcal{M}}(\lambda)P(\theta,\Theta) = 0 \\
\left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)P(\theta,\Theta) = l_{\mathcal{M}}(\lambda)P(\theta,\Theta) = 0
\end{gathered} \tag{B.13}$$

$P(\theta,\Theta)$ is an element of the basic cohomology of $\mathcal{M}$, cohomologous to $P(\omega,\Omega)$ within the equivariant cochain algebra and this correspondence is obviously defined at the level of cohomology. The same calculation shows that given two connections θ_1, θ_2 on $\Omega^*(\mathcal{M})$, $P(\theta_1,\Theta_1)$ and $P(\theta_2,\Theta_2)$ are cohomologous within the basic cohomology of $\Omega^*(\mathcal{M})$.

APPENDIX C

An Alternative construction of equivariant cohomology classes of $\mathcal{A}$.

The construction given in the text may look unnecessarily complicated. In the present case where $\mathcal{P}(\mathrm{M},\mathrm{G}) = \mathcal{A} \times \mathrm{P}(\Sigma,\mathrm{G})$ and $\mathcal{G}$ acts separately on $\mathcal{A}$ and $\mathrm{P}(\Sigma,\mathrm{G})$, the situation can be simplified as follows : construct equivariant cohomology classes of $\mathrm{P}(\Sigma,\mathrm{G})$ using a connection ω on $\mathcal{A}$, with curvature Ω. Now, choose $\hat{\Gamma} = \hat{a}$:

$$\mathcal{I}(\lambda)\hat{\Gamma} = 0 \tag{C.1}$$

$$\left(\mathcal{L} + l_{\mathrm{P}}\right)(\lambda)\hat{\Gamma} = 0 \tag{C.2}$$

The equivariant curvature of $\hat{a}$ in the intermediate scheme is given by :

$$\mathrm{F}^{\mathrm{eq}}_{\mathrm{int}}(\hat{a}) = \left(\delta + \mathrm{d}_{\mathrm{P}} + l_{\mathrm{P}}(\tilde{\omega}) - i_{\mathrm{P}}(\tilde{\Omega})\right)\hat{a} + \frac{1}{2}[\hat{a},\hat{a}] \tag{C.3}$$

One easily finds that this is the same as the equivariant curvature in the Weil scheme :

$$\mathrm{F}^{\mathrm{eq}}_{\mathrm{W}}(\hat{a}) = \left(\delta + \mathrm{d}_{\mathrm{P}}\right)\left(\hat{a} + i_{\mathrm{P}}(\omega)\hat{a}\right) + \frac{1}{2}\left[\left(\hat{a} + i_{\mathrm{P}}(\omega)\hat{a}\right),\left(\hat{a} + i_{\mathrm{P}}(\omega)\hat{a}\right)\right] \tag{C.4}$$

and they both coincide with $\mathrm{F}^{\mathrm{eq}}_{\mathrm{W}}(\hat{a},\omega,\Omega)$ of equation (3.17).

APPENDIX D
An action of $\mathcal{D}(\Sigma)$ on $T^{(1,0)}_{\{\mu\}}(\Sigma)$ and $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$.

If (x,μ,V_x) is a point of $T^{(1,0)}_{\{\mu\}}(\Sigma)$, we choose the following right-action of $\mathcal{D}(\Sigma)$:

$$\forall\varphi\in\mathcal{D}(\Sigma)\ ,\ (\mu,x,V_x)^{\varphi} = (\mu^{\varphi},\ \varphi^{-1}(x),\ d_x\varphi^{-1}(V_x)) \tag{D.1}$$

where $d_x\varphi^{-1} : T_x\Sigma \to T_{\varphi^{-1}(x)}\Sigma$ is the differential of $\varphi\in\mathcal{D}(\Sigma)$ at $x\in\Sigma$:

$$\forall V_x\in T_x\Sigma\ ,\ \forall f\in\mathcal{C}^{\infty}(\Sigma)\ ,\ d_x\varphi^{-1}V_x(f) = V_x(f\circ\varphi^{-1}) \tag{D.2}$$

and μ^{φ} is the element of $\mathcal{B}(\Sigma)$ with components :

$$\left(\mu^{\varphi}\right)^{w}{}_{\overline{w}} = \frac{\left(\partial_{\overline{w}}\varphi^{w}\right)+\left(\partial_{\overline{w}}\varphi^{\overline{w}}\right)\left(\mu^{z}{}_{\overline{z}}\circ\varphi\right)}{\left(\partial_{w}\varphi^{w}\right)+\left(\partial_{w}\varphi^{\overline{w}}\right)\left(\mu^{z}{}_{\overline{z}}\circ\varphi\right)} \tag{D.3}$$

where $(z,\overline{z})$ and $(w,\overline{w})$ are coordinates at x and $\varphi^{-1}(x)$ respectively, and $(\varphi^{w},\varphi^{\overline{w}})$ the local representative of φ with respect to $(z,\overline{z})$ and $(w,\overline{w})$. Equation (D.3) defines the natural right-action of $\mathcal{D}(\Sigma)$ on $\mathcal{B}(\Sigma)$.

From now on we shall consider infinitesimal diffeomorphisms represented by vector fields $\lambda = \lambda^{z}\partial_{z} + \lambda^{\overline{z}}\partial_{\overline{z}} \in\mathcal{V}(\Sigma)$:

$$z(x)\to z(\varphi(x)) = z(x) + \lambda^{z}(x)\ ,\ \text{and c.c.} \tag{D.4}$$

we get :

$$\mu^{\varphi} = \mu + \delta_{\lambda}\mu = \mu + \overline{D}_{\mu}\Lambda_{\mu} \tag{D.5}$$

with the notations of equation (4.5).

Now, at $x\in\Sigma$ with coordinates $(z,\overline{z})$ we can solve the Beltrami equation :

$$\left(\partial_{\overline{z}} - \mu^{z}{}_{\overline{z}}\partial_{z}\right)Z_{\mu} = 0$$

thus obtaining new complex coordinates $(Z_{\mu},\overline{Z}_{\mu})$ at x. The component $V^{Z_{\mu}}(x)$ of V_x with respect to the natural frame $\partial_{Z_{\mu}}$ associated to $(Z_{\mu},\overline{Z}_{\mu})$ are chosen to be coordinates of V_x. Similarly, at $\varphi^{-1}(x)\in\Sigma$ with coordinates $(w,\overline{w})$ we solve the Beltrami equation for $\mu+\delta_{\lambda}\mu$ and obtain complex coordinates $(Z_{\mu+\delta_{\lambda}\mu},\overline{Z}_{\mu+\delta_{\lambda}\mu})$ at $\varphi^{-1}(x)$, and the coordinates of $V_{\varphi^{-1}(x)}$ are the component $V^{Z_{\mu+\delta_{\lambda}\mu}}(\varphi^{-1}(x))$ of $V_{\varphi^{-1}(x)}$ with respect to the natural frame $\partial_{Z_{\mu+\delta_{\lambda}\mu}}$. This is how we define a complex analytic structure on $T^{(1,0)}_{\{\mu\}}(\Sigma)$. Hence, at the coordinates level, the infinitesimal action of $\mathcal{D}(\Sigma)$ is :

$$\left(\mu,Z_{\mu}(x),V^{Z_{\mu}}(x)\right) \longrightarrow \left(\mu+\delta_{\lambda}\mu, Z_{\mu+\delta_{\lambda}\mu}(\varphi^{-1}(x)), V^{Z_{\mu+\delta_{\lambda}\mu}}(\varphi^{-1}(x))\right) \tag{D.6}$$

Combining the Beltrami equations which define the coordinates $(Z_{\mu},\overline{Z}_{\mu})$ and $(Z_{\mu+\delta_{\lambda}\mu},\overline{Z}_{\mu+\delta_{\lambda}\mu})$ with equation (D.3), one can show that $Z_{\mu+\delta_{\lambda}\mu}(\varphi^{-1}(x))$ is an invertible holomorphic function of $Z_\mu(x)$ [L]. Hence, if we introduce the complex function $Z(z,\bar z,\mu) = Z_\mu(z,\bar z)$, we get the following, correct to first order :

$$Z_{\mu+\delta_\lambda\mu}(\varphi^{-1}(x)) = Z(w,\bar w,\mu^\varphi\,\delta_\lambda\mu) = Z(z-\lambda^z,\bar z-\lambda^{\bar z},\mu+\delta_\lambda\mu) \tag{D.7}$$

$$= Z(z,\bar z,\mu) + (\bar D_\mu\Lambda_\mu)\left.\frac{\delta Z}{\delta\mu}\right|_{z,\bar z}(z,\bar z,\mu) - \lambda^z\left.\frac{\partial Z}{\partial z}\right|_{\mu,\bar\mu}(z,\bar z,\mu) - \lambda^{\bar z}\left.\frac{\partial Z}{\partial \bar z}\right|_{\mu,\bar\mu}(z,\bar z,\mu)$$

Accordingly, for the coordinates $V^{Z_\mu}(x)$ of $V(x)$ we get :

$$V^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x)) = \left(V^{Z_\mu}\circ\varphi\right)(\varphi^{-1}(x))\frac{\partial Z_{\mu+\delta_\lambda\mu}}{\partial Z_\mu}(Z_\mu(x)) \tag{D.8}$$

Finally, it is straightforward to see that on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$:

$$E^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x)) = \left(E^{Z_\mu}\circ\varphi\right)(\varphi^{-1}(x))\frac{\partial Z_{\mu+\delta_\lambda\mu}}{\partial Z_\mu}(Z_\mu(x)) \tag{D.9}$$

with $E^{Z_\mu}(x)$ and $E^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x))$ coordinates for $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ (see main text).

Finally, to get the fundamental vector field $\underline{\lambda}$ of $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ associated to the action of $\lambda\in\mathcal{V}(\Sigma)$ with respect to the complex structure of $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$, we need to relate the dérivatives with respect to z, $\bar z$ and μ to those with respect to Z_μ, $\bar Z_\mu$ and μ, namely [KLS91] :

$$\left.\frac{\delta}{\delta\mu}\right|_{z,\bar z} = \left.\frac{\delta}{\delta\mu}\right|_{Z,\bar Z} + \left(\frac{\delta Z}{\delta\mu}\right)_{z,\bar z}\left.\frac{\partial}{\partial Z}\right|_{\mu,\bar\mu} \quad \text{and c.c} \tag{D.10}$$

so that :

$$\begin{aligned}\underline{\lambda} &= (\bar D_\mu\Lambda_\mu)\left.\frac{\delta}{\delta\mu}\right|_{Z,\bar Z} + \left((\bar D_\mu\Lambda_\mu)\left(\frac{\delta Z}{\delta\mu}\right)_{z,\bar z} - \Lambda^Z_\mu\right)\frac{\partial}{\partial Z}\\ &\quad + E^Z\frac{\partial}{\partial Z}\left((\bar D_\mu\Lambda_\mu)\left(\frac{\delta Z}{\delta\mu}\right)_{z,\bar z} - \Lambda^Z_\mu\right)\frac{\delta}{\delta E^Z} + \text{c.c.}\\ &= \underline{\lambda}^h + \underline{\lambda}^{\bar h}\end{aligned} \tag{D.11}$$

with :

$$\Lambda^Z_\mu = \left(\frac{\partial Z}{\partial z}\right)\Lambda^z_\mu = \left(\frac{\partial Z}{\partial z}\right)\left(\lambda^z + \mu^z_{\bar z}\lambda^{\bar z}\right) = \lambda^Z \tag{D.12}$$

APPENDIX E

Calculation of the equivariant curvature of $\hat{\Gamma}$.

$$\begin{aligned} R^{eq}_{W}(\hat{\Gamma}) &= \left(\tilde{\delta} + \mathcal{D} + \overline{\mathcal{D}}\right)\left(\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + D \ln E^{Z_\mu} + I(\tilde{\omega})\Gamma\right) \\ &= \left(\tilde{\delta} + \mathcal{D} + \overline{\mathcal{D}}\right)\left(\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + I(\tilde{\omega})\Gamma\right) \\ &= \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} - I(\tilde{\Omega})\Gamma + \frac{1}{2} I\left(\left[\tilde{\omega}, \tilde{\omega}\right]\right)\Gamma + I(\tilde{\omega})\overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \end{aligned} \tag{E.1}$$

where we have used the invariance of Γ. The third term is :

$$\begin{aligned} \frac{1}{2} I\left(\left[\tilde{\omega}, \tilde{\omega}\right]\right)\Gamma &= \frac{1}{2}\left[L(\tilde{\omega}), I(\tilde{\omega})\right]\Gamma \\ &= -\frac{1}{2} L(\tilde{\omega}) I(\tilde{\omega})\hat{\Gamma} \\ &= -\frac{1}{2} L(\tilde{\omega}) I(\tilde{\omega})\left(\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + D \ln E^{Z_\mu}\right) \\ &= \frac{1}{2} I(\tilde{\omega}) D I(\tilde{\omega}) \mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + \frac{1}{2} I(\tilde{\omega}) D I(\tilde{\omega}) D \ln E^{Z_\mu} \\ &= \frac{1}{2} I(\tilde{\omega}) I(\tilde{\omega}) \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + \frac{1}{2} I(\tilde{\omega}) L(\tilde{\omega}) \mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \\ &\quad + \frac{1}{2} I(\tilde{\omega}) L(\tilde{\omega}) D \ln E^{Z_\mu} \\ &= \frac{1}{2} I(\tilde{\omega}) I(\tilde{\omega}) \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \end{aligned} \tag{E.2}$$

Finally :

$$\begin{aligned} R^{eq}_{W}(\hat{\Gamma}) = {} & \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + I(\tilde{\omega})\overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + \frac{1}{2} I(\tilde{\omega}) I(\tilde{\omega}) \overline{\mathcal{D}}\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} \\ & - I(\tilde{\Omega})\left(\mathcal{D} \ln \rho_{Z_\mu \overline{Z}_\mu} + D \ln E^{Z_\mu}\right) \end{aligned} \tag{E.3}$$

Thus, as expected, $R^{eq}_{W}(\hat{\Gamma})$ is of type (1,1) for the natural complex structure of $\mathcal{B}(\Sigma) \times \Sigma$.

APPENDIX F
The action of $\mathcal{D}(\Sigma)$ on $\mathcal{P}F_{\{g\}}(\Sigma)$.

Let (x, E_x) be a point of F(Σ) the frame bundle of Σ, where, by definition, E_x is a frame (a basis) of $T_x\Sigma$: $E_x = (E_{x\,i})$. One defines coordinates for E_x as follows. One selects coordinates (x^k) for $x \in \Sigma$ and denoted by (∂_k) the natural basis of $T_x\Sigma$ defined by these coordinates : $\partial_k = \partial/\partial x^k$. Then, the coordinates of E_x are the components $A^j{}_k$ of the decomposition of E_x with respect to the natural basis (∂_k) :

$$E_{x\,k} = A^j{}_k \partial_j \tag{F.1}$$

Each vector $E_{x\,i}$ belongs to $T_x\Sigma$. As explained in Appendix C, there is a natural (left) action of $\varphi \in \mathcal{D}(\Sigma)$ on $T_x\Sigma$, given by $d_x\varphi : T_x\Sigma \to T_{\varphi(x)}\Sigma$ the differential of φ at x :

$$\forall V_x \in T_x\Sigma \ , \ \forall f \in \mathcal{C}^\infty(\Sigma) \ , \ d_x\varphi V_x(f) = V_x(f \circ \varphi) \tag{F.2}$$

In terms of coordinates, this gives :

$$\left(d_x\varphi(V_x)\right)^i = V_x^m\left(\partial_m \varphi^i\right) \tag{F.3}$$

where φ^i means the local representative of φ with respect to the coordinates (x^k) : $\varphi(x) = y = (y^i) = (\varphi^i(x^k))$. Applying equation (F.3) to the frame vectors $E_{x\,j}$, one gets :

$$A'^i{}_j = A^m{}_j\left(\partial_m \varphi^i\right) \tag{F.4}$$

and at the infinitesimal level, for $\lambda \in \mathcal{V}(\Sigma)$:

$$A'^i{}_j = A^m{}_j\left(\partial_m(x^i + \lambda^i)\right) = A^m{}_j\left(\delta^i_m + \partial_m\lambda^i\right) = A^i{}_j + (\partial_m\lambda^i)A^m{}_j \tag{F.5}$$

where $A'^i{}_j$ are the coordinates of the transformed frame at $\varphi(x)$.

Finally, at the coordinates level, the natural left-action of $\lambda \in \mathcal{V}(\Sigma)$ on F(Σ) is :

$$\left((x^k),(A^i{}_j)\right) \longrightarrow \left((x^k + \lambda^k),(A^i{}_j + (\partial_m\lambda^i)A^m{}_j)\right) \tag{F.6}$$

Hence, the fundamental vector field on F(Σ) defined by the action of $\lambda \in \mathcal{V}(\Sigma)$ reads :

$$\lambda^k\partial_k + A^m{}_j(\partial_m\lambda^i)\frac{\delta}{\delta A^i{}_j} \tag{F.7}$$

Now, if we consider $\mathcal{P}F_{\{g\}}(\Sigma)$ instead of F(Σ), we need a right-action of $\mathcal{D}(\Sigma)$ on $\mathcal{P}F_{\{g\}}(\Sigma)$ and thus a right-action on F(Σ) :

$$\left((g_{\mu\nu}),(x^\alpha),(A^\sigma{}_\tau)\right) \longrightarrow \left((g_{\mu\nu} + l_\Sigma(\lambda)g_{\mu\nu}),(x^\alpha - \lambda^\alpha),(A^\sigma{}_\tau - (\partial_\rho\lambda^\sigma)A^\rho{}_\tau)\right) \tag{F.8}$$

at the coordinates level, for $\lambda \in \mathcal{V}(\Sigma)$, and the corresponding fundamental vector field is given by :

$$\underline{\lambda} = \left((l_\Sigma(\lambda) g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}} - \lambda^\alpha \partial_\alpha - (\partial_\rho \lambda^\sigma) A^\rho{}_\tau \frac{\delta}{\delta A^\sigma{}_\tau} \right) \tag{F.9}$$

In particular :

$$\begin{aligned} l_{\mathcal{G}}(\lambda) g_{\lambda\gamma} \equiv l_{\mathcal{G}}(\underline{\lambda}) g_{\lambda\gamma} &= \mathcal{L}((l_\Sigma(\lambda) g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}}) g_{\lambda\gamma} + l_\Sigma(-\lambda) g_{\lambda\gamma} \\ &= l_\Sigma(\lambda) g_{\mu\nu} \delta^\mu_\lambda \delta^\nu_\gamma - l_\Sigma(\lambda) g_{\lambda\gamma} = 0 \end{aligned} \tag{F.10}$$

so that :

$$l_{\mathcal{G}}(\lambda) g = 0 \tag{F.10}$$

for any $g \in \mathcal{M}(\Sigma)$.

APPENDIX G

Calculation of the equivariant curvature $\hat{R}_W^{eq}$

and of the corresponding Euler class $\mathcal{E}_W^{eq}$.

Recall :

$$\Gamma^\lambda{}_\mu = {}^{LC}\Gamma^\lambda{}_\mu + \frac{1}{2}(g^{-1}\delta g)^\lambda{}_\mu \equiv {}^{LC}\Gamma^\lambda{}_\mu + \frac{1}{2}g^{\lambda\nu}\delta g_{\nu\mu} \tag{G.1}$$

where ${}^{LC}\Gamma$ is the Levi-Civita connection :

$$^{LC}\Gamma^\lambda{}_\mu = \frac{1}{2}g^{\lambda\nu}\left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}\right)dx^\rho \tag{G.2}$$

Now :

$$\begin{aligned} R(\Gamma) &= (\delta + d_\Sigma)\left({}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right) + \frac{1}{2}\left[{}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g, {}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right] \\ &= {}^{LC}R + {}^{LC}D\left(\frac{1}{2}g^{-1}\delta g\right) + \delta\, {}^{LC}\Gamma - \left(\frac{1}{2}g^{-1}\delta g\right)^2 \end{aligned} \tag{G.3}$$

where ${}^{LC}R$ stands for the Levi-Civita curvature :

$$^{LC}R = d_\Sigma {}^{LC}\Gamma + \frac{1}{2}\left[{}^{LC}\Gamma, {}^{LC}\Gamma\right] \tag{G.4}$$

and ${}^{LC}D$ stands for the Levi-Civita covariant derivative. We also recall that, by definition :

$$^{LC}Dg = 0 \tag{G.5}$$

Differentiating equation (G.5), one gets :

$$^{LC}D_\lambda\left(\delta g_{\mu\nu}\right) - \left(\delta\, {}^{LC}\Gamma^{\rho}_{\lambda\nu}\right)g_{\rho\mu} - \left(\delta\, {}^{LC}\Gamma^{\rho}_{\lambda\mu}\right)g_{\rho\nu} = 0 \tag{G.6}$$

from which one deduces that :

$$\delta\, {}^{LC}\Gamma^{\nu}_{\lambda\mu} = \frac{1}{2}g^{\rho\nu}\left({}^{LC}D_\lambda \delta g_{\rho\mu} + {}^{LC}D_\mu \delta g_{\lambda\rho} - {}^{LC}D_\rho \delta g_{\mu\lambda}\right) \tag{G.7}$$

Since δ and d_Σ anticommute, it follows that :

$$\begin{aligned} \left(\delta\, {}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right)^\nu_\mu &= \frac{1}{2}g^{\rho\nu}\left({}^{LC}D_\mu \delta g_{\lambda\rho} - {}^{LC}D_\rho \delta g_{\mu\lambda}\right)dx^\lambda \\ &\underset{\text{Def}}{=} \frac{1}{2}\left({}^{LC}D \wedge \delta\bar{g}\right)^\nu_\mu \end{aligned} \tag{G.8}$$

where :

$$\bar{g}_\mu = g_{\mu\lambda}dx^\lambda \tag{G.9}$$

In the same way, using :

$$\mathcal{I}(\tilde{\omega})\left(g^{-1}\delta g\right)=g^{-1}\mathcal{I}(\tilde{\omega})\delta g=-g^{-1}\mathcal{L}(\tilde{\omega})g=-g^{-1}l_{\Sigma}(\tilde{\omega})g \tag{G.10}$$

one obtains :

$$\begin{aligned}\left(\mathcal{I}(\tilde{\omega})(\delta^{LC}\Gamma+\frac{1}{2}g^{-1}\delta g)\right)^{\nu}_{\mu} &= -\frac{1}{2}g^{\rho\nu}\left({}^{LC}D_{\mu}l_{\Sigma}(\tilde{\omega})g_{\lambda\rho}-{}^{LC}D_{\rho}l_{\Sigma}(\tilde{\omega})g_{\mu\lambda}\right)dx^{\lambda}\\ &= -\frac{1}{2}\left({}^{LC}D\wedge l_{\Sigma}(\tilde{\omega})\overline{g}\right)^{\nu}_{\mu}\end{aligned} \tag{G.11}$$

Going over to the Weil scheme, the same construction occurs in the transformation of the quadratic term $\left(\frac{1}{2}g^{-1}\delta g\right)^{2}$, leading to the term : $\frac{1}{4}\tilde{\psi}\tilde{\psi}$, with $\tilde{\psi}$ defined in equation (4.49).

Finally, one needs the property that $\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}$ is covariant constant for the connection Γ.

First, for the Levi-Civita part :

$$\begin{aligned}{}^{LC}D_{\lambda}\left(\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}\right) &= g_{\rho\nu}\,{}^{LC}D_{\lambda}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}\\ &= \frac{1}{\sqrt{g}}\left(\Gamma^{\mu}_{\lambda\nu}\varepsilon^{\nu\rho}+\Gamma^{\rho}_{\lambda\nu}\varepsilon^{\mu\nu}\right)-\frac{1}{2}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}\left(g^{\alpha\beta}\partial_{\lambda}g_{\alpha\beta}\right)\end{aligned} \tag{G.12}$$

Using the identity :

$$V^{\mu}\varepsilon^{\nu\rho}+V^{\rho}\varepsilon^{\mu\nu}+V^{\nu}\varepsilon^{\rho\mu}=0 \tag{G.13}$$

together with :

$$\Gamma^{\nu}_{\lambda\nu}=\frac{1}{2}g^{\alpha\beta}\partial_{\lambda}g_{\alpha\beta} \tag{G.14}$$

the property follows.

Finally, for the second part of the connection, one needs :

$$\delta\left(\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}\right)+\frac{1}{2}\left(g^{-1}\delta g\right)^{\mu}_{\lambda}\frac{\varepsilon^{\lambda\rho}}{\sqrt{g}}g_{\rho\nu}-\frac{1}{2}\left(g^{-1}\delta g\right)^{\lambda}_{\nu}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\lambda}=0$$

which again follows from the identity (G.13).

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