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Metaplectic Quantization of the Moduli Spaces of Flat and Parabolic Bundles

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Metaplectic Quantization of the Moduli Spaces of Flat and Parabolic Bundles(*)

(after Peter Scheinost)

Martin Schottenloher

1. Introduction

The aim of this note is to give a survey on some recent results of P. Scheinost [**Sch**] and to explain some related background material and motivations.

These results concern primarily the rigorous quantization of the Chern-Simons theory with Wilson loops but they are also related to a number of other interesting subjects in Mathematics and Physics. First of all the results constitute another step in giving the Jones polynomial [**Jo1**] a rigorous geometric interpretation. Note that the Alexander polynomial of a knot can be obtained via the skein relation but also is an invariant of the knot space and therefore is well-understood from the geometric viewpoint. This is not yet true for the Jones polynomial although Witten made a beautiful suggestion of a geometric interpretation [**Wit**] leading to a topological quantum field theory. This approach, however, depends seriously on path integral methods. To overcome the difficulties in using the path integral Witten has also suggested another route in understanding the quantization of the Chern-Simons theory employing the method of geometric quantization. The first rigorous steps in this direction has been undertaken in the articles of Hitchin [**Hi2**] and Axelrod, della Pietra and Witten [**APW**], and the dissertation of P. Scheinost [**Sch**] can be understood as presenting the next step.

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In addition to this contribution to the Jones-Witten theory, the results of P. Scheinost can also be understood as giving examples of a geometric quantization where the quantization does not depend on the polarisations chosen. Moreover, his approach to the quantization of the Chern-Simons theory with Wilson loops makes it necessary to introduce the metaplectic correction of geometric quantization already because of purely mathematical reasons.

Since the metaplectic quantization is a rigorous quantization scheme the results may also contribute to the actual discussion of quantum gravity using the Ashtekar variables, where the Wilson loops play an important role (e.g. **[Ash]**, **[Ish]**, **[Lol]**).

Moreover, there is a strong connection to non-abelian Hodge theory. Indeed Scheinost's results depend essentially on non-abelian Hodge theory as developed by Corlette **[Cor]**, Hitchin **[Hi1]** and Simpson **[Si1, Si3, Si4]**. In this context P. Scheinost also contributes to the theory of moduli spaces of Higgs bundles since some of the results known for moduli spaces over curves are generalized to the corresponding moduli spaces over smooth projective varieties over $\mathbb{C}$.

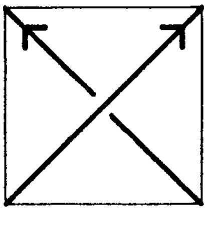
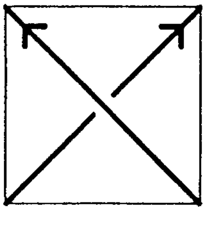
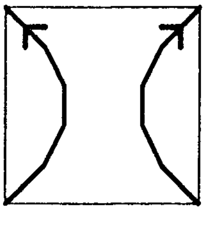
Finally, the main technical goal in the dissertation is to construct a projectively flat connection on certain natural vector bundles over the Teichmüller space $T_{g,n}$ of conformal structures on a surface of genus g with n punctures. Therefore the results are also strongly related to conformal field theory and in particular to similar constructions of flat connections as e.g. in **[BeL]**, **[Fa1]** or **[TUY]**.

2. The Jones Polynomial

To every oriented knot or link diagram D in the plane there corresponds a unique Laurent polynomial $V_D \in \mathbb{Z}[q,q^{-1}]$ with the following three properties:

1) $V_U = 1$ for any diagram U representing the unknot.

2) If D_+, D_- and D_0 are diagrams related by the *skein relation*

D_+  D_-  D_0 

then the corresponding polynomials satisfy

$$\boxed{\text{(J)} \qquad q^{-2}V_{D_+} - q^2V_{D_-} = (q - q^{-1})V_{D_0}.}$$

Here the skein relation means that the three diagrams are identical outside the indicated square and that they differ exactly by what is described within the squares.

3) If two knot diagrams are ambient isotopic then they have the same Jones polynomial. In other words the Jones polynomial V_D is invariant under Reidemeister moves and defines a knot invariant.

The *Jones polynomial* V_D has been discovered by Jones **[Jo1]** in 1984, and it is a much stronger invariant then the Alexander polynomial. For example, it can distinguish between the trefoil knot and its mirror image. This is a consequence of the fact that for a diagram D and its mirror image D^* one has the general identity

$$V_D(q) = V_{D^*}(q^{-1})$$

and since the Jones polynomial of the trefoil knot K can easily be calculated: $V_K(q) = q^2 + q^6 - q^8$. For the same reasons the Jones polynomial distinguishes the torus knots and its mirror images.

Nevertheless, the nature of the Jones polynomial is not well-understood. Many problems concerning the Jones polynomial are open and it is not only Jones **[Jo2]** who attributes these difficulties to the fact that there is no truely three dimensional topological or geometrical interpretation for the Jones polynomial. All proofs for the existence of the Jones polynomial only use the plane knot diagrams and they are based on certain combinatorial manipulations with these diagrams. This becomes particularly apparent in the simplest of the existence proofs given by Kauffman **[Kau]**. The same holds true for the companion knot invariants like the Homfly polynomial, the Kauffman polynomial and all the invariants coming from quantum groups using the R-matrix in order to obtain suitable representations of Artin's braid groups.

This is in contrast to the Alexander polynomial. The *Alexander polynomial* $A_D(q)$ can be defined by using the same skein relation for the diagrams as above but replacing (J) by

$$\boxed{\text{(A)} \qquad A_{D_+} - A_{D_-} = qA_{D_0}.}$$

However, the Alexander polynomial is at the same time a topological invariant of the *knot space*, i.e. the complement of the knot in the space $\mathbb{S}^3$. It arises from the homology of the infinite cyclic cover of the knot space. Equivalently it can be obtained by considering the cohomology of a locally constant sheaf of rank 1 (or one-dimensional local system) on the knot space.

Of course, the Jones polynomial cannot be explained as a topological invariant of the knot space since for the trefoil knot and its mirror image the knot spaces are homeomorphic to each other by a reflection at a plane.

3. Witten's Interpretation of the Jones Polynomial

In his seminal article [**Wit**] E. Witten gives a physical interpretation of the nature of the Jones polynomial by means of quantizing the Chern-Simons theory. The Chern-Simons theory on a three dimensional compact oriented differentiable manifold Y (without boundary) is governed by the *Chern-Simons action*

$$S(A) = (4\pi)^{-1}\int_Y \mathrm{Tr}(A\wedge dA - \tfrac{2}{3}A\wedge A\wedge A) .$$

Here, A is a connection in a fixed SU(N)-principal fibre bundle P over Y given as a $\mathfrak{g} = \mathrm{Lie}\ SU(N)$-valued 1-form on Y. One remarkable feature of the Chern-Simons action is that it does not depend on any metric structure on Y or P. Therefore, the quantum numbers or other quantum objects which occur after a suitable quantization of the theory have to be topological invariants of the differentiable manifold Y. The same is true if one incorporates knots and links into the discussion: A closed curve C in Y gives rise to the following functional on the space $\mathscr{A} = \Gamma(Y,\mathrm{Ad}P)$ of connections in P:

$$W_{R,C}(A) = \mathrm{Tr}_R(\mathrm{Hol}_C(A)) , \ A \in \mathscr{A} ,$$

where $\mathrm{Hol}_C(A)$ is the holonomy of the connection A around the loop C, i.e. an element of SU(N) up to conjugation, and where Tr_R is the trace with respect to a given representation R of SU(N). Again, the functional $W_{R,C}$, which is called a *Wilson loop*, is independent of any metric. In particular, general covariance is maintained.

In order to interpret knot invariants within the quantized Chern-Simons theory with Wilson loops one considers an *oriented link* L in the manifold Y which consists of finitely many non-intersecting knots represented by suitable loops C_i , $i = 1, 2, \ldots , m$. Moreover, one fixes a *level* k which is simply a positive natural number and one fixes a finite number of irreducible representations R_i , $i = 1, 2, \ldots , m$, of SU(N) which can be viewed as to be certain *decorations* of the link components C_i. Now for $C = (C_1, C_2, \dots, C_m)$ and $R = (R_1, R_2, \dots, R_m)$ the Feynman path integral

$$Z(Y,C,R) = \int \mathscr{D}[A] \exp^{ikS(A)} \prod_{i=1}^{m} W_{R_i,C_i}(A)$$

yields an entity (the unnormalized "*expectation value*") which is in particular an invariant of the link L if one believes that the integral gives sense at all. The formal integration $\mathscr{D}[A]$ has to be carried through over all connections A on Y up to gauge equivalence. If one assumes that these integrals are well-defined then by varying k, N and the representations one obtains various knot polynomials, in particular the Jones polynomial as Witten has shown in a convincing manner (**[Wit]**, see also **[Ati]**). As an example, the two variable generalization of the Jones polynomial – the Homfly polynomial (cf. **[Hom]**) – will be given by the values of the above expectation values for k, N $\in \mathbb{N}$ and R_i the standard representation of SU(N) in GL(N,$\mathbb{C}$).

However, since the path integral is not well-defined to our present knowledge, it cannot be the basis of an existence proof for the knot invariants. In particular, although Witten's suggestion provides an important and interesting idea how to look at the new knot invariants, this approach cannot be regarded as to give a satisfactory geometric interpretation of the Jones polynomial, unless the use of the path integral has been justified completely. Since it is presently out of scope to give a rigorous foundation of the path integral methods, Witten suggests in his article **[Wit]** to replace the path integral quantization by a suitable Hamiltonian quantization.

4. Hamiltonian Approach

In order to explain the Hamiltonian approach to the quantization of the Chern-Simons theory we first concentrate on the case without knots, i.e. the Chern-Simons theory on the three-manifold Y without Wilson loops. The basic strategy to obtain solutions of the Euler-Lagrange equations belonging to the Chern-Simons action on an arbitrary compact oriented manifold Y and to quantize them is to cut Y in pieces, solve the problem on the pieces and gluing the solutions back together.

In the immediate neighborhood of a "cut" of Y, which is represented by an embedded surface S in Y, the three-manifold looks like $J \times S$ for an intervall J in $\mathbb{R}$. Choosing the gauge $A_0 = 0$ (in the direction of the intervall) the system which we want to quantize is governed by the constraint equation

$$F_A = 0,$$

("Gauß law") where F_A is the curvature of the SU(N)-connection A on the surface S. Let $\mathscr{A}$ now denote the affine space of SU(N)-connections on S with the subspace $\mathscr{A}_0$ of flat SU(N)-connections: $\mathscr{A}_0 := \{A \in \mathscr{A} : F_A = 0\}$. As a result of the above equation, the *classical phase space* is

$$M := \mathscr{A}_0/\mathscr{G},$$

the space of flat connections modulo gauge transformations. Here, $\mathscr{G}$ is the group of unitary gauge transformations (i.e. the automorphism group of the principal SU(N)-bundle) on S acting on $\mathscr{A}$ in the usual way: $A \longmapsto g^{-1}Ag$ for $(g, A) \in \mathscr{G} \times \mathscr{A}$. $\mathscr{A}$ has a natural symplectic form which is given by

$$\omega(A,B) := -(2\pi)^{-2}\int_S \mathrm{Tr}(A \wedge B)$$

for the $\mathfrak{g}$-valued connection forms A and B on S. The action of the gauge group leaves the symplectic form invariant, hence the form descents to the quotient $\mathscr{A}_0/\mathscr{G}$ and therefore defines a symplectic form ω_M on the regular part of M. Note, that the symplectic action of the infinite dimensional Lie group $\mathscr{G}$ of gauge transformations on the affine symplectic space $\mathscr{A}$ induces a moment map $m : \mathscr{A} \longrightarrow (\mathrm{Lie}\,\mathscr{G})^*$ on $\mathscr{A}$ which can be described as $m(A) = F_A$ for $\mathfrak{g}$-valued 1-forms A (cf. **[AtB]**), and that the classical phase space M consequently can be understood as the Marsden-Weinstein quotient with respect to the moment map m: $M = m^{-1}(0)/\mathscr{G}$.

The classical phase space M has also an interpretation as the space of equivalence classes of flat unitary vector bundles of rank N on S. Therefore, M can be identified with the space of equivalence classes of SU(N)-representations of the fundamental group of S:

$$M \cong \mathrm{Hom}(\pi_1(S), SU(N))/SU(N),$$

where the action of SU(N) on the set of representations is conjugation. Last not least, M has also the description of the non-abelian (Čech) cohomology space $H^1(S, SU(N))$. Altogether,

$$M \cong \mathrm{Hom}(\pi_1(S), SU(N))/_{SU(N)} \cong H^1(S, SU(N)),$$

with the symplectic structure given by ω_M.

Assume now, that the quantization of the classical phase space (M, ω_M) leads to a natural vector space $Z(S)$ of the quantum state vectors. In order to come to a solution of the Chern-Simons theory on the original manifold Y one has to glue the partial solutions on the surfaces in an appropriate manner. This can be best formulated by the notion of a topological quantum field theory. Note that $Z(S)$ will be a finite dimensional vector space over $\mathbb{C}$ since M is compact.

5. Topological Field Theory

In order to emphazise the topological nature of the approach of cutting and gluing we briefly explain the essential content of a two dimensional topological quantum field theory (cf. [**Ati**] for more details): A *two-dimensional topological quantum field theory* consists of the following assignments:

- To each two-dimensional oriented compact manifold S without boundary there corresponds a finite dimensional $\mathbb{C}$-vector space $Z(S)$.
- To each three dimensional oriented compact manifold Y with boundary ∂Y there corresponds a vector $Z(Y) \in Z(\partial Y)$ where ∂Y obtains the induced orientation.

These assignments are subject to a number of natural properties, cf. functoriality, and they satisfy in particular the following axioms:

- $Z(S^{opp}) = Z(S)^*$

where S^{opp} is the surface S equipped with the opposite orientation and where V^* denotes the dual of the vector space V.

- $Z(\emptyset) = \mathbb{C}$

where $\emptyset$ is regarded as to be a surface, and

• $Z(S_1 \amalg S_2) = Z(S_1) \otimes Z(S_2)$

for the disjoint union $S_1 \amalg S_2$ of two surfaces. Hence, $S \longmapsto Z(S)$ is a kind of a multiplicative homology theory. In particular, $Z(\emptyset) = \mathbb{C}$ can be deduced from the latter axiom.

The gluing of solutions is encoded in the axiom

• $Z(Y) = \langle Z(Y_1), Z(Y_2) \rangle$,

where "$\langle\ ,\ \rangle$" denotes the evaluation of $Z(Y_1) \in Z(\partial Y_1) = Z(\partial Y_2)^*$ at the vector $Z(Y_2) \in Z(\partial Y_2)$. In particular, since $Z(Y) \in Z(\emptyset) = \mathbb{C}$ one gets a complex number which corresponds to the expectation value expressed by the path integral.

A similar axiom holds in the case of $\partial Y \neq \emptyset$. As normalisation one requires $Z(\emptyset) = 1 \in \mathbb{C}$, where $\emptyset$ now is regarded to be a three-manifold with boundary $\emptyset$.

6. Geometric Quantization

In general, the program of geometric quantization of a finite dimensional symplectic manifold (M, ω) requires the choice of

• a hermitian complex line bundle $\mathscr{L}$ over M with curvature $-\frac{1}{2\pi i}\operatorname{curv}(\mathscr{L}) = \omega$ ($\mathscr{L}$ is called the *prequantum line bundle*),

• and a *polarization* $F \subset TM \otimes \mathbb{C}$

cf. e.g. [Woo]. The *space of quantized states* (or "*quantum Hilbert space*") is then

$$Q(M, \omega, \mathscr{L}, F) := \{ s \in \Gamma(M, \mathscr{L}) : \nabla_X s = 0 \text{ for all } X \in \Gamma(M, F) \}$$

i.e. the vector space of the sections in $\mathscr{L}$ which are covariant constant along F.

There are a number of serious mathematical problems which arise in the framework of geometric quantization. One of these problem is the construction of a convenient scalar product on Q or on a suitable subspace of Q in such a way that the relevant observables can be represented as self-adjoint operators. Another problem, which is of considerable importance in the context of topological quantum field theory, is the obvious dependence of the quantization on the geometric data $\mathscr{L}$ and F. Moreover, examples show that in general one has to introduce a metaplectic correction which is given in our situation by fixing a spin structure on the phase space.

In order to obtain the necessary geometric data $\mathscr{L}$ and F on the space M of flat connection on a given surface Witten **[Wit]** suggests to introduce a complex structure J on the connected compact surface S of genus g. Let S_J denote the surface S equipped with the complex structure J. S_J is now a compact Riemann surface, and by a fundamental result of Narasimhan and Seshadri [NaS] the space M has the interpretation of the moduli space of semi-stable holomorphic vector bundles on S_J of rank N with vanishing first Chern class (i.e. degree 0) and trivial determinant. In particular, in this description the space M has a natural complex structure as a complex manifold M_J (with mild singularities at those bundles which are not stable). On M_J there exists a natural holomorphic vector bundle namely the determinant line bundle $\mathscr{L}$ which automatically satisfies the prequantization condition $\mathrm{curv}(\mathscr{L}) = \omega_M$, where ω_M is the above mentioned symplectic form on the space M of flat connections on S. Of course, the line bundle $\mathscr{L}^{\otimes k}$ ($\mathscr{L}$ k-times tensored with itself) is then a prequantum bundle of the phase space $(M, k\,\omega_M)$, since we have $\mathrm{curv}(\mathscr{L}^{\otimes k}) = k\,\omega_M$. A suitable polarization for all these holomorphic line bundles is simply the *holomorphic polarization* on M_J and therefore, the space of quantum vectors

$$Z_J := Z(S_J) := \Gamma(M_J, \mathscr{L}^{\otimes k})$$

is nothing else than the complex vector space of the holomorphic sections on M_J with values in the holomorphic line bundle $\mathscr{L}^{\otimes k}$.

So far, everything can be found in Witten's article **[Wit]** where he also claims that the Z_J are essentially independent of the complex structure J and thus define the quantum vector spaces Z(S) which one wants to construct for the topological quantum field theory. The argument for this independence is the following: The complex vector spaces define a fibration $\pi : \mathscr{Z} \longrightarrow T_g$ over the Teichmüller space T_g of complex structures of a surface of genus g with $\pi^{-1}(J) = Z_J$. On the basis of heuristic arguments Witten concludes in [Wit]:

THEOREM: *Under the above assumptions:*

1) The fibration $\pi : \mathscr{Z} \longrightarrow T_g$ *is a vector bundle.*

2) There exists a natural projectively flat connection on $\mathscr{Z}$.

Consequently the fibres Z_J of the vector bundle $\mathscr{Z}$ can be identified canonically via the parallel transport given by this connection. Since the Teichmüller space is simply connected and by the projective flatness of the connection the parallel transport is independent of the paths connecting the points $J, J' \in T_g$ up to a constant. This means that the projective spaces of the Z_J can be identified. But this is all what is wanted for in Quantum Mechanics.

The task of proving the theorem is quite a difficult one. It has been carried through by Hitchin [**Hi2**] and by Axelrod, Della Pietra and Witten [**APW**]. Hitchin constructs the connection by means of deformation theory of complex structures working directly on the moduli space $M_J = \mathscr{A}_0/\mathscr{G}$ while Axelrod, Della Pietra and Witten work in the infinite dimensional setting on the space $\mathscr{A}_0$ to construct a connection which finally can be pushed forward to the quotient.

7. Incorporation of Knots and Links

In the presence of links in the manifold Y one has to modify the topological quantum field theory if one wants to carry through a program similar to what has been described in the sections 4-6. Fix a level k and a group SU(N), and let $SU(N)^\wedge$ denote the set of irreducible representations of SU(N). A *two-dimensional topological quantum field theory with links* now consists of the following assignments

- To each two-dimensional oriented compact manifold S with finitely many marked points $P = (x_1, x_2, \dots, x_p) \in S^p$ and irreducible representations $R = (R_1, R_2, \dots, R_p) \in (SU(N)^\wedge)^p$ there corresponds a finite dimensional $\mathbb{C}$-vector space $Z(S,P,R)$.
- To each three-dimensional oriented compact manifold Y with boundary ∂Y, equipped with an oriented "link" L consisting of m components and with $R = (R_1, R_2, \dots, R_m) \in (SU(N)^\wedge)^m$ there corresponds a vector

$$Z(Y,L,R) \in Z(\partial Y, L\cap\partial Y, R\cap\partial Y).$$

Here, ∂Y is endowed with the orientation induced from Y, and a component K_i, $i = 1, 2, \dots, m$, of a "link" L is either an ordinary oriented knot in the interior $Y\setminus\partial Y$ of the manifold Y (given by a non-intersecting loop) or a non-intersecting oriented curve K_i in Y joining two points of ∂Y but sitting in the interior $Y \setminus \partial Y$ except for the endpoints (where K_i is transversal to the boundary ∂Y). The set (or sequence) of representations $R^{\sim} := R \cap \partial Y$ decorating the marked points $P = L \cap \partial Y = (x_1, x_2, \dots, x_p)$ is chosen in the following way: If $x_j \in L \cap \partial Y$ is a point on the "knot" K_i (i.e. an endpoint of the i-th curve joining two points of ∂Y) then $R^{\sim}_j$ is the representation R_i if at x_j the orientations of ∂Y and K_i match to give the orientation of Y. Otherwise, $R^{\sim}_j$ is simply the conjugate of the representation R_i.

Similar to the situation in section 5 these assignments have to satisfy various functoriality properties and axioms in order to give, for example, for a link L in the three-manifold Y without boundary the number

$$Z(Y,L,R) \in Z(\emptyset) = \mathbb{C}$$

We do not explain the details of these axioms (see e.g. [**Ati**]) but mention only the important rule which represents the gluing in case of a cut: Let the three-manifold Y with its link L be cut in the two pieces Y_1 and Y_2 with the common boundary $S := \partial Y_1 = \partial Y_2^{opp}$ (in particular, $\partial Y = \emptyset$, $Y = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = S$) and let $L_1 := L \cap Y_1$ and $L_2 := Y_2 \cap Y$ be the corresponding links with induced orientations and decorations R_1 and R_2. Then

$$\begin{aligned} &Z(Y_1, L_1, R_1) \in Z(S, L_1 \cap \partial Y_1, R_1 \cap \partial Y_1) \text{ and} \\ &Z(Y_2, L_2, R_2) \in Z(S^{opp}, L_2 \cap \partial Y_2, R_2 \cap \partial Y_2) \text{ with} \\ &Z(S^{opp}, L_2 \cap \partial Y_2, R_2 \cap \partial Y_2) = Z(S, L_1 \cap \partial Y_1, R_1 \cap \partial Y_1)^* \end{aligned}$$

in such a way that $Z(Y,L,R) \in Z(\emptyset) = \mathbb{C}$ can be obtained by the evaluation

$$Z(Y,L,R) = \langle Z(Y_2, L_2, R_2), Z(Y_1, L_1, R_1) \rangle.$$

The approach of Witten [**Wit**] to yield at least the finite dimensional vector spaces $Z(S,P,R)$ of the topological quantum field theory with links requires the following modifications for the geometric quantization: Heuristically, the space $Z(S,P,R)$ is the quantization of the space of singular $SU(N)$-connections on the surface S which are flat outside the marked points in P and which have a prescribed holonomy around these marked points. Thus, in the presence of knots the space of flat $SU(N)$-connections on S modulo gauge equivalence has to be replaced by the following classical phase space

$$M := \mathcal{A}_{\mathcal{C}} / \mathcal{G}.$$

Here, $\mathscr{A}_{\mathscr{C}}$ is the space of *singular connections* on S the curvature of which at the marked points x_i is locally of the form $m(A) = \sum R_i \delta(x_i - x)$ up to conjugation (with $\delta(x_i - x)$ being the Dirac δ-functional at x_i). Therefore, $\mathscr{A}_{\mathscr{C}}$ can be understood as to be the inverse image $m^{-1}(\mathscr{C})$ with respect to the moment map m where $\mathscr{C}$ is a certain coadjoint orbit of the dual of the Lie algebra of $\mathscr{G}$. Hence, the classical phase space is again a (generalized) Marsden-Weinstein quotient. Another interpretation of M is the following: The conjugation classes of the representations R_i determine certain orders m_i and therefore, M can be identified with the representation space of the orbifold fundamental group $\pi_1^{orb}(S)$:

$$M \cong \mathrm{Hom}(\pi_1^{orb}(S), SU(N))/SU(N),$$

where the *orbifold fundamental group* $\pi_1^{orb}(S)$ in terms of generators and relators is given by the generators $a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g, c_1, c_2, \dots, c_p$ and the $p+1$ relators:

$$\prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{p} c_j = 1 \quad \text{and} \quad c_i^{m_i} = 1, \text{for } i = 1, 2, \dots, p.$$

It is not evident how to define a natural symplectic structure on the (regular part of the) space M. If one tries to imitate the situation of the case where no knots are present one runs into difficulties because of the problem of how to multiply distributional connections.

Disregarding this important point for a moment one has at least a well-known interpretation of M in the realm of algebraic geometry once a holomorphic structure J on the surface S is picked. By a result of Mehta and Seshadri **[MeS]** the space $M \cong \mathrm{Hom}(\pi_1^{orb}(S), SU(N))/SU(N)$ is the moduli space of semistable parabolic holomorphic vector bundles on S_J of rank N with parabolic degree 0 and trivial determinant. As such a moduli space M acquires at least the structure of a complex manifold M_J (with mild singularities).

A *parabolic structure* on a holomorphic vector bundle E on a Riemann surface Σ with the marked points $x_1, x_2, \dots, x_p \in \Sigma$ is given by

• a *flag* of proper subspaces at each of the fibers E_i of E over x_i: $E_i = F_i^{(0)} \subset F_i^{(1)} \subset \dots F_i^{(r_i)} \subset \{0\}$ with the $k_i^{(s)} := \dim F_i^{(s)}/F_i^{(s+1)}$ as multiplicities,

• and a sequence of *weights* $\alpha_i^{(s)}$ attached to each of the flags with $0 \le \alpha_i^{(0)} < \alpha_i^{(1)} < \dots < \alpha_i^{(r_i)} < 1$.

The *parabolic degree* of such a parabolic bundle E is

$$\mathrm{paradeg}\, E := \deg(E) + \sum_i d_i, \text{ where } d_i := \sum_s \alpha_i^{(s)} k_i^{(s)}.$$

A parabolic bundle E is *semi-stable* if for all parabolic subbundles F of E the following holds: $(\mathrm{rk}(F))^{-1}\mathrm{paradeg}\, F \leq (\mathrm{rk}(E))^{-1}\mathrm{paradeg}\, E$. E is *stable* if the same holds when " $\leq$ " always can be replaced by a strict inequality.

8. The Elliptic Surface Trick

So far, this survey has presented some features of Witten's attempt [**Wit**] to provide a truely three-dimensional geometric interpretation of the Jones polynomial including the rigorous results of [**Hi2**] and [**APW**]. Note however, that in these two articles the geometric quantization is carried through only under the assumption that no knots or links are contained in the three-manifold in question. We now come to the results of P. Scheinost where knots and links are incorporated into the investigations. In the presence of knots, the first problem to address is to find a suitable symplectic structure on the classical phase space M of the singular connections. (The phase space M has been described in the previous section.)

This symplectic structure on M has been obtained by P. Scheinost by a trick which has also been used by S. Bauer [**Bau**] in a similar context:

Let Σ denote a compact connected Riemann surface (which should better be called a smooth projective curve over $\mathbb{C}$ in the sequel) with marked points $x_1, x_2, \ldots, x_p$ of multiplicities $m_1, ,m_2, \ldots, m_p$. These data determine an orbifold structure on Σ by the orbifold fundamental group $\pi_1^{orb}(\Sigma)$ (cf. the previous section). Then there always exists an *elliptic surface* X over Σ, i.e. a surjective holomorphic map

$$\varphi : X \longrightarrow \Sigma$$

from a compact connected Kähler manifold Σ of complex dimension 2 (hence a "surface") such that

- the general fiber of φ is a smooth elliptic curve, i.e. outside a finite number of points of Σ the fiber $\varphi^{-1}(x)$ is a compact Riemann surface of genus 1 (of varying complex structure),
- over the marked points x_i one has the multiple fibers $\varphi^{-1}(x_i)$ of multiplicity m_i,
- φ induces an isomorphism $\varphi_* : \pi_1(X) \longrightarrow \pi_1^{orb}(\Sigma)$,
- there is at least one singular fiber and $\chi(X) > 0$.

(See [Ue] for the existence of the fibration $\varphi : X \longrightarrow \Sigma$ or [BPV].)

This elliptic fibration allows it to understand the classical phase space M of singular connections on the curve Σ with prescribed holonomy around the marked points as a certain moduli space of semi-stable holomorphic vector bundles on the surface X.

PROPOSITION [Bau]: *Let* $\varphi : X \longrightarrow \Sigma$ *be an elliptic surface over the curve* Σ *with* $b_1(X)$ *even,* $\chi(X) > 0$ *and* $\mathrm{kod}(X) = 1$. *Then the direct image functor* φ_* *induces an isomorphism between*

- *the moduli space of semi-stable rank* N *parabolic bundles on the curve* $(\Sigma, x_1, x_2, \dots, x_p)$ *of paradegree* 0 *with rational weights* $\alpha_i^{(s)} = \frac{1}{m_i} t_i^{(s)}$, $s = 1, 2, \dots, r_i$, *at each marked point such that* $d_i = \sum_s \alpha_i^{(s)} k_i^{(s)} \in \mathbb{N}$ *(where the* $k_i^{(s)}$ *are the multiplicities of the flags) and with determinant equal to the line bundle* $\mathcal{O}(-\Sigma\, d_i x_i)$ *on* Σ.

  and

- *a corresponding component of the moduli space* M(X,N,0) *of the semi-stable holomorphic rank* N *bundles on* X *with degree* 0 *and trivial determinant.*

Now, on any component M of M(X,N,0) with sufficiently many regular points one has the natural symplectic form ω_M given by

$$\omega_M(A,B) := -(2\pi)^{-2}\int_X \mathrm{Tr}(A\wedge B)\wedge \omega_X,$$

where ω_X is the Kähler form on the surface X.

In a similar manner as in the one dimensional case one gets a holomorphic line bundle $\mathcal{L}$ on $M \subset M(X,N,0)$ with $-\frac{1}{2\pi i}\mathrm{curv}(\mathcal{L}) = \omega_M$, the generalized theta bundle. Therefore, by using again the holomorphic structure as a suitable polarization, the procedure of geometric quantization leads to the spaces

$$Z' = \Gamma(M, \mathcal{L}^{\otimes k})$$

of holomorphic sections of the line bundle $\mathcal{L}^{\otimes k}$.

This finite dimensional complex vector space is a candidate for the quantum Hilbert space Z(S,P,R) in the topological quantum field theory with links. However, the geometric quantization can only work if S is equipped with a complex structure J such that $\Sigma = S_J$, and then the discussion of this section depends on this complex structure. As a consequence, the above space Z' should better be written as

$$Z'_J = Z'(S_J, P, R) = \Gamma(M_J, \mathscr{L}^{\otimes k})$$

to indicate the dependence on the complex structure J.

As before in section 6 one wants to get rid of the dependence on the chosen complex structure and this can in fact be done under some mild restrictions as is explained in the next three sections.

Before we start with the general discussion of this problem let us mention that the trick to consider a siutable elliptic surface over the curve with marked points also provides a simplification of a proof of a result of Simpson **[Si2]** on noncompact curves. The result in question is the one-to-one correspondence of the category of stable parabolic Higgs bundles of rank N and paradegree 0 (which are called filtered Higgs bundles in **[Si2]**) on a smooth projective curve Σ with marked points and the stable parabolic local systems of rank N and paradegree 0 on Σ. This can be deduced from a suitable extension of the above proposition to parabolic Higgs bundles as will be explained in a forthcoming paper. The extension of the proposition is due to P. Scheinost.

9. The Main Theorem

In order to show that the spaces $\Gamma(M_J, \mathscr{L}^{\otimes k})$ of holomorphic sections are essentially independent of the complex structure J on the surface S and of other choices made in the above construction it suffices to prove that these spaces are independent of deformations of the holomorphic structure on the elliptic surfaces X which arise in the discussion of the previous section. More generally, starting with a compact Kähler manifold X one can study smooth components M of the space $H^1(X, SU(N))$ with suitable prequantum line bundles $\mathscr{L}$ on M and discuss in which way the spaces $\Gamma(M, \mathscr{L})$ of holomorphic sections depend on the deformations of the complex structure of X. It can be shown that under suitable assumptions these spaces are independent of the deformations up to a constant as will be explained in the following.

Let X be a compact algebraic Kähler manifold with Kähler form ω_X. According to results of Donaldson **[Do1]** and Uhlenbeck and Yau **[UhY]** the space $H^1(X, SU(N))$ can be understood as the moduli space of semi-stable rank N holomorphic vector bundles on X with vanishing Chern classes and trivial determinant. Hence, $H^1(X, SU(N))$ has a natural analytic structure. Let M be a smooth component of this moduli space. As in the cases of $\dim X = 1, 2$ the Kähler form ω_X together with the Ad-invariant trace on $\mathfrak{g} = \mathrm{Lie}\, SU(N)$ induces a natural symplectic structure on M given by the symplectic form

$$\omega_M(A, B) := -(2\pi)^{-2} \int_X \mathrm{Tr}(A \wedge B) \wedge \omega_X \wedge \omega_X \wedge \ldots \wedge \omega_X ,$$

where ω_X has to be taken $n-1$ times if n is the (complex) dimension of X. In the situation of $b_1(M) = 0$ there exists a unique line bundle $\mathscr{L}$ on M with $-\frac{1}{2\pi i}\mathrm{curv}(\mathscr{L}) = \omega_M$ (which comes from a suitable power of the determinant line bundle, cf. [BiF] or [Do2]). Hence, for any level k there is a prequantization bundle $\mathscr{L}^{\otimes k}$ with $-\frac{1}{2\pi i}\mathrm{curv}(\mathscr{L}^{\otimes k}) = k\,\omega_M$. In contrast to the one-dimensional situation one has to take care of a metaplectic correction of the quantization. Thus, one has to fix a metaplectic structure on M which is given by a spin structure on M which in turn is given by a square root $\mathscr{K}^{\frac{1}{2}}$ of the canonical line bundle $\mathscr{K}$ on M (if it exists). Instead of the ordinary quantization with $\Gamma(M, \mathscr{L}^{\otimes k})$ as the space of quantized vectors the appropriate quantum vector space in the metaplectic quantization is now the space

$$\Gamma(M, \mathscr{L}^{\otimes k} \otimes \mathscr{K}^{\frac{1}{2}})$$

of holomorphic sections in $\mathscr{L}^{\otimes k} \otimes \mathscr{K}^{\frac{1}{2}}$.

Now let $(I_t, \omega_t)_{t \in B}$ be a holomorphic family of complex structures on M, induced by a holomorphic family on X given by a holomorphic map

$$p : \mathfrak{M} \longrightarrow B$$

between complex manifolds: $M_t = (M, I_t) = p^{-1}(t), t \in B$. Assume $\omega_t = \omega_M$ to be constant. Furthermore, let $\mathfrak{L}$ be a holomorphic line bundle on $\mathfrak{M}$ such that the restriction to M_t is the prequantum line bundle for the symplectic manifold (M_t, ω_M) and let $\mathfrak{K}^{\frac{1}{2}}$ be a spin structure on $\mathfrak{M}$.

MAIN THEOREM [Sch]: *Let* M *be a non-singular compact component of the moduli space* $H^1(X, SU(N))$ *satisfying* $b_1(M) = 0$ *and assume that* M *has a spin structure. Let* $p : \mathfrak{M} \longrightarrow B$ *be a holomorphic family of complex structures on* M *induced from a deformation of the complex structures on* X *and adapted to the given geometric data* $\mathscr{L}$, ω_M *and* $\mathscr{K}^{\frac{1}{2}}$ *in the manner just described. Then:*

1) The sheaf $\mathcal{H} := p_*(\mathfrak{L}^{\otimes k} \otimes \mathfrak{K}^{\frac{1}{2}})$ *on* B *is locally free, i.e. a holomorphic vector bundle,*

2) There exists a projectively flat connection on $\mathcal{H}$,

provided a certain technical condition with respect to the moduli space of Higgs bundles is satisfied. (This condition will be explained at the end of the next section. It is always satisfied for elliptic surfaces X. *The spin condition, however is not always satisfied. This will be discussed in the last section.)*

As a consequence, the vector spaces $\mathcal{H}_t$, i.e. the fibers of $\mathcal{H}$ over $t \in B$, can locally be identified projectively by parallel transport..

Concerning the construction of the spaces Z(S,P,R) of a topological quantum field theory (cf. section 7) one can now apply the main theorem in the following way: Varying the complex structure J on the surfaces S with marked points P gives a deformation of the Kähler structure on the elliptic surface X_J over S_J (cf. section 8) and a deformation of any component M_J of $H^1(X_J, SU(N))$ corresponding to the space of parabolic vector bundles on S_J. Under the assumption of a spin structure on M_J one then gets the J-independent space Z(S,P,R) by identifying the various $\Gamma(M_J, \mathcal{L}^{\otimes k} \otimes \mathcal{K}^{\frac{1}{2}})$ up to a constant. In particular, varying the complex structure by taking B to be the Teichmüller space $T_{g,p}$ of Riemann surfaces of genus g with p marked points leads to the corresponding bundle $\mathcal{H}$ of quantum Hilbert spaces over $T_{g,p}$ endowed with a projectively flat connection. The fibers of $\mathcal{H}$ are the conformal blocks of conformal field theory.

To a large extent the strategy of the proof of the main theorem follows the article of Hitchin **[Hi2]** where the case of $\dim X = 1$ is treated. At a number of occasions, however, the higher dimensional case causes additional difficulties and it is not obvious how to overcome these difficulties. I will try to give a sketch of the proof without explaining too much of the details but rather by presenting the main lines of the proof.

SKETCH OF PROOF OF THE MAIN THEOREM:

The proof starts with the description of the deformation tensor. Let I_t denote the integrable almost complex structure of the fiber $p^{-1}(t)$ over the point $t \in B$. For any curve $t = t(s)$ in the base manifold B the *infinitesimal deformation* is given by

$$\delta I = \frac{\partial}{\partial s}\left(I_{t(s)}\right)\Big|_{s=0}.$$

Since I_t is an endomorphism of the tangent bundle TM the infinitesimal deformation δI is an element of the tangent space of the space of integrable almost complex structures, i.e.

$$\delta I \in \mathrm{End}(T^{0,1}M, T^{1,0}M) = \Omega^{0,1}(M, TM)$$

with the integrability condition $\partial''\delta I = 0$ and its cohomology class

$$[\delta I] \in H^1(M,TM),$$

the *Kodaira-Spencer class* of the deformation. (Here and in the following, TM denotes the tangent bundle and at the same time the sheaf of germs of holomorphic sections with values in TM.) By the compatibility of the complex structures with the fixed symplectic form $\omega = \omega_M$ one gets

$$\delta I^a{}_{\bar{c}} = G^{ab}\omega_{b\bar{c}},$$

with a symmetric tensor $G \in H^0(M, S^2TM)$, called the *deformation tensor.* It is rather easy to check:

STEP 1: *The deformation tensor* G *is holomorphic.*

Next, one considers the sheaf $\mathscr{D}^m$ of holomorphic m^{th} order differential operators on the line bundle $\mathscr{L}^{\otimes k} \otimes \mathscr{K}^{\frac{1}{2}}$. For a fixed complex structure I and $s \in H^0(M_I, \mathscr{L}^{\otimes k} \otimes \mathscr{K}^{\frac{1}{2}})$ one has the *double Dolbeault complex* on $M = M_I$:

$$\Omega_I{}^q = \Omega^{0,q}(M, \mathscr{D}^1) \oplus \Omega^{0,q-1}(M, \mathscr{L}^{\otimes k} \otimes \mathscr{K}^{\frac{1}{2}})$$

with the *coboundary operator* $\delta_s(D \oplus u) = (\nabla''D, \nabla''u + (-1)^{q-1}Ds)$ and the *hypercohomology group*

$$\mathbb{H}^q_{I,s}(M, \mathscr{D}^1).$$

Here, $\nabla'' := \frac{1}{2}(1 + iI)\nabla$ where ∇ is the connection of the prequantum line bundle $\mathscr{L}$. As in [Hi2] one can show:

STEP 2: *A connection on the sheaf* $\mathcal{Z} := p_*(\mathcal{L}^{\otimes k} \otimes \mathcal{K}^{\frac{1}{2}})$ *is given by a class*

$$A = A(\delta I, s) \in \mathbb{H}^1_{I,s}(M, \mathcal{D}^1)$$

depending smoothly on the parameters such that the symbol map is $\sigma(A) = [\frac{1}{2} i \delta I]$.

In our situation such a class can be found by exploiting the exact cohomology sequence

$$\mathbb{H}^0_{I,s}(M, \mathcal{D}^1) \longrightarrow \mathbb{H}^0_{I,s}(M, \mathcal{D}^2) \longrightarrow H^0(M, S^2TM) \xrightarrow{\delta^1} \mathbb{H}^1_{I,s}(M, \mathcal{D}^1)$$

and taking A to be $\frac{1}{4ki}\delta^1(G)$ with G the above holomorphic deformation tensor G. An elaborate calculation shows:

STEP 3: *The cohomology class* $\delta^1(G)$ *satisfies* $\sigma(\delta^1 G) = [-2k\delta I]$. *Hence, according to Step 2,* $\frac{1}{4ki}\delta^1(G)$ *defines a connection* D *on* $\mathcal{Z}$.

One reason for this condition to hold is the fact that the metaplectic version of the geometric quantization is used. It can be shown that without the "twist" of the prequantum line bundle $\mathcal{L}^{\otimes k}$ by $\mathcal{K}^{\frac{1}{2}}$ the corresponding terms of the Ricci tensor do not in general cancel so that the candidate $\delta^1 G$ does not provide a class of the required type in $\mathbb{H}^1_{I,s}(M, \mathcal{D}^1)$. This explains why the metaplectic quantization must be taken instead of the uncorrected geometric quantization.

It is a pure coincidence that in the dimension $n = 1$ ($= \dim X$) one does not need to consider the metaplectic correction. In fact, one knows $\mathcal{K} = (\mathcal{L}^{\vee})^{\otimes 2N}$ in that case, where $\mathcal{L}^{\vee}$ denotes the dual of $\mathcal{L}$, and therefore

$$\mathcal{L}^{\otimes k} \cong \mathcal{L}^{\otimes(k+N)} \otimes \mathcal{K}^{\frac{1}{2}}$$

As a consequence, ignoring the metaplectic correction in the one-dimensional case just amounts to a shift $k \longmapsto k + N$ (cf. [**Wit**], p. 362 with $N = 2$). Reversing the arguments, the shift occuring in the above mentioned article of Witten and in other publications can be explained by an incomplete quantization procedure. This is in contrast to the usual explanations in which the "shift" is attributed to an "anomaly" in the regularization of the corresponding quantum field theory. In a combination of these arguments one could view the metaplectic quantization as the correct quantization procedure which automatically avoids a possible anomaly.

Having found a connection D on the sheaf $\mathscr{X}$, the first part of the main theorem is already established: $\mathscr{X}$ has to be a holomorphic vector bundle. It remains to show that D is projectively flat. This is the main part of the proof. First of all, as in [Hi2] the connection D has a local description by holomorphic heat operators with the deformation tensors determining the part of order 2. Hence, the commutators are of order 3 at first sight. But by the follwing step they are only of order 2.

STEP 4: *Given any two infinitesimal deformations of the family* $p : \mathfrak{M} \longrightarrow B$, *the corresponding deformation tensors* $G, G' \in H^0(M, S^2TM)$ *Poisson commute as (quadratic) functions on the cotangent bundle* T^*M: $\{G, G'\} = 0$.

Finally, the connection is projectively flat since there exist no global second order operators except for the constant operators:

STEP 5: $H^0(M, \mathscr{D}^2) = \mathbb{C}$, *if the aforementioned technical condition is satisfied.*

In order to show Step 5 one studies the convolution with ω_M on the symmetric tensors on M: $\Lambda_k : H^0(M, S^kTM) \longrightarrow H^1(M, S^{k-1}TM)$ given by $\vartheta \longmapsto k[\omega_M] * \vartheta$. The Step 5 and hence the proof of the Main Theorem can now be reduced to the following

STEP 6: $H^0(M, TM) = 0$ *and the convolutions* Λ_k *are injective for* $k > 0$ *(again under the assumption of the technical condition).*

To show Step 5 under the assumption of the results of Step 6, consider the exact sequence

$$H^0(M, \mathscr{D}^1) \longrightarrow H^0(M, \mathscr{D}^2) \longrightarrow H^0(M, S^2TM) \xrightarrow{\delta^1} H^1(M, \mathscr{D}^1).$$

Similar to Step 3 one can check that $\sigma \circ \delta^1 = k \Lambda_2$. Thus, the connecting homomorphism δ^1 is injective by Step 6. By the exactness of the sequence this implies that every global second order operator is of first order already. But by the exact sequence

$$H^0(M, \mathscr{O}_M) \longrightarrow H^0(M, \mathscr{D}^1) \longrightarrow H^0(M, TM) \longrightarrow H^1(M, \mathscr{O}_M)$$

and the vanishing of $H^0(M,TM)$, any global first order operator is constant.

Last not least, the essential step 6 can be derived from a general reinterpretation of the moduli space M and its cotangent bundle T^*M as part of a moduli space of Higgs bundles on X. This will be described in the next section.

10. Moduli Spaces of Higgs Bundles as Completely Integrable Systems.

Let again (X,ω_X) be a compact algebraic Kähler manifold and P a differentiable SU(N)-principal bundle such that the complex vector bundle E associated to P by the standard matrix representation has vanishing Chern classes. Let $\mathscr{A}$ be the affine space of SU(N)-connections with translation vector space $\mathscr{A}^1(X,\mathrm{Ad}\,P)$, the space of $\mathfrak{g}$-valued 1-forms. $\mathscr{A}$ is endowed with the natural complex structure $\mathbf{I} : T_{A'}\mathscr{A} = \mathscr{A}^1(X,\mathrm{Ad}\,P) \longrightarrow \mathscr{A}^1(X,\mathrm{Ad}\,P)$ given by $A \longmapsto A\circ I_X^{-1} = -A\circ I_X$, when I_X is the integrable almost complex structure $I_X : TX \longrightarrow TX$ of X. Let $M \neq \emptyset$ be a non-singular compact component of the moduli space of semi-stable holomorphic vector bundles on X of topological type E and trivial determinant $\mathcal{O}_X$.

Motivated by the one-dimensional situation **[Hi2]** one wants to have a presentation of the cotangent bundle T^*M as a Marsden-Weinstein quotient of $T^*\mathscr{A}$ with respect to the complexified gauge symmetry $\mathscr{G}^{\mathbb{C}}$, where $\mathscr{G}$ is, as before, the group $\mathscr{G} = \mathrm{Aut}\,P$ of (unitary) gauge transformations.

In order to do this let $P^{\mathbb{C}}$ denote the usual complexification of P as a principal bundle with structure group $SL(N,\mathbb{C})$ and consider the affine space $\mathscr{C} = \mathscr{A}^{\mathbb{C}}$ of connections on $P^{\mathbb{C}}$. Any connection D in $\mathscr{C}$ splits uniquely into

$$D = d_A + i\eta,$$

with d_A the unitary part (according to the decomposition of Lie $SL(N,\mathbb{C})$ into $\mathfrak{g} + i\mathfrak{g}$ for $\mathfrak{g} = \mathrm{Lie}\,SU(N)$). Therefore, the real tangent space $T_D\mathscr{C}$ of $\mathscr{C}$ can be represented as $T_D\mathscr{C} = \{(\alpha,\beta) : (\alpha,\beta) \in \mathscr{A}^1(X,\mathrm{Ad}\,P)\}$ and $\mathscr{C}$ has the natural Riemannian metric

$$g((\alpha\,\beta),(\alpha',\beta')) := -\int_X \mathrm{Tr}(\alpha\wedge *\alpha' + \beta\wedge *\beta').$$

As an additional special feature $\mathscr{C}$ has a natural Hyperkähler structure (cf. [HKLR] for basic properties of Hyperkähler manifolds): There are three complex structure **I**, **J**, **K** on the affine space $\mathscr{C}$ with $\mathbf{I}\,\mathbf{J} = \mathbf{K}$ and corresponding Kähler forms $\omega_\mathbf{I}$, $\omega_\mathbf{J}$ and $\omega_\mathbf{K}$ with respect to g:

$$\begin{aligned}\mathbf{I}(\alpha,\beta) &= (-\alpha\circ I_X, \beta\circ I_X)\\ \mathbf{J}(\alpha,\beta) &= (\beta,-\alpha)\\ \mathbf{K}(\alpha,\beta) &= \mathbf{I}\,\mathbf{J}(\alpha,\beta) = -(\beta\circ I_X, \alpha\circ I_X).\end{aligned}$$

PROPOSITION: *The action of the (unitary) gauge group $\mathscr{G}$ on the affine space $\mathscr{C}$ is compatible with all three complex structures and induces three different moment maps which describe various interesting constraints:*

$$\begin{aligned}m_\mathbf{I}(D) &= (-F_A + \tfrac{1}{2}\eta\wedge\eta)\wedge{\omega_X}^{n-1} &&= -\mathrm{Re}\, D^2\wedge{\omega_X}^{n-1}\\ m_\mathbf{J}(D) &= d_A{*}\eta &&= 2\mathrm{Re}\, D''^2\wedge{\omega_X}^{n-1}\\ m_\mathbf{K}(D) &= d_A\eta\wedge{\omega_X}^{n-1} &&= 2\mathrm{Im}\, D''^2\wedge{\omega_X}^{n-1},\end{aligned}$$

Here $D = d_A + i\eta = \partial_A + \Theta + \bar\partial_A + \bar\Theta$, *with* $F_A = F(d_A)$ *the curvature of* d_A *and* $D'' = \bar\partial_A + \Theta$ *as in Simpson's papers (cf. e.g. in* [Si3]).

The quotient $\mathscr{C}_0/\mathscr{G}^{\mathbb{C}}$ with $\mathscr{C}_0 := \{D\in\mathscr{G}:\ D^2 = 0\}$ is the moduli space of flat connections or local systems. According to a result of Corlette, the semi-stable part of this quotient is the Marsden-Weinstein quotient with respect to the moment map $m_\mathbf{J}$. (Here a flat SL(N,$\mathbb{C}$)-connection is *stable* (*semi-stable*) if its holonomy representation is irreducible (semi-simple).)

PROPOSITION [Cor]: *There is a homeomorphism* $(\mathscr{C}_0/\mathscr{G}^{\mathbb{C}})^{ss} \cong m_\mathbf{J}^{-1}(0)/\mathscr{G}$. *On any semi-stable flat* SL(N,$\mathbb{C}$)*-connection there exists a (harmonic) metric such that for the above decomposition of* D *one has*

$$D''^2\wedge{\omega_X}^{n-1} = 0.$$

Now let us consider the condition $D''^2 = 0$. This is equivalent to the three equations $\bar\partial_A{}^2 = 0$, $\bar\partial_A(\Theta) = 0$ and $\Theta\wedge\Theta = 0$. By definition, a *Higgs bundle* is a holomorphic vector bundle $\mathscr{E}$ on X together with a holomorphic $\mathrm{End}\,\mathscr{E}$-valued 1-form $\Theta\in H^0(X,\mathrm{End}\,\mathscr{E}\otimes T^*X)$ satisfying $\Theta\wedge\Theta = 0$. Then ${\mathscr{C}_0}'' := \{D\in\mathscr{C}:\ D'^2 = 0\}$ is the set of all Higgs bundle structures on E. The notion of *stability* (*semi-stability*) of ordinary vector bundles carries over to the case of Higgs bundles. The following is similar to Corlette's result:

PROPOSITION [**Si3**]: *The moduli space* $(\mathscr{E}_0''/\mathscr{G}^{\mathbb{C}})^{ss}$ *of semi-stable Higgs bundles with* E *the underlying differentiable bundle is homeomorphic to* $m_I^{-1}(0)/\mathscr{G}$. *On any semi-stable Higgs bundle there exists a metric such that for the corresponding operator* D *one has*

$$D^2 \wedge \omega_X{}^{n-1} = 0.$$

A comparison of all these construction leads to

PROPOSITION [**Sch**]: *There are homeomorphisms*

$$\mathrm{Hom}^{ss}(\pi_1(X), SL(N,\mathbb{C}))/SL(N,\mathbb{C}) \cong (\mathscr{E}_0''/\mathscr{G}^{\mathbb{C}})^{ss} \cong (\mathscr{E}_0/\mathscr{G}^{\mathbb{C}})^{ss}$$

and the corresponding space, denoted by $\mathbb{M}$ *in the following, inherits a Hyperkähler structure from* $\mathscr{E}$.

The existence of the Hyperkähler structur has also been shown by Fujiki [**Fuj**]. To explain the proposition in a more detailed manner, let us consider that component of $(\mathscr{E}_0''/\mathscr{G}^{\mathbb{C}})^{ss}$ which contains the cotangent bundle T^*M to the smooth component M of the Main Theorem. This component is itself smooth (at least in the elliptic surface case) and is denoted by $\mathbb{M}$. T^*M is then a dense and open subspace of $\mathbb{M}$. Let Ω denote the sheaf of germs of holomorphic 1-forms on X.

DEFINITION: The *Hitchin map* is the map

$$H : \mathbb{M} \longrightarrow V := H^0(X, S^2\Omega) \oplus H^0(X, S^3\Omega) \oplus \dots \oplus H^0(X, S^N\Omega),$$

$$(\mathscr{E}, \Theta) \longmapsto (a_2(\Theta), \dots, a_N(\Theta)),$$

assigning to each Higgs bundle the coefficients of the characteristic polynomial of the (traceless) endomorphism Θ, i.e. $a_2(\Theta) = \frac{1}{2}((\mathrm{Tr}\,\Theta)^2 - \mathrm{Tr}(\Theta^2))$, ... , $a_N(\Theta) = \det\Theta$.

The important Step 6, for which all this machinary has been developed, is essentially a consequence of the following key result:

PROPOSITION [**Sch**]: $\mathbb{M}_I$ *with the holomorphic 2-form* $\omega := \omega_K - i\omega_J$ *is a completely integrable system with the components* H_j *of the Hitchin map* $H = (H_1, H_2, \dots, H_m)$ *as the Poisson commuting functions (i.e. the constants of motion in involution), at least* $\frac{1}{2}\dim\mathbb{M}$ *of which are generically independent.*

From this proposition one can deduce the property of Step 4 in the previous section. Moreover, if the codimension of T^*M in $\mathbb{M}$ is small, one also obtains the required vanishing results of Step 6 as a consequence of the following result.

PROPOSITION [**Sch**]: *Assume*

$$\textbf{(cod1)}\quad \mathrm{codim}\,(T^*M, \mathbb{M}) > 1.$$

Then one gets the natural isomorphisms

$$H^0(\mathbb{M},\mathcal{O}) \cong \bigoplus_{k\geq 0} H^0(M,S^kTM)$$
$$H^1(\mathbb{M},\mathcal{O}) \cong \bigoplus_{k\geq 0} H^1(M,S^kTM)$$

by extension. Similarly, the "convolution" $\Lambda : H^0(\mathbb{M},\mathcal{O}) \longrightarrow H^1(\mathbb{M},\mathcal{O})$ *mapping* $f \in H^0(\mathbb{M},\mathcal{O})$ *to the class* $[i_{H_f}\omega_I]$ *where* H_f *is the Hamiltonian vector field with respect to the form* ω, *is induced by the corresponding convolutions* Λ_k *on* M, *and satisfies* $\mathrm{Ker}\,\Lambda = \mathbb{C}$. *In particular,* Λ_k *is injective for* $k > 0$, *and* $H^0(M,TM)$ *vanishes.*

The *technical condition* referred to in the main theorem and in the steps 5 and 6 is

$$\textbf{(cod1)}\quad \mathrm{codim}\,(T^*M, \mathbb{M}) > 1.$$

11. Final Remarks

The formulation of the Main Theorem suggests several natural questions:

1. Do there exist non-singular compact components M of the moduli space $H^1(X,SU(N))$?

2. Under which conditions does the first Betti number $b_1(M)$ vanish?

3. Does M possess a spin structure?

4. Under which circumstances is the technical condition **(cod1)** $\mathrm{codim}\,(T^*M, \mathbb{M}) > 1$ satisfied?

5. Can the results be extended to non-singular components of the moduli space?

These questions are investigated in detail by Scheinost [**Sch**]. As a general rule, in the elliptic surface case X coming from a smooth curve Σ with marked points, the questions 1, 2, 4, 5 have a positive answer in the sense that the Main Theorem (or a generalization thereof) applies to the topological field theory such that the quantum Hilbert spaces $Z(S,P,R)$ which one wants to construct are well-defined for most of the cases.

There is, however, a serious exception, namely M does not in general have a spin structure. The existence of a square root of the canonical bundle on M depends essentially on relations among the numbers N (= rank of the bundles), p (= number of marked points) and d ($= \sum_i d_i$ cf. sections 7, 8). For example M does not possess a spin structure for an even rank N if $p = d = 1$. On the other hand, a spin structure does exist e.g. for N odd if d and N are coprime (and g is not too small). For the construction of the knot polynomials from the topological field theory the exceptions for which there is no spin structure on M presumably will not matter, since there are enough cases for which M has a spin structure and the Main Theorem applies. Note, that one can e.g. avoid the case $p = 1$ by considering only a suitable subclass of all "cuts" of a three-manifold containing a link.

There seems to be also a procedure to construct the quantum Hilbert spaces $Z(S,P,R)$ without using the elliptic surface trick and, hence, avoiding the higher dimensional theory altogether, by employing the description of a symplectic structure on the moduli spaces of parabolic bundles recently given by Biswas and Guruprasad [**BiG**].

Concerning the fifth question there is again a codimension condition under which there is a rather straightforward answer. Let M be a (possibly non-singular) component of the moduli space $H^1(X,SU(N))$. If the codimension of the regular part of M is not too small (namely > 2) then the Main Theorem holds for M. This can be shown by analytic continuation from the regular part of M to all of M. In particular, in the elliptic surface case X over a marked curve Σ the convolution $\Lambda : H^0(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O})$ on the Higgs moduli space (cf. the last proposition in the previous section) is not only injective but even an isomorphism for all components M of the moduli space.

CONCLUSION:

The goal of presenting a rigorous foundation of Witten's geometric interpretation of the Jones polynomial has been achieved in parts by the Main Theorem of this paper: The quantum spaces $Z(S,P,R)$ are essentially a result of the metaplectic quantization procedure. However, what is completely missing so far is a rigorous quantization method to obtain the ("expectation value") vectors $Z(Y,L,R)$ contained in the vector spaces $Z(\partial Y, L\cap\partial Y, R\cap\partial Y)$ (cf. section 7).

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