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Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1993, tome 45
« Conférences de P. Cartier, P. Di Francesco, J. Fröhlich, P. Hello, Ch. Kassel, V. Kharlamov, B. Khesin, J. Magnen, M. Rabaud, M. Schottenloher », , exp. nº 2, p. 11-25

<http://www.numdam.org/item?id=RCP25_1993__45__11_0>

Observables in the Kontsevich Model

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Kontsevich introduced a hermitian random matrix model to compute the generating function of intersection numbers of the moduli space of (punctured) Riemann surfaces. He showed that this generating function is also a τ–function for the Korteveg–de Vries (KdV) hierarchy of differential equations. This model is fundamentally different from the usual double scaling limit of random matrix models known to yield analogous τ–functions. Our aim is to clarify the notion of "observables" in both pictures, as related to KdV time evolutions. As a result we prove two conjectures by Kontsevich and Witten about the form of these observables, which involve polynomial matrix averages.

* Conférence de la R.C.P. No 25, I.R.M.A. Strasbourg, 3-5 décembre 92.

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Ce texte reprend une prépublication du Service de Physique Théorique de Saclay de Décembre 1992 - SPhT 92/174

1. Introduction

The matrix model formulation of two dimensional quantum gravity coupled to matter (with central charge c less than 1) enables to express the physical properties of the solutions in a very simple and elegant way [1]. Let us recall the original definition. One starts with a $N \times N$ hermitian matrix model, with partition function :

$$\mathcal{Z}(\mathcal{V}, N) = \int dM \exp -Ntr\mathcal{V}(M) \tag{1.1}$$

where $dM = (2\pi)^{-N^2/2} \prod_{i<j} dRe(M_{ij}) \ dIm(M_{ij}) \prod_i dM_{ii}$ is the Haar measure over hermitian matrices, and $\mathcal{V}$ some polynomial potential. Roughly speaking the Feynman diagrammatic expansion of (1.1) simulates all possible polygon decompositions of Riemann surfaces (i.e. gravitational fluctuations) while the precise form of the potential indicates which type of matter interactions are involved. The partition function (1.1) can be calculated explicitly using orthogonal polynomial techniques, but the main progress was to consistently define the so called double scaling limits of the orthogonal polynomial solution. On the one hand the large N limit was already known to concentrate on surfaces with spherical topology, wheras a large N expansion would give access order by order to surfaces with higher genera [2]. The idea is to combine a large N limit with a critical limit in which the parameters of the potential are tuned to make it (multi)critical as N grows. This has the advantage of capturing the whole genus expansion into a single asymptotic series of a rescaled variable x (the renormalized cosmological constant), kept fixed while N is sent to infinity and the potential is taken to a (multi)critical value. The orthogonal polynomial solution becomes in this limit an ordinary differential equation for the double scaled string susceptibility $u(x) = \partial_x^2 \log \tau(x)$, where $\log \tau(x)$ is the double scaled free energy :

$$\log \tau(x) = \lim_{\substack{N \to \infty \\ \mathcal{V} \to \mathcal{V}^*; x \ fixed}} -\frac{1}{2N^2} \log \mathcal{Z}(\mathcal{V}, N) \tag{1.2}$$

If instead of taking a special m–critical potential $\mathcal{V}_m^*$, we consider a linear combination of them for various m's with coefficients t_m, we end up with a function $u(x, t_i) = \partial_x^2 \log \tau(x, t_i)$. The double-scaled solution is then simply characterized by the fact that (i) $\tau(x, t_i)$ is a τ–function for the Korteveg–de Vries (KdV) hierarchy and (ii) $\tau(x, t_i)$ satisfies an ordinary differential equation (the so called "string equation"). Upon introducing the differential operator $Q = d^2 + u(x, t_i)$, where d stands for ∂_x, and the fractional powers of Q truncated to their differential piece $Q_+^{m+1/2}$, the two above properties become :

$$\begin{aligned} &(i) \qquad \partial_{t_m} Q = [Q_+^{m+1/2}, Q] \\ &(ii) \;\; P = \sum_{j>0} (j+1/2) t_j Q_+^{j+1/2} \; ; \; [P, Q] = 1 \end{aligned} \tag{1.3}$$

The string equation (ii) can be viewed as an interpolating equation of motion between the m–critical points $t_i = \delta_{i,m}$, which correspond to $c = 1 - 3(2m-1)^2/(2m+1)$ conformal matter coupled to 2D quantum gravity. More generally (multi)critical (multi)matrix models are known to interpolate between conformal points with $c < 1$, the KdV flows (i) being replaced by generalized KdV flows and (ii) modified accordingly [3] (Q is now a differential operator of order p, for a model of $p-1$ matrices, and one considers all its fractional powers of the form $Q_+^{k/p}$ where k is not a multiple of p).

The KdV flows (i) and their generalizations enable therefore to move in the space of $c < 1$ matter coupled to 2D gravity, along RG trajectories. The latter are obtained in ordinary conformal theory by adding to the action perturbations by relevant operators, it is therefore tempting to identify the "dressed" operators of the conformal theory coupled to gravity as dual to the "KdV times" t_i. More precisely, the insertion of a dressed operator ϕ_m in a correlator will be generated by differentiation w.r.t. t_m : $\partial_{t_m}\langle ... \rangle = \langle \phi_m ... \rangle$. This definition of observables was successfully applied to one and more matrix models [4] and shown to confirm predictions from the continuum theory such as KPZ scaling dimensions of dressed operators [5], Liouville correlators [6], etc... Going back to the original matrix model for a while, we see that these observables correspond to the insertion of very specific polynomials $\mathcal{V}_m^*(M)$ into the defining integral. On the other hand, such polynomial insertions correspond in the Feynman expansion to the insertion of sources with vertices of a well defined order, or in the dual picture to the creation of microscopic holes in the (discretized) Riemann surface. In fact, the properties (i) and (ii) can be rephrased into equations of motion (or loop equations) for these so called loop operators, and take the form of Virasoro constraints [7] $L_m \tau = 0$, $m \geq -1$, L_m certain Virasoro generators constructed in terms of bilinears of t_i and ∂_{t_i}. The simplest of those observables is the "puncture" operator dual to $t_0 = x$, the renormalized cosmological constant.

Meanwhile after introducing topological gravity [8] and uncovering its relations to KdV hierarchies [9], Witten conjectured that the one matrix model partition function $\tau(x, t_i)$

could also be interpreted as the generating function for intersection numbers of the moduli space of (punctured) Riemann surfaces [10]. These intersection numbers $\langle \sigma_0^{n_0}..\sigma_p^{n_p} \rangle$ have a precise definition as integrals over a compactification of the moduli space of Riemann surfaces of genus g with n marked points of exterior powers of the first Chern class of the line bundle defined by the cotangent spaces at these points. We will not emphasize the topological aspect of this model here and refer the reader to ref. [10] for more details.

M. Kontsevich made this statement even deeper by introducing yet another matrix model of a very different nature [11], enabling to compute these intersection numbers directly. By interpreting his cell decomposition of the moduli space of (punctured) Riemann surfaces in terms of "fat graphs" he was able to write directly an ad-hoc hermitian matrix model whose connected partition function is exactly the generating function for the intersection numbers. Let $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_N)$ be a real diagonal matrix, the partition function reads :

$$\Xi_N(\Lambda) = \frac{\int dY \exp\ tr(iY^3/6 - \Lambda Y^2/2)}{\int dY \exp\ -tr(\Lambda Y^2/2)} \tag{1.4}$$

Kontsevich established that when expressed in the variables

$$t_j = -(2j-1)!!\ tr(\Lambda^{-2j-1}) \tag{1.5}$$

this function is an asymptotic series whose truncation to terms of degree less than N is a universal polynomial of the $t_.$'s. It admits a universal $N \to \infty$ limit $\Xi(t_.)$, whose connected piece $\log \Xi(t_.)$ is equal to the generating function $F(t_.)$ for intersection numbers :

$$F(t_.) = \sum_{n_j} \prod_j \frac{t_j^{n_j}}{n_j!} \langle \sigma_0^{n_0} ... \sigma_p^{n_p} \rangle \tag{1.6}$$

Using this correspondence, Witten was able to show using topological arguments [12] that $\Xi(t_.)$ satisfies the properties (i) and (ii) of (1.3), and is therefore equivalent to $\tau(x, t_i)$ $(x = t_0)$ [1]. Some generalizations of the Kontsevich model were also introduced and shown to satisfy generalized KdV time evolutions [11] [15]. Their topological interpretation was given in [16].

The obvious question in view of this equivalence is what about the observables? We just clarified their interpretation in the (double scaled) one matrix model as dual to the KdV times. Their topological counterpart is given by the first Chern class of the line

[1] See also [13] and [14] for alternative proofs.

bundle of the cotangent spaces at marked points. But Witten noticed on a few examples that the disconnected partition function $\Xi(t_.)$ could also give rise to interesting objects by differentiation w.r.t. KdV times. In a number of cases, he found that the action of some differential polynomials of the KdV times $R(\partial_{t_.})$ on $\Xi(t_.)$ could be rewritten as a kind of polynomial expectation value in the form :

$$R(\partial_{t_.})\ \Xi(t_.) = \frac{\int d\mu_\Lambda(Y) P(Y)\ \exp\ tr(iY^3/6)}{\int d\mu_\Lambda(Y)} = \ll P(Y) \gg \tag{1.7}$$

where $P(Y)$ is a certain polynomial of traces of odd powers of Y (odd traces for short), and $d\mu_\Lambda(Y) = dY \exp -tr(\Lambda Y^2/2)$ is the natural Gaussian measure of the problem. This led him to the conjecture that there exists a general mapping $R \to P$ defined on $\mathbb{C}[\partial_{t_0}, \partial_{t_1}, ...]$. Let us first discuss a few implications of this fact. First of all if the mapping can be made explicit, this gives in principle a straightforward way of computing any intersection number using also (1.6). Another important consequence is the definition of yet another kind of observables in the topological model. Those are very much like the ones of the one matrix model before the double scaling limit, and correspond to the insertion of vertices with a well defined number of legs in the Feynman diagrammatic expansion of $\Xi(t_.)$. The explicit mapping $R \to P$ yields rules for computing correlations of such observables.

The rest of the lecture will be dedicated to the proof of this conjecture and the explicit construction of the mapping $R \to P$ (see also [17] for a more detailed version). We will need a few definitions and preliminaries, and will first prove a weaker statement on Gaussian polynomial averages (sect.2), due to Kontsevich, who proved it by topological arguments. It involves the construction of another mapping defined on polynomials $P(Y)$ of odd traces of Y, by taking the average over the Gaussian measure $d\mu_\Lambda(Y)$. The result is that as a function of $t_.$ this average is still polynomial :

$$\langle P(Y) \rangle = \frac{\int d\mu_\Lambda(Y) P(Y)}{\int d\mu_\Lambda(Y)} = Q(t_.) \tag{1.8}$$

We will compute the mapping $P \to Q$ explicitly in a purely algebraic way, and be naturally led to introduce a new set of polynomials which generalize the ordinary Schur polynomials. These polynomials will be used in sect.3 to describe the Witten mapping $R \to P$. The essence of these proofs is extremely simple and relies mainly on comparisons between integrals over $N+1 \times N+1$ matrices and over $N \times N$ matrices.

2. The Kontsevich mapping

In this section, we construct the Kontsevich mapping $P \to Q$ explicitly, where $Q(\Lambda^{-1}) = \langle P(Y) \rangle$.

The main tool for working with matrix models is the formula of integration over the "angular variables" U, when one diagonalizes the hermitian matrix integration variable $Y = UyU^\dagger$, with $y = diag(y_1, .., y_n)$. The Haar measure decomposes into $dY = (2\pi)^{-N^2/2} \prod dy_i \; dU \; \Delta(Y)^2$, where $\Delta(Y) = \prod_{i>j}(y_i - y_j)$ is the Vandermonde determinant of Y. The Harish–Chandra–Itzykson–Zuber formula [18] reads, for any two diagonal matrices x and y :

$$\int dU \exp \; tr(UxU^\dagger y) \propto \frac{\det[e^{x_i y_j}]_{1 \leq i,j \leq N}}{\Delta(x)\Delta(y)} \tag{2.1}$$

up to an irrelevant numerical factor depending on N only (in the following we omit most of these cumbersome factors and use the symbol $\propto$ to indicate their presence). This formula happens to be a simple case of the Duistermaat–Heckman integration formula [19] : (2.1) expresses nothing but the fact that the semi–classical approximation to the integral is exact (the classical solutions for the potential $tr(UxU^\dagger y)$ are just permutations, and the inverse Vandermonde determinants arise from the Gaussian integral)[2]. This result enables to restrict most of the interesting matrix integrals to integrals over eigenvalues. Let us use it to rewrite the Gaussian integral :

$$Z_N = \int d\mu_\Lambda(Y) \propto \int \frac{\prod dy_i}{\Delta(\Lambda)} \prod_{i<j} \frac{y_i + y_j}{y_i - y_j} e^{-\sum \lambda_i y_i^2/2} \tag{2.2}$$

where we dropped the determinant symbol by noticing the antisymmetry of the prefactor. On the other hand this integral is easily computed by direct integration :

$$Z_N \propto \det(\Lambda)^{-1/2} \frac{\Delta(\Lambda)}{\Delta(\Lambda^2)} \tag{2.3}$$

Consider now the $N+1 \times N+1$ version of (2.2), this amounts to introducing additional eigenvalues y and λ to Y and Λ respectively and we compute :

$$\frac{Z_{N+1}}{Z_N} \det(\lambda - \Lambda) = \lambda^{-1/2} \det \frac{1 - \lambda\Lambda^{-1}}{1 + \lambda\Lambda^{-1}} = \int \frac{dy}{(2\pi)^{1/2}} e^{-\lambda y^2/2} \langle \det \frac{y - Y}{y + Y} \rangle \tag{2.4}$$

[2] I thank J.-B. Zuber for explaining this unpublished result to me.

Let us take a closer look to the integrand of (2.4). It involves the determinant :

$$f_Y(y) = \det \frac{y-Y}{y+Y} = \exp -2\sum_{k=0}^{\infty} y^{-2k-1} tr(Y^{2k+1})/(2k+1) = \sum_{m=0}^{\infty} y^{-m}\, p_m(Y) \tag{2.5}$$

where p_m denote the Schur polynomials of the odd variables :

$$\theta_{2i+1}(Y) = -\frac{2}{2i+1} tr(Y^{2i+1}) \tag{2.6}$$

$$p_m = \sum_{\substack{\nu_{2j+1}\geq 0 \\ \sum_{j\ odd} j\nu_j = m}} \prod_{j\ odd} \frac{\theta_j^{\nu_j}}{\nu_j!} \tag{2.7}$$

Therefore (2.4) gives us some expressions for the Gaussian averages of these polynomials. But due to the divergence of the y integral, we need to perform some analytic continuation.

To avoid this difficulty, let us compute directly as a function of the formal variable y the following Gaussian average over Y :

$$\langle f_Y(y)\rangle = 1 - 2\sum_{i=1}^{N} \langle \frac{y_i}{y+y_i} \prod_{j\neq i} \frac{y_i+y_j}{y_i-y_j} \rangle \tag{2.8}$$

where we performed a decomposition of f defined in (2.5) into fractions with simple poles at $y = -y_i$. Integrating over the angular variables, we are left with

$$\langle f_Y(y)\rangle = 1 + 2\sum_{m=1}^{\infty} (-y)^{-m} \sum_{i=1}^{N} \int dy_i (\frac{\lambda_i}{2\pi})^{1/2} (-1)^{i-1} y_i^m e^{-\lambda_i y_i^2/2} \frac{\Delta(\Lambda_i) Z_{N-1}(\Lambda_i)}{\Delta(\Lambda) Z_N(\Lambda)} \tag{2.9}$$

where we denoted by $X_i = diag(x_1, .., x_{i-1}, x_{i+1}, .., x_N)$ for any diagonal matrix X. Note that we singled out the integration over y_i and that the integration over the other y's just recombined to yield $Z_{N-1}(\Lambda_i)$. Using the one dimensional integral $\int dz/(2\pi)^{1/2} z^{2m} e^{-\lambda z^2/2} = \lambda^{-m-1/2}(2m-1)!!$, we get :

$$\langle f_Y(y)\rangle = 1 + 2\sum_{m=1}^{\infty} (-1)^{N-1} (2m-1)!! y^{-2m} \sum_{i=1}^{N} \lambda_i^{-m} \prod_{j\neq i} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \tag{2.10}$$

We now use the decomposition of $f_{\Lambda^{-1}}(\lambda)$ into fractions with single poles at $\lambda = -\lambda_i$ to identify for $m \geq 1$:

$$p_m(\Lambda^{-1}) = 2(-1)^{N+m-1} \sum_{i=1}^{N} \lambda_i^{-m} \prod_{j\neq i} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \tag{2.11}$$

and finally :

$$\langle f_Y(y)\rangle = \sum_{k=0}^{\infty} y^{-k}\langle p_k(Y)\rangle = 1 + \sum_{m=1}^{\infty} y^{-2m}(-1)^m(2m-1)!!p_m(\Lambda^{-1}) \tag{2.12}$$

So we get the first elements of the mapping $P \to Q$, namely

$$\begin{aligned} (i) \quad & \langle p_{2m+1}(Y)\rangle = 0 \\ (ii) \quad & \langle p_{2m}(Y)\rangle = (-1)^m(2m-1)!!\; p_m(\Lambda^{-1}) \end{aligned} \tag{2.13}$$

To proceed, we need to consider more complicated polynomials, like products of p_m's, known to generate the space of polynomials of odd variables $\theta_.$'s. One can think of performing an average of the form (2.8), but in the presence of a "spectator" insertion of $p_k(Y)$. Actually, it is easy to see that it is more useful to consider :

$$\langle f_Y(y) p_k(\theta_.(Y) + \theta_.(y))\rangle \tag{2.14}$$

If one decomposes f into fractions with single poles at $y = -y_i$ as in (2.8), then we see that the "spectator" $p_k(\theta_.(Y) + \theta_.(y)) = p_k(\theta_.(Y_i)) + O(y + y_i)$ can be replaced with $p_k(\theta_.(Y_i))$ if we retain only terms y^{-k}, $k > 0$ in the formal y expansion. Therefore, we can go through the previous steps, with the only modification :

$$\begin{aligned} \langle f_Y(y) p_m(\theta_.(Y) + \theta_.(y))\rangle_{<0} = 2\sum_{m=1}^{\infty}(-y)^{-m}\sum_{i=1}^{N}\int dy_i \left(\frac{\lambda_i}{2\pi}\right)^{1/2}(-1)^{i-1} y_i^m e^{\lambda_i y_i^2/2} \times \\ \times \frac{\Delta(\Lambda_i) Z_{N-1}(\Lambda_i)\langle p_k(\theta_.(Y_i))\rangle}{\Delta(\Lambda) Z_N(\Lambda)} \end{aligned} \tag{2.15}$$

where the subscript < 0 indicates that we truncate the y expansion to negative powers only. We use the result (2.13) to compute $\langle p_k(\theta_.(Y_i))\rangle = (k-1)!!(-1)^{k/2} p_{k/2}(\Lambda_i^{-1})$, with the convention that $(k-1)!! = 0$ when k is odd. Performing the integration over y_i, we get a generalization of (2.10) :

$$2\sum_{m=1}^{\infty}(-1)^{N-1}(2m-1)!! y^{-2m}(k-1)!!(-1)^{k/2}\sum_{i=1}^{N}\lambda_i^{-m} p_{k/2}(\Lambda_i)\prod_{j\neq i}\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \tag{2.16}$$

We recognize the general term of the series generated by expanding the decomposition of $f_{\Lambda^{-1}}(\lambda) p_{k/2}(\theta_.(\Lambda^{-1}) + \theta_.(\lambda^{-1}))$ into rational fractions with single poles at $\lambda = -\lambda_i$. If we introduce a generating function (for $z > y$) :

$$f_Y(z,y) = \sum_{m\in\mathbb{Z}} z^{-m} p_m(\theta_.(Y)+\theta_.(y)) \det \frac{y-Y}{y+Y}$$
$$= \frac{z-y}{z+y} \det \frac{(y-Y)(z-Y)}{(y+Y)(z+Y)} \tag{2.17}$$
$$= \sum_{m,n\in\mathbb{Z}} z^{-m} y^{-n} \varphi_{m,n}(Y)$$

then the result takes the simple form :

$$\begin{aligned} &(i) \quad \langle \varphi_{m,n}(Y)\rangle = 0 \quad \text{if } m \text{ or } n \text{ is odd} \\ &(ii) \quad \langle \varphi_{2m,2n}(Y)\rangle = (-1)^{m+n}(2m-1)!!(2n-1)!!\ \varphi_{m,n}(\Lambda^{-1}) \end{aligned} \tag{2.18}$$

Strictly speaking, we only proved it for $m,n \geq 1$, as we concentrated on the negative power expansion in y. But from the definition (2.17) it is easy to see that $\varphi_{m,0} = \varphi_m = p_m$ if $m \geq 0$, 0 if $m < 0$, that $\varphi_{m,-n} = 0$ for $n > 0$, and that $\varphi_{-m,n} = 2(-1)^m \delta_{m,n}$ for $m,n > 0$. Then if we define $(-2m-1)!! \equiv (-1)^m/(2m-1)!!$ for $m > 0$, (2.18) holds for any $m,n \in \mathbb{Z}$.

In view of the above, it is clear that the general Kontsevich mapping will follow from an analogous treatment of the Gaussian average over Y of

$$f_Y(z_1,..,z_p) = \prod_{a<b} \frac{z_a - z_b}{z_a + z_b} \det \prod_{a=1}^{p} \frac{z_a - Y}{z_a + Y} = \sum_{m_1,..,m_p\in\mathbb{Z}} \prod_a z_a^{-m_a}\ \varphi_{m_1,..,m_p}(Y) \tag{2.19}$$

understood as a formal series of the variables z_a in a domain where, say, $z_1 > z_2 > .. > z_p$. The polynomials $\varphi_{m_1,..,m_p}(Y)$ generalize the Schur polynomials, and generate the whole set of polynomials of the odd variables $\theta_.(Y)$. Actually a basis is formed by the φ's with ordered indices $m_1 > m_2 > .. > m_p \geq 1$, $p \geq 0$. After some algebra one gets the Gaussian averages of these basis elements, defining the Kontsevich map [3] :

$$\begin{aligned} &(i) \quad \langle \varphi_{m_1,..,m_p}(Y)\rangle = 0 \quad \text{if at least one of the } m_i \text{ is odd} \\ &(ii) \quad \langle \varphi_{2m_1,..,2m_p}(Y)\rangle = \prod_{i=1}^{p} (-1)^{m_i}(2m_i-1)!!\ \varphi_{m_1,..,m_p}(\Lambda^{-1}) \end{aligned} \tag{2.20}$$

[3] See [17] for a detailed proof. An inductive proof can also be made along the lines of the case $p=2$ treated above, involving a "spectator" insertion of $\varphi_{m_1,..,m_p}(\theta_.(Y)+\theta_.(y))$ into the Gaussian average of $f_Y(y)$.

The φ's are easily computed from the definition (2.19), which can be recast into the following recursive formula (by convention, the φ with no index is the constant 1) :

$$\varphi_{m_1,..,m_p,m} = \sum_{s\geq 0} \varphi_{m-s} \sum_{\substack{r_1,..,r_p\geq 0 \\ r_1+..+r_p=s}} \left(\prod_{i=1}^{p} \alpha_{r_i}\right) \varphi_{m_1+r_1,..,m_p+r_p} \tag{2.21}$$

where $\alpha_r = (-1)^r(2-\delta_{r,0})$ are the coefficients of the expansion $(1-y)/(1+y) = \sum_{r\geq 0} \alpha_r y^r$, and $\varphi_m = p_m$ if $m \geq 0$, and vanishes for $m < 0$. We list below the first few φ's with positive ordered indices.

φ_1	$= \theta_{[1^1]}$
φ_2	$= \theta_{[1^2]}$
φ_3	$= \theta_{[1^3]} + \theta_{[3^1]}$
$\varphi_{2,1}$	$= \theta_{[1^3]} - 2\theta_{[3^1]}$
φ_4	$= \theta_{[1^4]} + \theta_{[1^1 3^1]}$
$\varphi_{3,1}$	$= 2\theta_{[1^4]} - \theta_{[1^1 3^1]}$
φ_5	$= \theta_{[1^5]} + \theta_{[1^2 3^1]} + \theta_{[5^1]}$
$\varphi_{4,1}$	$= 3\theta_{[1^5]} - 2\theta_{[5^1]}$
$\varphi_{3,2}$	$= 2\theta_{[1^5]} - \theta_{[1^2 3^1]} + 2\theta_{[5^1]}$
φ_6	$= \theta_{[1^6]} + \theta_{[1^3 3^1]} + \theta_{[1^1 5^1]} + \theta_{[3^2]}$
$\varphi_{5,1}$	$= 4\theta_{[1^6]} + \theta_{[1^3 3^1]} - \theta_{[1^1 5^1]} - 2\theta_{[3^2]}$
$\varphi_{4,2}$	$= 5\theta_{[1^6]} - \theta_{[1^3 3^1]} + 2\theta_{[3^2]}$
$\varphi_{3,2,1}$	$= 2\theta_{[1^6]} - \theta_{[1^3 3^1]} + 2\theta_{[1^1 5^1]} - 4\theta_{[3^2]}$

Table I : the φ polynomials up to degree 8. The notation $\theta_{[1^{\nu_1}3^{\nu_3}\ldots(2k+1)^{\nu_{2k+1}}..]}$ is a short hand for $\frac{\theta_1^{\nu_1}}{\nu_1!}\frac{\theta_3^{\nu_3}}{\nu_3!}\cdots\frac{\theta_{2k+1}^{\nu_{2k+1}}}{\nu_{2k+1}!}\cdots$.

3. The Witten mapping

We want to investigate the mapping $R \to P$, where, hopefully for any polynomial $R(\partial_{t_.})$ one can find a polynomial P of odd traces of Y, such that $R(\partial_{t_.})\Xi(t_.) =\ll P(Y) \gg$, where the double bracket denotes the weighted average (1.7). Starting from the partition function $\Xi(\Lambda^{-1})$ (this notation is just to recall that the $t_.$'s are themselves normalised odd traces of Λ^{-1}, and indicates that Ξ is also a function of the odd variables $\theta_{2j+1}(\Lambda^{-1})$), one can generate differentiations w.r.t. odd traces of Λ^{-1} by just expanding :

$$\Xi(\lambda^{-1}\oplus\Lambda^{-1}) = \exp\left(\sum_{i=0}^{\infty} -\frac{2\lambda^{-2j-1}}{2j+1}\frac{\partial}{\partial\theta_{2j+1}}\right)\Xi(\Lambda^{-1}) = \sum_{k=0}^{\infty}\lambda^{-k}p_k(\partial_.)\,\Xi \tag{3.1}$$

where $\lambda^{-1}\oplus\Lambda^{-1} = diag(\lambda^{-1},\lambda_1^{-1},\lambda_2^{-1},..)$, and the r.h.s. is a formal power series of λ^{-1}, which is the generating function of the Schur polynomials of the odd derivatives $\partial_{2j+1} = -(2/2j+1)\partial_{\theta_{2j+1}}$ acting on Ξ, as a function of the infinitely many variables θ_{2j+1}. To evaluate this action in terms of matrix averages, we will have to compare $\Xi_{N+1}(\lambda^{-1}\oplus\Lambda^{-1})$ to $\Xi_N(\Lambda^{-1})$. To get the most general action on Ξ of polynomials of derivatives w.r.t. $\theta_.$'s, we can just add p eigenvalues $\lambda_1,..,\lambda_p$ to Λ, and expand Ξ as a formal series of λ_i^{-1}. By analogy with the situation of previous section, it is very natural to consider the generating function :

$$\prod_{1\leq i<j\leq p}\frac{\lambda_i^{-1}-\lambda_j^{-1}}{\lambda_i^{-1}+\lambda_j^{-1}}\,\Xi(\lambda_1^{-1}\oplus..\oplus\lambda_p^{-1}\oplus\Lambda^{-1}) = \sum_{m_k\in\mathbb{Z}}\prod_{k=1}^{p}\lambda_k^{-m_k}\,\varphi_{m_1,..,m_p}(\partial_.)\,\Xi(\Lambda^{-1}) \tag{3.2}$$

where the polynomials φ are now considered as functions of the odd differentials ∂_{2j+1} (substituted for the odd traces θ_{2j+1} of (2.19)), and the function is expanded in the domain $\lambda_1^{-1} > .. > \lambda_p^{-1}$. The reexpression of (3.2) as matrix averages is achieved through the following formula, where we decompose $\Lambda^{-1} = \Lambda_1^{-1}\oplus\Lambda_2^{-1}$, with $\Lambda_1 = diag(\lambda_1,..,\lambda_p)$ and $\Lambda_2 = diag(\lambda_{p+1},...,\lambda_{N+p})$:

$$\begin{aligned}\prod_{1\leq i<j\leq p}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\,\Xi_{p+N}(\Lambda_1^{-1}\oplus\Lambda_2^{-1}) = \int\prod_{k=1}^{p}d\nu_{\lambda_k}(y_k)\times\\ \prod_{1\leq m<n\leq p}\frac{2i(\lambda_m-\lambda_n)+y_m-y_n}{2i(\lambda_m+\lambda_n)+y_m+y_n}\ll\prod_{l=1}^{p}\det\left(\frac{2i\lambda_l+y_l-Y_2}{2i\lambda_l+y_l+Y_2}\right)\gg(\Lambda_2^{-1})\end{aligned} \tag{3.3}$$

where $d\nu_\lambda(y) = (\lambda/2\pi)^{\frac{1}{2}}\exp(iy^3/6-\lambda y^2/2)dy$ is the measure of integration over the eigenvalues y adapted to our problem, and the double bracket denotes the integral over the $N\times N$ matrix Y_2 as defined in (1.7). The expansion of both sides of (3.3) in series of $\lambda_1^{-1} > .. > \lambda_p^{-1}$ will characterize the Witten mapping completely. Let us turn to the proof of formula (3.3).

At first the matrices $\Lambda,\Lambda_1,\Lambda_2$ involve diagonal real positive elements, but if we introduce a cut in the complex plane along the negative real axis, the integrals make sense for each eigenvalue having a positive real part – as absolutely convergent integrals; as semi-convergent ones we can even extend them to the imaginary axis except the origin. To give a meaning to the following operations we will first continue analytically the λ_j to imaginary non vanishing values. Similar techniques were implicit in both [11] and [13]. We consider :

$$\Xi_{p+N}(\Lambda^{-1}) = \frac{1}{Z_{p+N}(\Lambda)} \int dY e^{\frac{i}{6}\mathrm{tr}(Y^3) - \frac{1}{2}\mathrm{tr}(\Lambda Y^2)} \tag{3.4}$$

where $Z_{p+N}(\Lambda)$ is defined in (2.2). We perform the change of variables $Z = Y + 2i(\Lambda_1 \oplus 0)$, with the obvious definition for the $(p+N)\times(p+N)$ matrix $\Lambda_1 \oplus 0 = \mathrm{diag}(\lambda_1, .., \lambda_p, 0, .., 0)$. Due to the relation

$$(\Lambda_1 \oplus 0)(0 \oplus \Lambda_2) = (0 \oplus \Lambda_2)(\Lambda_1 \oplus 0) = 0$$

the trace in the exponential becomes

$$\frac{i}{6}\mathrm{tr}(Y^3) - \frac{1}{2}\mathrm{tr}(\Lambda Y^2) = \frac{i}{6}\mathrm{tr}(Z^3) - \frac{1}{2}\mathrm{tr}([(0 \oplus \Lambda_2) - (\Lambda_1 \oplus 0)]Z^2) + \frac{2}{3}\mathrm{tr}(\Lambda_1^3) \tag{3.5}$$

We see that except for a constant term, the form of the exponential term is conserved, up to the substitution $\Lambda = \Lambda_1 \oplus \Lambda_2 \to \widetilde{\Lambda} = (0 \oplus \Lambda_2) - (\Lambda_1 \oplus 0)$. Let us now perform the angular average over Z using (2.1), which results in

$$\Xi_{p+N}(\Lambda^{-1}) = \prod_{1 \le i < j \le N+p} \frac{\lambda_j + \lambda_i}{\widetilde{\lambda_j} - \widetilde{\lambda_i}} \int \prod_{k=1}^{p+N} d\nu_{\widetilde{\lambda_k}}(z_k) \prod_{n=1}^{p} e^{\frac{2}{3}\lambda_n^3} \prod_{1 \le l < m \le N+p} \frac{z_l - z_m}{z_l + z_m} \tag{3.6}$$

where the $\tilde{\lambda}$'s are the diagonal elements of $\widetilde{\Lambda}$, i.e. $\widetilde{\lambda_k} = -\lambda_k$ for $1 \le k \le p$, $\widetilde{\lambda_k} = \lambda_k$ for $p+1 \le k \le p+N$ (recall that the $\tilde{\lambda}$'s are purely imaginary, so that the minus sign causes no harm in the integral). The antisymmetry of the integrand in z's in (3.6) automatically takes care of the denominators $z_l + z_m$, by antisymmetrizing the measure. We proceed and perform the opposite change of variables, but this time on the eigenvalues z by setting

$$\begin{aligned} z_k &= y_k + 2i\lambda_k \qquad 1 \le k \le p \\ z_k &= y_k \qquad p+1 \le k \le p+N \end{aligned} \tag{3.7}$$

which leads to

$$\begin{aligned} \Xi_{p+N}(\Lambda_1^{-1} \oplus \Lambda_2^{-1}) = &\prod_{1 \le i < j \le p} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \prod_{p+1 \le l < m \le p+N} \frac{\lambda_l^{-1} + \lambda_m^{-1}}{\lambda_l^{-1} - \lambda_m^{-1}} \int \prod_{k=1}^{N+p} d\nu_{\lambda_k}(y_k) \times \\ &\times \prod_{1 \le l < m \le p} \frac{y_l - y_m + 2i(\lambda_l - \lambda_m)}{y_l + y_m + 2i(\lambda_l + \lambda_m)} \prod_{p+1 \le i < j \le p+N} \frac{y_i - y_j}{y_i + y_j} \prod_{\substack{1 \le l \le p \\ p+1 \le j \le p+N}} \frac{y_l + 2i\lambda_l - y_j}{y_l + 2i\lambda_l + y_j} \end{aligned} \tag{3.8}$$

and amounts to (3.3) since

$$\prod_{p+1\leq l<m\leq p+N}\frac{\lambda_l^{-1}+\lambda_m^{-1}}{\lambda_l^{-1}-\lambda_m^{-1}}\int\prod_{k=p+1}^{p+N}d\nu_{\lambda_k}(y_k)\prod_{p+1\leq i<j<p+N}\frac{y_i-y_j}{y_i+y_j}\times$$
$$\times\prod_{\substack{1\leq l\leq p\\ p+1\leq j\leq p+N}}\frac{y_l+2i\lambda_l-y_j}{y_l+2i\lambda_l+y_j}=\ll\prod_{1\leq l\leq p}\det\left(\frac{y_l+2i\lambda_l-Y_2}{y_l+2i\lambda_l+Y_2}\right)\gg(\Lambda_2^{-1})\tag{3.9}$$

Let us rewrite the content of (3.3) for $p=1$:

$$\Xi_{N+1}(\lambda^{-1}\oplus\Lambda^{-1})=\int d\nu_\lambda(y)\ll\det\frac{y+2i\lambda-Y}{y+2i\lambda+Y}\gg=\int d\nu_\lambda(y)\ll f_{Y/2i}(\lambda+(y/2i))\gg\tag{3.10}$$

where we identified the generating function f for the odd Schur polynomials (2.5). In the sense of asymptotic series of λ^{-1}, we are allowed to expand f and integrate term by term over y to get :

$$\Xi_{N+1}(\lambda^{-1}\oplus\Lambda^{-1})=\sum_{m=0}^{\infty}\ll p_m(Y/2i)\gg\int d\nu_\lambda(y)(\lambda-\frac{iy}{2})^{-m}\tag{3.11}$$

so that comparing with (3.1), we find the first elements of the Witten mapping :

$$p_k(\partial.)\ \Xi(\theta.)=\sum_{0\leq s\leq[k/3]}(-1)^s c_{s,k}\ll p_{k-3s}(Y/2i)\gg\tag{3.12}$$

where

$$c_{s,k}=\sum_{l=0}^{2s}\frac{1}{2^l}\frac{(k-3s+l-1)!}{l!(k-3s-1)!}\frac{(6s-2l-1)!!}{6^{2s-l}(2s-l)!}\tag{3.13}$$

and $[x]$ denotes the integral part of x.

For p generic,we are left with the easy task of expanding the r.h.s. of (3.3) in $\lambda_1^{-1}>..>\lambda_p^{-1}$ and identifying term by term with (3.2). Noting that the integrand in (3.3) is again the generating function for φ's (2.19), but with the identification $Y\to Y_2/2i$ and $y_k\to\lambda_k+(y_k/2i)$, we can integrate over $y_1,..,y_p$ to get the general Witten mapping in the form :

$$\varphi_{m_1,..,m_p}(\partial.)\ \Xi(\theta.)=\sum_{\substack{s_1,\dots,s_p\geq 0\\ 3s_j\leq m_j+m_{j+1}+..+m_p}}\prod_{i=1}^{n}(-1)^{s_i}c_{s_i,m_i}\ll\varphi_{m_1-3s_1,\dots,m_p-3s_p}(Y/2i)\gg\tag{3.14}$$

We list the first few images of φ's below.

$\varphi_1(\partial_.)\Xi$	$= \ll \varphi_1 \gg$
$\varphi_2(\partial_.)\Xi$	$= \ll \varphi_2 \gg$
$\varphi_3(\partial_.)\Xi$	$= \ll \varphi_3 - \frac{5}{24} \gg$
$\varphi_{2,1}(\partial_.)\Xi$	$= \ll \varphi_{2,1} - \frac{1}{12} \gg$
$\varphi_4(\partial_.)\Xi$	$= \ll \varphi_4 - \frac{17}{24}\varphi_1 \gg$
$\varphi_{3,1}(\partial_.)\Xi$	$= \ll \varphi_{3,1} + \frac{5}{24}\varphi_1 \gg$
$\varphi_5(\partial_.)\Xi$	$= \ll \varphi_5 - \frac{35}{24}\varphi_2 \gg$
$\varphi_{4,1}(\partial_.)\Xi$	$= \ll \varphi_{4,1} \gg$
$\varphi_{3,2}(\partial_.)\Xi$	$= \ll \varphi_{3,2} + \frac{5}{24}\varphi_2 \gg$
$\varphi_6(\partial_.)\Xi$	$= \ll \varphi_6 - \frac{59}{24}\varphi_3 + \frac{385}{1152} \gg$
$\varphi_{5,1}(\partial_.)\Xi$	$= \ll \varphi_{5,1} - \frac{35}{24}\varphi_{2,1} + \frac{35}{576} \gg$
$\varphi_{4,2}(\partial_.)\Xi$	$= \ll \varphi_{4,2} + \frac{17}{24}\varphi_{2,1} - \frac{35}{576} \gg$
$\varphi_{3,2,1}(\partial_.)\Xi$	$= \ll \varphi_{3,2,1} - \frac{5}{24}\varphi_{2,1} - \frac{1}{12}\varphi_3 + \frac{5}{288} \gg$

Table II : the derivatives of the Kontsevich partition function with respect to the $\theta_.$'s expressed as averages over polynomials in odd traces. The notation $\partial_.$ stands for $\{-\frac{2}{2k+1}\frac{\partial}{\partial\theta_{2k+1}}\}$, $\theta_. \equiv \theta(\Lambda^{-1})$, while on the r.h.s. the matrix argument of the $\varphi_.$'s is $Y/2i$.

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