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Selection Rules for Topology Change

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There has recently been interest in the question of whether, in principle, one can construct, in the laboratory, a wormhole and use it for time-travel. Mathematically this may be taken to mean, among other things; **Does there exist a compact 4-manifold M with everywhere non-singular Lorentz metric g_L (which we shall assume time-orientable) such that**

(1)$\partial M = S^3 \sqcup \; S^1 \times S^2$

where $\sqcup$ denotes disjoint union and

(2) S^3 and $S^1 \times S^2$ **are spacelike with S^3 in the past of M and $S^1 \times S^2$ in the future.**

The answer to this question is contained in a celebrated theorem of Geroch which implies that:

There exist many such pairs (M, g_L) but they all contain closed timelike curves.

Because of these closed timelike curves these topology changing spacetimes were thought to be of limited interest. However, if the wormhole is to be used for time travel anyhow this objection hardly seems decisive. In this talk I want to describe a further, and I believe potentially more serious objection. Hawking and I have recently shown that:

No spacetime interpolating between S^3 and $S^1 \times S^2$ admits an $SL(2, \mathbb{C})$ spinor structure.

In other words we cannot construct such a time-machine from ordinary matter made from fermions. However there do exist time orientable Lorentzian 4-manifolds admitting $SL(2, \mathbb{C})$ spinors which interpolate between S^3 and $S^1 \times S^2 \sharp S^1 \times S^2$, where $\sharp$ denotes connected sum. In other words wormholes must be created or destroyed in pairs. More generally we can **define** a new $\mathbb{Z}_2$valued invariant $u(\Sigma)$ where $u\Sigma$ in a closed orientable but not necessarily connected 3-manifold such

$$u(\Sigma) = 0 \tag{1}$$

if Σ is the boundary of some compact time-orientable 4-manifold M which admits $SL(2, \mathbb{C})$ spinors and such that Σ is spacelike and

$$u(\Sigma) = 1 \tag{2}$$

otherwise.

The invariant $u(\Sigma)$ has the following properties:

Theorem 1 Under disjoint union, $\sqcup$,

$$u(\Sigma_1 \sqcup \Sigma_2) = u(\Sigma_1) + u(\Sigma_2) \bmod 2 \tag{3}$$

Theorem 2 Under connected sum $\sharp$,

$$u(\Sigma_1 \sharp \Sigma_2) = u(\Sigma_1) + u(\Sigma_2) + 1 \bmod 2. \tag{4}$$

These follow from:

Theorem 3

$$u(\Sigma) = \dim_{\mathbf{Z}_2}\{H_0(\Sigma; Z_2) \oplus H(\Sigma; \mathbf{Z}_2)\} \bmod 2. \tag{5}$$

Theorem 3 gives:

$$u(S^3) = 1, \tag{6}$$

$$u(S^1 \times S^2) = 0 \tag{7}$$

and hence our result about creating wormholes in pairs. Theorem 3 also shows that one cannot "create a single S^3 universe from nothing"or indeed one cannot have a transition between an odd number of S^3 universes.

To prove theorem 3 we need the following known propositions.

Prop 1 M admits a time-orientable Lorentz metric g_L with ∂M spacelike iff M admits a non-vanishing vector field transverse to ∂M.

A theorem of Hopf gives us

Prop 2 M admits a non-vanishing vector field transverse to ∂M iff $\chi(M) = 0$, where $\chi(M)$ is the Euler characteristic.

Prop 3 (M, g_L) admits an $SL(2, \mathbf{C})$ spinor structure iff

$$w_2(M) = 0 \tag{8}$$

where w_2 is the second Stiefel-Whitney class of M vanishes.

To proceed we note that Equation (8) is equivalent to:

$$x \frown x = 0 \tag{9}$$

$\forall x \in H^2(M;\ \mathbf{Z}_2)$, where $\frown$ denotes the cup product. Now the intersection form $Q : H^2(M, \Sigma; \mathbf{Z}_2) \times H^2(M, \Sigma; \mathbf{Z}_2) \to \mathbf{Z}_2$ is non-degenerate. Working mod 2, symmetric forms may also be regarded as skew symmetric, thus since Q vanishes on the diagonal it may be regarded as a symplectic form on $H^2(M, \Sigma; \mathbf{Z}_2)$. It follows that $\dim H^2(M, \Sigma; \mathbf{Z}_2)$ must be even.

The exact cohomology sequence (with $\mathbf{Z}_2$ coefficients)

$$0 \to H^0(\Sigma) \to H^0(M) \to H^1(M, \Sigma) \to H^1(\Sigma) \to H^1(M) \to H^2(M, \Sigma) \to H^2(M) \to ...$$

yields

$$\chi - s = \dim W \bmod 2$$

where

$$s(\Sigma) = \dim_{\mathbb{Z}_2}\{H_0(\Sigma;\mathbb{Z}_2) \oplus H_1(\Sigma;\mathbb{Z}_2)\} \tag{10}$$

and W is the image of $H^2(M,\Sigma)$ in $H^2(M)$ under the last homomorphism. We therefore have:

Prop 4, If M is a spin-manifold then

$$\chi(M) = u(\Sigma) \bmod 2 \tag{11}$$

Combining Prop 4 with Prop 1 and 2 yields the proof of Theorem 3.

Since giving the lecture Hawking and I have obtained a generalization of Theorem 3 to the case that the boundary Σ is not necessarily timelike. One can associate an invariant, $\text{kink}(\Sigma_a, g_L)$, the kink number of the Lorentz metric g_L with respect to the boundary component Σ_a. The kink number is a measure of how many times the light cone tips over on Σ_a. Finkelstein and Misner defined it as follows. Give Σ_a an orthonormal framing $\{\mathbf{e}_i\}, i = 1,2,3$ (it doesn't matter which one). Augment $\{\mathbf{e}_i\}$ by adding to it the unit inward directed unit normal $\mathbf{n}$. The normalization is done with respect to an arbitrary auxilliary Riemannian metric g_R on M. The vector field $\mathbf{V}$ mentioned in Prop. 1 may be taken to be that eigenvector of g_L with respect to g_R which has negative eigenvalue, normalized to unit length. However in the present case $\mathbf{V}$ is no longer everywhere transverse to ∂M. The restriction of $\mathbf{V}$ to each connected component Σ_a of ∂M allows one to construct a map ϕ_a: $\Sigma_a \to S^3$ given by the components of the vector field $\mathbf{V}$ in the frame $(\mathbf{n}, \mathbf{e}_i)$. The degree of this map is the kink number,$\text{kink}(\Sigma_a, g_L)$. The sum over connected components Σ_a of the boundary ∂M is the total kink number $\text{kink}(\partial M, g_L)$.

Props 1 and 2 are now replaced by

Prop 5 For any Lorentz metric g_L on M

$$\chi(M) = \text{kink}(\partial M, g_L) \tag{12}$$

Props 3 and 4 remain unchanged. It follows that Theorem 3 is replaced by

Theorem 4 - For a Lorentz spin manifold M boundary $\partial M = \Sigma$ we have:

$$\text{kink}(\Sigma, g_L) = \dim_{\mathbb{Z}_2}(H_0(\Sigma;\mathbb{Z}_2) \oplus H_1(\Sigma;\mathbb{Z}_2)) \ \ mod\ 2 \tag{13}$$

Acknowledgements

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References

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