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Quantum symmetry of rational conformal models

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Abstract

Recent progress in understanding the $U_q(s\ell_2)$ symmetry of $su(2)$ current algebra and minimal conformal models is reported. The review is updated by an account of work done by Yassen Stanev, L. Hadjiivanov and the author after the 1990 Strasbourg Rencontre.

Contents

1. Introduction

Single valued two dimensional (2D) local fields $\phi(\tau,\xi)$ on a cylinder (satisfying the periodicity condition $\phi(\tau,\xi+2\pi) = \phi(\tau,\xi)$) in a rational conformal field theory can be split into a finite sum of products

$$\phi(\tau,\xi) = \sum_i \varphi_i(\tau-\xi)\bar{\varphi}_i(\tau+\xi) \tag{1.1}$$

of multivalued "chiral fields" depending on a single light cone variable. The periodicity of ϕ is restored if φ and $\bar{\varphi}$ have inverse monodromies:

$$\varphi(x_R-2\pi) = \varphi(x_R)M \quad \left(\text{i.e.} \quad \varphi_i(x_R-2\pi) = \sum_j \varphi_j(x_R)M^j{}_i \right) \tag{1.2a}$$

$$\bar{\varphi}(x_L+2\pi) = M^{-1}\bar{\varphi}(x_L) \quad \left(x_R = \tau-\xi ,\ x_L = \tau+\xi \right) \tag{1.2b}$$

The chiral components φ_i and $\bar{\varphi}_i$ are not determined by the observable 2D field ϕ. The representation (1.1) gives room to a (first kind) gauge freedom:

$$\varphi \to \varphi V \quad , \quad \bar{\varphi} \to V^{-1}\bar{\varphi} \tag{1.3}$$

(which leaves ϕ invariant). The monodromy matrix M in (1.2) can be viewed at as the square of the statistic operator (or braid group generator) $\mathcal{B}$ that exchanges two neighbouring fields in a correlation function. Demanding that the extension of the gauge transformation (1.3) to a tensor product of fields commutes with the statistic operator would imply that it is realized by a (twisted) coproduct $\Delta(V)$, a characteristic feature of a quantum symmetry. In a large family of cases its Lie algebraic counterpart is given by a quantum universal enveloping (QUE) algebra.

The hidden quantum symmetry of (the chiral constituent of) 2D conformal models is being gradually understood at both the classical [1] and the quantum level [2-6]. We outline here recent progress in the axiomatic approach to the problem (started in a joint effort by D.Buchholz, G.Mack and I.Todorov back in 1989 and reflected in [3] and [6]). The physical state space of a chiral model admits a non-associative "weak quasi-Hopf" symmetry algebra [6]. Here we follow [3] in considering instead an extended indefinite metric chiral state space that gives room to an associative (Hopf) QUE symmetry algebra. The correlation functions of physical 2D fields are recovered by using a semidefinite sesquilinear form that contracts the quantum group's degree of freedom of the left and right sectors, the non-unitary contributions being absorbed by its kernel.

We consider the familiar $\hat{su}(2)_k$ current algebra model with QUE symmetry algebra

$$U_q = U_q(s\ell_2) \text{ for } q = \exp\left(\frac{i\pi}{k+2}\right) \tag{1.4}$$

generated by elements E, F, H (and $q^{\pm H}$) satisfying

$$[H,E] = 2E, [H,F] = -2F, \ [E,F] = [H] := \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{1.5}$$

(A representation in the tensor product of two irreducible U_q modules of) Drinfeld's universal R-matrix - followed by a permutation - gives rise to a QUE algebra representation of the braid group (on top of the one obtained by exchanging neighbouring factors in a correlation function through analytic continuation). Our chief objective is to establish the invariance of (U_q extended) chiral correlation functions under the diagonal action of the braid group. A new feature in the argument presented here is the introduction of a regular basis of n-point U_q invariants [7.

(different from the standard one expressed in terms of Clebsch-Gordan coefficients [8]) and a corresponding basis of conformal blocks (sec. 3) which allows to work with a deformation parameter q at a root of unity (like in (1.4)) without having to resort to a regularization.

We start in Sec. 2 with a selfcontained exposition of the theory of the $\mathcal{A}_k(su_2)$ local chiral current algebra and its representations (combining the approach of Borcherds, Frenkel, Lepowsky, Meurman and Goddard [9] with that of Lüscher, Mack and the author [10]). The positive energy (lowest weight) representations of $\mathcal{A}_k$ are described in Sec. 2B where the Knizhnik-Zamolodchikov (KZ) equation [11,10,12] is derived for the $\mathcal{A}_k$-primary chiral vertex operators (CVO). In Secs. 3A and 4A we derive integral representations for current algebra and minimal models 4-point conformal blocks of the type first written by Dotsenko and Fateev [13] (see also [12,14]).

In Sec. 3C we spell out the derivation (outlined in [7]) of the regular basis of U_q invariant polynomials of n variables. The q-deformed e_+-exponent (Sec. 3D) has been used in writing the universal R-matrix for more general (simple) QUE algebras in [15]. The treatment of quantum group symmetry of diagonal minimal conformal models (sec. 4) follows [16].

Sec. 3 reproduces essentially part of my 1992 lecture notes [17] (written in collaboration with Yassen Stanev).

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2. *Conformal current algebra and chiral vertex operators.*

2A. *The su_2 chiral current algebra.*

We describe the local chiral algebra $\mathcal{A}_k = \mathcal{A}_k(su_2)$ generated by a level k su_2 current as on operator field algebra acting in a vacuum Hilbert space $\mathcal{H}_0$ which carries a (unitary) positive energy representation of the Möbius Lie algebra $su(1,1)$. More precisely, there is a distinguished positive (selfadjoint) operator L_0, the chiral conformal energy, of integer spectrum of finite multiplicity which defines a grading in $\mathcal{H}_0$:

$$\mathcal{H}_0 = \bigoplus_{n=0}^{\infty} \mathcal{H}_0^{(n)}, (L_0 - n)\mathcal{H}_0^{(n)} = 0, \tag{2.1a}$$

the space $\mathcal{H}_0^{(0)}$ being 1-dimensional (uniqueness of the vacuum)

$$\dim\mathcal{H}_0^{(0)} = 1, \quad \dim\mathcal{H}_0^{(n)} < \infty. \tag{2.1b}$$

Neighbouring eigenvalues of L_0 are intertwined by a pair of conjugate operators $L_{\pm 1}$ which, together with L_0, span $su(1,1)$:

$$[L_0, L_{\pm 1}] = \pm L_{\mp 1}, [L_1, L_{-1}] = 2L_0, L_{\pm}^* = L_{\mp 1}. \tag{2.2}$$

It follows from the fact that the vacuum $|0> \in \mathcal{H}_0^{(0)}$ is the lowest energy state for L_0, that it is annihilated by L_1. It is then a consequence of (2.2) that L_{-1} also annihilates $|0>$:

$$\| L_{-1}|0<\|^2 = \langle 0|L_1 L_{-1}|0\rangle = 2\langle 0|L_0|0\rangle = 0 \quad (\text{for } L_1|0\rangle = 0 = L_0|0\rangle). \tag{2.3}$$

We define a system of local chiral vertex operators ($LCVO$) $\mathcal{A}_k$ by assigning to each finite energy state v a $LCVO$ $Y(v,z)$, $z \in \mathbb{C}$, such that the vector valued function

$$z \to Y(v,z)|0> = e^{zL_{-1}}v \in \mathcal{H}_0 \tag{2.4}$$

is analytic (in the norm topology) in the unit circle $|z| < 1$. (We shall use interchangeably the ket notation, like $|0>$, and "vector notation", such as v.) Assuming that Y is L_0 and L_{-1} covariant,

$$[L_{-1}, Y(v,z)] = \frac{d}{dz}Y(v,z) \tag{2.5a}$$

$$e^{iL_0 t}Y(v,z)e^{-iL_0 t} = Y(e^{iL_0 t}v, e^{it}z), \tag{2.5b}$$

we derive Eq. (2.4) from the existence of the limit

$$\lim_{z \to 0} Y(v,z)|0> = v. \tag{2.6}$$

Local commutativity can be formulated as follows. For each pair v_1, v_2 of finite energy vectors there exists an integer $N_{v_1 v_2}$ such that

$$z_{12}^N [Y(v_1, z_1), Y(v_2, z_2)] = 0 \text{ for } N \geq N_{v_1 v_2}, z_{12} = z_1 - z_2. \tag{2.7}$$

The following Reeh-Schlieder type uniqueness theorem is a direct consequence of (2.4) and of locality : If two $LCVO$ $Y_i(v, z)$ $i = 1, 2$ satisfy (2.4) (with the same v) then they coincide.

It follows that $Y(v, z)$ is linear in v and that its derivatives are also $LCVO$ (for a review of these and related properties and for further references, see, e.g., Dolan et al. [9]). An impressive result which justifies the above definition of a local chiral algebra is Borcherds operator product expansion (OPE) formula [9].

PROPOSITION 2.1. — *Eqs. (2.5, 6, 7) imply the product formula*

$$Y(v_1, z_1) Y(v_2, z_2) = Y(Y(v_1, z_{12}) v_2, z_2). \tag{2.8}$$

If v_i are eigenvectors of the conformal energy,

$$(L_0 - \ell_i) v_i = 0, \quad Y(v_i, z) = \sum_{n \in \mathbb{Z}} Y_{n-\ell_i}(v_i) z^{-n} \tag{2.9}$$

then the (first) composite argument of Y in the right hand side of (2.8) has a finite number of singular terms for $z_{12} \to 0$:

$$Y(v_1, z_{12}) v_2 = \sum_{n=0}^{\infty} z_{12}^{n-\ell_1-\ell_2} Y_{\ell_2 - n}(v_1) v_2. \tag{2.10}$$

It is assumed that there is a $v^{(2)}$ in $\mathcal{H}_0^{(2)}$ corresponding to the stress-energy tensor :

$$T(z) :=< Y(v^{(2)}, z) = \sum_n L_n z^{-n-2}. \tag{2.11}$$

According to the Lüscher-Mack theorem [10] L_n generate the Virasoro algebra Vir.

We now come to a specific requirement for the su_2 current algebra. In this case the space $\mathcal{H}_0^{(1)}$ is 3-dimensial and we can choose in it an orthonormal basis v_a, $a = 1, 2, 3$ and set

$$J^a(z) := Y(v_a, z) = \sum_n J_n^a z^{-n-1}. \tag{2.12}$$

The su_2 charges J_0^a commute with Vir and satisfy

$$[J_0^a, J^b(z)] = i\sqrt{2}\varepsilon^{abc} J^c(z), \quad J_0^a|0>= 0, \tag{2.13}$$

where ε^{abc} is the totally antisymmetric unit tensor. Whit this normalization the eigenvalue of the Casimir operator for the adjoint (3-dimensional) representation is equal to 4 (twice the dual Coxeter number for su_2) :

$$(\vec{J}_0^2 - 4)\mathcal{H}_0^{(1)} = 0 \qquad \vec{J}_0^2 = (J_0^1) + (J_0^2)^2 + (J_0^3)^2. \tag{2.14}$$

The possible algebras $\mathcal{A}(su_2)$ are classified by the following analogue of the Lüscher-Mack theorem [10].

PROPOSITION 2.2. — *The commutation relations of two currents (2.12) satisfying (2.13) are given by*

$$[J^a(z_1), J^b(z_2)] = i\sqrt{2}\varepsilon^{abc} J^c(z_2)\delta(z_{12}) - K\delta_{ab}\delta'(z_{12}) \tag{2.15}$$

where the complex variable δ-function is defined by

$$\oint_{S^1} \delta(z_{12}) f(z_2)\, \frac{dz_2}{2\pi i} = f(z_1) \tag{2.16a}$$

or, equivalently, by its Fourier-Laurent expansion

$$\delta(z_{12}) = \frac{1}{z_1} \sum_{n\in\mathbb{Z}} \left(\frac{z_2}{z_1}\right)^n. \tag{2.16b}$$

Wightman positivity and integrability imply that k is a positive integer (labeling the different current algebras $\mathcal{A}_k$).

The proof is an easy application of Proposition 2.1.

PROPOSITION 2.3. — *The stress energy tensor in $\mathcal{A}_k(su_2)$ is given by teh Sugawara formula*

$$2(k+2)T_J(z) =: \vec{J}^2(z) :, \tag{2.17}$$

where the normal product is defined by the nonsingular part of the expansion (2.8) (2.10) :

$$J_a^2(z) := \lim_{\varepsilon \to 0} \{J_a(z+\varepsilon)J_a(z) - \frac{k}{\varepsilon^2}\}. \tag{2.18}$$

The corresponding Virasoro central charge is

$$c = \frac{3k}{k+2}. \tag{2.19}$$

The simple proof of this theorem is presented in [10] [17].

2B. *Lowest weight states and primary CVO.*

The positive energy (ground state) representations of $\mathcal{A}_k$ are in one-to-one correspondence with the irreducible representations of su_2 for weights (twice isospin) λ not exceeding the level k.

We regard the $(\lambda+1)$-dimensional su_2 representation space $\mathcal{H}_\lambda^{(0)}$ as the lowest weight subspace of an (infinite dimensional) graded $\mathcal{A}_k$ module

$$\mathcal{H}_\lambda = \bigoplus_{n=0}^{\infty} \mathcal{H}_\lambda^{(n)}. \tag{2.20}$$

Setting

$$J_n^a \mathcal{H}_\lambda^{(0)} = 0 (= L_n \mathcal{H}_\lambda^{(0)}) \text{ for } n = 1, 2, \ldots \tag{2.21}$$

we deduce that $\mathcal{H}_\lambda^{(n)}$ are eigensubspaces of the chiral Hamiltonian

$$(L_0 - \Delta_\lambda - n)\mathcal{H}_\lambda^{(n)} = 0, \quad 4(k+2)\Delta_\lambda = \lambda(\lambda+2). \tag{2.22}$$

Each $\mathcal{H}_\lambda$, $\lambda = 0, \ldots, k$ defines a superselection sector of the theory with observable algebra $\mathcal{A}_k$. The 2-dimensional charged Bose fields can be written as finite sums of products of CVO :

$$\phi_{\lambda\bar{\lambda}}(z, \zeta \bar{z}, \bar{\zeta}) = \sum_\alpha V_{\lambda\alpha}(z, \zeta) \overline{V}_{\bar{\lambda}\alpha}(\bar{z}, \bar{\zeta}). \tag{2.23}$$

We shall not specify here the role of the dummy index α. (In the standard approach to CVO - see, e.g., Tsuchiya and Y. Kanie [12] - it is taken as a pair (λ_f, λ_i)-indicating the weight λ_i of the domain of V_λ and the weight λ_f of its image (final state); in Sec. 3 below we shall interpret α as a $U_q(s\ell_2)$ quantum group index.)

To display the $su(2)$ transformation properties of CVO, instead of using a discrete index like the third isospin projection, we write a field of weight λ as a polynomial of degree λ in a formal variable ζ (cf. Zamolodchikov and Fateev [11] as well as [16]). The resulting CVO $V_\lambda(z,\zeta)$ transforms under the "coherent state" action of $su(2)$. Assuming that V_λ is $\mathcal{A}_k$-*primary*, we shall write

$$\begin{aligned} \left[J_n^+, V_\lambda\right] &= z^n \partial_\zeta V_\lambda, & J_n^\pm &= J_n^1 \pm \mathrm{i} J_n^2 \quad \partial_\zeta = \frac{\partial}{\partial \zeta}, \\ 2\left[J_n^3, V_\lambda\right] &= z^n (2\zeta\partial_\zeta - \lambda) V_\lambda, & \left[J_n^-, V_\lambda\right] &= z^n \left(\lambda\zeta - \zeta^2 \partial_\zeta\right) V_\lambda. \end{aligned} \tag{2.24}$$

(The $su(2)$ transformation law is recovered for $n = 0$.) The current $\vec{J}(z)$ can also be written as a second degree polynomial:

$$J(z,\zeta) = J^-(z) + 2\zeta J^3(z) - \zeta^2 J^+(z), \quad J^\pm = J^1 \pm \mathrm{i} J^2. \tag{2.25}$$

Although it is a primary field with respect to Vir,

$$[L_n, J(z,\zeta)] = \frac{\partial}{\partial z}\left(z^{n+1} J(z,\zeta)\right), \tag{2.26}$$

it is only a quasiprimary field with respect to $\mathcal{A}_k$. Indeed, Eq. (2.15) implies, e.g.,

$$\left[J_n^+, J(z,\zeta)\right] = z^n \partial_\zeta J(z,\zeta) + knz^{n-1}. \tag{2.27}$$

Proposition 2.4 *If V_λ is a primary CVO for both $\mathcal{A}_k$ and Vir, of weight λ and Δ_λ respectively, then*

$$\Delta_\lambda = \frac{1}{4}\frac{\lambda(\lambda+2)}{k+2} \tag{2.28}$$

and V_λ satisfies the Knizhnik-Zamolodchikov (KZ) equations

$$(k+2)\partial_z V_\lambda =: \frac{\lambda}{2}(\partial_\zeta J) V_\lambda - J \partial_\zeta V_\lambda : \tag{2.29}$$

where the normal product is defined by the non-singular term in the OPE $J(z_1)V(z_2)$*:*

$$2J_3(z_1)V_\lambda(z_2,\zeta) = \frac{1}{z_{12}}(2\zeta\partial_\zeta - \lambda)V_\lambda(z_2,\zeta) + :2J_3(z_1)V_\lambda(z_2,\zeta): \; etc.\; for\; z_{1,2}\to z. \tag{2.30}$$

Proof. Eq. (2.28) is a consequence of Eq. (2.22) and of the intertwining property of V_λ; in fact,

$$V_\lambda(z,\zeta)\,|0\rangle = V_{\lambda 0}^{\lambda}(z,\zeta)\,|0\rangle = e^{zL_{-1}}V_\lambda(0,\zeta)\,|0\rangle, \tag{2.31a}$$

where

$$V_\lambda(0,\zeta)\,|0\rangle = \zeta^\lambda|\lambda\rangle + \ldots + |-\lambda\rangle \in \mathcal{H}_\lambda^{(1)} \quad (\dim \mathcal{H}_\lambda^{(1)} = \lambda+1). \tag{2.31b}$$

Eq. (2.29) follows from the commutator $[L_n, V_\lambda]$ and the Sugawara expression (2.17) for T (see Furlan et al. [10]), Chapter 5 for details).

2C Ward identities, null vectors and fusion rules

In order to derive the KZ equations for correlation functions from the operator equation (2.29), we need to express the expectation value of a product of currents and fields in terms of the corresponding correlation function of fields alone. To do this one derives *Ward identities* which we display on the example of a 3-point function:

$$\langle 0|\, V_{\lambda_1}(z_1,\zeta_1) : V_{\lambda_2}(z_2,\zeta_2)J(z,\zeta) : V_{\lambda_3}(z_3,\zeta_3)\,|0\rangle =$$
$$\left\{\frac{1}{z_1-z}\Big((\zeta_1-\zeta)^2\partial_{\zeta_1} - \lambda_1(\zeta_1-\zeta)\Big) - \frac{1}{z-z_3}\Big((\zeta-\zeta_3)^2\partial_{\zeta_3} + \lambda_3(\zeta-\zeta_3)\Big)\right\} W_3 \tag{2.32a}$$

where

$$W_3 := \langle 0|\, V_{\lambda_1}(z_1,\zeta_1)V_{\lambda_2}(z_2,\zeta_2)V_{\lambda_3}(z_3,\zeta_3)\,|0\rangle. \tag{2.32b}$$

The general structure of the KZ equation for an arbitrary n-point function becomes already apparent in the simple case of $n = 3$:

$$\left\{(k+2)\partial_{z_2} + \frac{\Omega_{12}}{z_{12}} - \frac{\Omega_{23}}{z_{23}}\right\} W_3 = 0. \tag{2.33}$$

Here Ω_{ij} are the $SU(2)$ invariant operators

$$\Omega_{ij} = \frac{1}{2}\lambda_i\lambda_j + \zeta_{ij}(\lambda_i\partial_j - \lambda_j\partial_i) - \zeta_{ij}^2\partial_i\partial_j, \quad \partial_i = \partial_{\zeta_i}. \tag{2.34}$$

It turns out that the $SU(2)$ and conformal invariant 3-point function (Eq. (2.38) below – with λ and Δ_λ related by (2.28)) satisfies the KZ equation automatically.

We postpone the study of the KZ equations for a general 4-point function until Sec. 3 and proceed to extract further restrictions on 3-point functions from a current algebra null vector condition.

We shall see that not all 3-point functions consistent with $SU(2)$ and conformal invariance are admissible. If they were, we would have come to a contradiction with the restriction $\lambda \leq k$ for unitary (integrable) representations. The vanishing of some 3-point functions is implied by the *null vector condition*

$$\| \left(J_{-1}^{+}\right)^{k+1} |0\rangle\|^2 = \langle 0| \left(J_{1}^{-}\right)^{k+1} \left(J_{-1}^{+}\right)^{k+1} |0\rangle = 0, \tag{2.35}$$

which follows from the easily verifiable identity

$$J_1^{-} \left(J_{-1}^{+}\right)^{k+1} |0\rangle = \left(k - 2J_0^3 + J_{-1}^{+}J_1^{-}\right) \left(J_{-1}^{+}\right)^{k} |0\rangle = 0, \tag{2.36}$$

valid for $\mathcal{A}_k(su_2)$.

Indeed, in a *positive metric theory*, the vanishing of the norm square (2.35) implies the vanishing of all matrix elements of the vector $\left(J_{-1}^{+}\right)^{k+1}|0\rangle$; hence

$$\begin{aligned} 0 &= \langle 0|\, V_{\lambda_1}(z_1,\zeta_1)V_{\lambda_2}(z_2,\zeta_2)V_{\lambda_3}(z_3,\zeta_3)\left(J_{-1}^{+}\right)^{k+1}|0\rangle \\ &= \left(-\frac{1}{z_1}\partial_{\zeta_1}-\frac{1}{z_2}\partial_{\zeta_2}-\frac{1}{z_3}\partial_{\zeta_3}\right)^{k+1} W_3. \end{aligned} \tag{2.37}$$

Given that the $\{\zeta_i\}$ dependence of W_3 is contained in a z- independent factor, that is a homogeneous polynomial in ζ of degree $I_1+I_2+I_3$:

$$W_3 = C_{\lambda_1\lambda_2\lambda_3}\frac{\zeta_{12}^{\lambda_{12}}\zeta_{13}^{\lambda_{13}}\zeta_{23}^{\lambda_{23}}}{z_{12}^{\Delta_{12}}z_{13}^{\Delta_{13}}z_{23}^{\Delta_{23}}},\qquad 2\lambda_{12}+2\lambda_{13}+2\lambda_{23}=\lambda_1+\lambda_2+\lambda_3, \tag{2.38}$$

$$(\Delta_{12}=\Delta_{\lambda_1}+\Delta_{\lambda_2}-\Delta_{\lambda_3}\text{ etc.}),$$

we conclude, by varying the coefficients z_1, z_2 and z_3 in the differential operator in the right-hand side of (2.37), that the (integer) sum of isospins should not exceed k:

$$I_1+I_2+I_3=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)\le k. \tag{2.39a}$$

The spin addition formula supplemented by this constraint provides the celebrated KZ *fusion rule* (derived in these lines by Gepner and Witten [18])

$$V_{2I_1}\times V_{2I_2}\sim\sum_{I=|I_1-I_2|}^{\min(I_1+I_2,k-I_1-I_2)} V_{2I}. \tag{2.39b}$$

(The last equation can be viewed as an OPE formula with nonzero coefficients set equal to 1.)

We end up with a remarkable class of QFT, each having a finite number of superselection sectors closed under a well-defined composition law.

The basic data of a RCFT is thus the algebra of observables ($\mathcal{A}_k\otimes\overline{\mathcal{A}}_k$ in the case at hand), its positive energy representations (generated by "charged" fields) which give rise to (a finite number of) superselection sectors, and the fusion rules.

3 Braid group statistics and quantum symmetry

There are two ways of approaching extended chiral RCFT with a quantum group symmetry. One is to start with monodromy (and braiding) properties of conformal blocks and to match them with corresponding properties of quantum group multipoint invariants. The other begins, conversely, at the quantum group end. We are adopting here the first approach which is, clearly, the natural one from the point of view expounded in these notes. It should be noted, however, that the choice of a regular basis of solutions of the KZ equations, which plays a crucial role in what follows, was originally suggested to us by our preceding study of quantum group invariants and their properties under braid and duality transformations.

3A Integral representations for 4-point current algebra blocks

The analysis of the KZ equations for an arbitrary 4-point amplitude requires some work. Its understanding is hindered by the necessity of making some seemingly arbitrary choices at the very first steps. In order to ease the reader's bewilderment, we accompany our choices by some comments that could only be fully appreciated at a later stage.

As is already manifest from the form (2.38) of a general 3-point function, expectation values of products of primary CVO are multivalued functions of z_i. Therefore, the monodromy group* $\mathcal{M}_4$ ($\subset \mathcal{B}_4$) will act in a nontrivial way on the solutions of the KZ equations for 4-point blocks. The fact that every conformal block yields a (multivalued) analytic continuation on the Riemann sphere implies that it is sufficient to consider the braid (and monodromy) properties with respect to the first $n - 1$ arguments of each n-point block (keeping z_n fixed). In the case of 4-point blocks, we shall be interested in the action of the generators M_1 and M_2 of $\mathcal{M}_4$, that rotate the point z_1 around z_2 and z_2 around z_3, as well as that of the duality (or fusion) transformation F (to be defined in Sec. 3B), which intertwines the corresponding exchange matrices. These are treated in a most symmetric manner if the differentiation in the KZ equation is taken with respect z_2:

$$\left\{(k+2)\partial_{z_2} + \frac{\Omega_{12}}{z_{12}} - \frac{\Omega_{23}}{z_{23}} - \frac{\Omega_{24}}{z_{24}}\right\} W_4 = 0, \tag{3.1}$$

where Ω_{ij} are defined by (2.34).

Here W_4 is, a priori, any Möbius and $SU(2)$ invariant function of $z_1, \ldots, z_4$, $\zeta_1, \ldots, \zeta_4$ that is a polynomial of degree λ_i in ζ_i, translation-invariant and homogeneous in the set of all ζ of degree $\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$ and corresponds to conformal weights

$$\Delta_i (\equiv \Delta_{\lambda_i}) = \frac{1}{4} \frac{\lambda_i(\lambda_i + 2)}{k+2}, \quad i = 1, 2, 3, 4. \tag{3.2}$$

Every Möbius invariant (i.e. invariant under fractional linear $SU(1,1)$ transformations) 4-point function can be written in the form

$$W_4 = h(z_{13}, z_{14}, z_{24}, z_{34}) f\left(\eta \middle| \begin{matrix} \lambda_i \\ \zeta_i \end{matrix}\right) \tag{3.3a}$$

where the homogeneous prefactor h is given by

$$h = \frac{z_{14}^{\Delta_2+\Delta_3-\Delta_1-\Delta_4} \, z_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4}}{z_{13}^{\Delta_1+\Delta_2+\Delta_3-\Delta_4} \quad z_{24}^{2\Delta_2}}, \tag{3.3b}$$

and η is the (unique) Möbius invariant cross ratio,

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \left(= 1 - \frac{z_{14} z_{23}}{z_{13} z_{24}}\right). \tag{3.4}$$

The prefactor is determined uniquely from the following three conditions:

(i) it is a translation (L_{-1}-)invariant homogeneous function of z_i of degree $-\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4$ (L_0-invariance);

(ii) it is a product of powers of z_{i4} and z_{13} (z_{12} and z_{23} are excluded);

(iii) for large $|z_i|$ (and finite z_j for $j \neq i$) it behaves as $|z_i|^{-2\Delta_i}$ (L_1-invariance).

Inserting (3.3) into (3.1) we find

$$(k+2)\frac{\mathrm{d}f}{\mathrm{d}\eta} = \left(\frac{\Omega_{12}}{\eta} - \frac{\Omega_{23}}{1-\eta}\right) f. \tag{3.5}$$

The absence of z_{12} and z_{23} in the prefactor in Eq. (3.3) and the related choice of Eq. (3.5) is singled out by the symmetry of (3.5) with respect to the "duality transformation" (see Sec. 3B below),

$$I_1 \leftrightarrow I_3, \quad z_1 \leftrightarrow z_3 \;\; (\eta \leftrightarrow 1-\eta,\; \frac{\mathrm{d}}{\mathrm{d}\eta} \to -\frac{\mathrm{d}}{\mathrm{d}\eta}).$$

Thus we reduce Eq. (3.1) to a system of ordinary differential equations for the coefficients $f_l(\eta)$ in the expansion

$$f\left(\eta\big|{}^{\lambda_i}_{\zeta_i}\right) = \sum_l f_l(\eta) J_l(\zeta), \tag{3.6a}$$

in terms of a basis of $SU(2)$ invariant polynomials (both J_l and f_l depending on the weights)

$$J_l(\zeta) = J_l\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{pmatrix}. \tag{3.6b}$$

Of course, the explicit form of the solution of the KZ equations depends on the choice of J_l. Here we depart from the tradition ([2, 3, 8, 11–14]) consisting in diagonalizing either Ω_{12} or Ω_{23} (then the other one appears as a matrix with nonzero elements on the main diagonal and the two adjacent ones). Following [16], we impose instead the following more symmetric (with respect to the exchange of the two Ω) condition:

$$\Omega_{12} J_l = a_l J_l + b_l J_{l-1}, \quad \Omega_{23} J_l = c_l J_{l+1} + d_l J_l. \tag{3.7a}$$

This alteration turns out to be more substantial than one could have foreseen: the resulting basis of solutions of the KZ equation is regular even at points (of interest) at which the standard basis is ill-defined.

In order to specify the coefficients $a_l, \ldots, d_l$ in Eq. (3.7a), we must restrict the order of the weights λ_i. We shall choose the weight λ_4 to be the minimal one. More precisely, setting

$$I_{ij} = I_i - I_j \qquad \text{for} \quad \lambda_i = 2I_i, \tag{3.8a}$$

we assume

$$\lambda_4 = \min \lambda_i \equiv m, \qquad |I_{12}| \le I_{34}, \quad |I_{23}| \le I_{14}. \tag{3.8b}$$

(This choice ensures that the minimal isospin is I_{34} in the "s-channel" $12 \to 34$, and I_{14} in the "u-channel" $23 \to 14$. It simplifies subsequent calculations, in particular for the quantum group case considered in Sec. 3C, without affecting the results.) Then we can set

$$J_l = \zeta_{12}^{\mu_{12}} \zeta_{23}^{\mu_{23}} \zeta_{34}^{\mu_{34}} \zeta_{13}^{\mu_{13}} \zeta_{14}^{\mu_{14}}, \tag{3.9a}$$

where

$$\begin{aligned} \mu_{12} &= I_1 + I_{23} + I_4 - l, \quad \mu_{14} = l, \;\; \mu_{34} = m - l \\ \mu_{23} &= I_{21} + I_{34} + l, \qquad \mu_{13} = I_{12} + I_{34}, \end{aligned} \tag{3.9b}$$

thus recovering (3.7a) with

$$\begin{aligned} a_l &= (I_{34} + l)(I_{34} + l + 1) - I_1(I_1 + 1) - I_2(I_2 + 1), \\ b_l &= l\mu_{23}, \qquad\qquad c_l = \mu_{12}\mu_{34}, \\ d_l &= (I_{14} + m - l)(I_{14} + m - l + 1) - I_2(I_2 + 1) - I_3(I_3 + 1). \end{aligned} \tag{3.7b}$$

Inserting (3.6) and (3.9) into (3.5), we find that f_l satisfy the system of equations

$$(k+2)f_l' = \left(\frac{a_l}{\eta} - \frac{d_l}{1-\eta}\right) f_l + \frac{b_{l+1}}{\eta} f_{l-1} - \frac{c_{l-1}}{1-\eta} f_{l+1}, \quad f' \equiv \frac{\mathrm{d}f}{\mathrm{d}\eta}, \tag{3.10}$$

where $a_l, \ldots, d_l$ are given by (3.7b). This system of $m+1$ ordinary differential equations (for the $m+1$ functions f_l, $l = 0, \ldots, m$) has $m+1$ linearly independent solutions which we shall label by an additional index λ $(= 0, \ldots, m)$. The solutions are obtained by analytic continuation from the interval $0 < \eta < 1$ in which

$$f_{\lambda l}(\eta) = \eta^{l-\Delta_1-\Delta_2+\Delta_{34}} (1-\eta)^{m-l-\Delta_2-\Delta_3+\Delta_{14}} \frac{g_{\lambda l}(\eta)}{l!(m-l)!} \tag{3.11a}$$

$$g_{\lambda l}(\eta) = N \int_0^\eta \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{\lambda-1}} \mathrm{d}t_\lambda \int_\eta^1 \mathrm{d}t_{\lambda+1} \int_{t_{\lambda+1}}^1 \mathrm{d}t_{\lambda+2} \ldots \int_{t_{m-1}}^1 \mathrm{d}t_m P_{\lambda l}(t_i; \alpha, \beta, \gamma), \tag{3.11b}$$

where $N = N_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is independent of λ and l, the kernel $P_{\lambda l}$ is a product of fractional powers of coordinate differences

$$\begin{aligned} P_{\lambda l}(t_i;\alpha,\beta,\gamma) &= \prod_{i=1}^{m} t_i^{\alpha}(1-t_i)^{\beta}\prod_{i=1}^{\lambda}(\eta-t_i)^{\gamma-1}\times \\ &\qquad \prod_{j=\lambda+1}^{m}(t_j-\eta)^{\gamma-1}\prod_{i<j}(\epsilon_{\lambda j}t_{ij})^{2\delta}\sum_{\sigma}\prod_{s=1}^{l}t_{i_s}^{-1}\prod_{r=l+1}^{m}(1-t_{i_r})^{-1}, \\ \epsilon_{\lambda j} &= \begin{cases} 1 & \text{for } \lambda\le j \\ -1 & \text{for } \lambda>j \end{cases}, \qquad t_{ij}=t_i-t_j, \end{aligned} \tag{3.12a}$$

and the sum extends over all $m!$ permutations $\sigma : (1,\ldots,m) \to (i_1,\ldots,i_m)$; here

$$\begin{aligned} &(k+2)\alpha = I_{34}-I_{12}+1, && (k+2)\beta = I_{14}+I_{23}+1 \\ &(k+2)\gamma = I_{12}+I_{34}+1 = I_{14}-I_{23}+1, && (k+2)\delta = 1. \end{aligned} \tag{3.12b}$$

The integrand (3.12) has the same structure as in [12] and [13]. The main difference lies in the integration contours in (3.11b). Its significance and the advantages of our choice will be discussed below in connection with the properties of exchange operators. We note that both the KZ equation and its solutions make perfect sense for any real $k > -2$.

3B Braid relations for KZ amplitudes

As expected, the KZ amplitudes

$$W_\lambda(z,\zeta) = h(z_{13},z_{14},z_{24},z_{34})f_\lambda(\eta,\zeta), \tag{3.13a}$$

where h is given by (3.3b) and

$$f_\lambda(\eta,\zeta) = \sum_{l=0}^{m} f_{\lambda l}(\eta)J_l(\zeta), \tag{3.13b}$$

are multivalued functions of z_i. The exchange of two neighbouring factors in a vacuum expectation value requires analytic continuation along a definite path. We shall work out the action on W_λ of the exchange operators B_1 and B_2. To define B_i one first performs analytic continuation in z_i, z_{i+1} along the path

$$C_i: \quad \begin{pmatrix} z_i \\ z_{i+1} \end{pmatrix} \Rightarrow \frac{1}{2}(z_i + z_{i+1}) \pm \frac{1}{2} z_{ii+1} e^{i\pi t} \quad 0 \leq t \leq 1 \tag{3.14}$$

(more precisely, along any path in the homotopy class of C_i that does not go around or touch any other point z_j $j \neq i, i+1$; one starts with a W_λ defined in a simply connected complex neighbourhood of the primitive real domain of analyticity $z_1 > z_2 > z_3 > z_4 > -z_3$). Secondly, one permutes (λ_i, ζ_i) and $(\lambda_{i+1}, \zeta_{i+1})$ with the result:

$$B_1: \; h \to h(1-\eta)^{\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3}, \qquad \eta \to \frac{e^{i\pi}\eta}{1-\eta}, \; \alpha \leftrightarrow \gamma,$$

$$J_l(\zeta) \equiv J_l \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{pmatrix} \to J_l \begin{pmatrix} \lambda_2 & \lambda_1 & \lambda_3 & \lambda_4 \\ \zeta_2 & \zeta_1 & \zeta_3 & \zeta_4 \end{pmatrix} = b_1^{ll'} J_{l'}(\zeta) \tag{3.15a}$$

where summation is to be carried over the repeated index l' from 0 to m and

$$b_1^{ll'} = (-1)^{I_1 + I_2 - I_{31} - l} \begin{pmatrix} l \\ l' \end{pmatrix}, \qquad \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}; \tag{3.15b}$$

$$B_2: h \to h\eta^{\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3}, \qquad \eta \to \frac{1}{\eta} \tag{3.15c}$$

(the path from η to $\frac{1}{\eta}$ going around 1 from below), $\beta \leftrightarrow \gamma$,

$$J_l \to b_2^{ll'} J_{l'}, \qquad b_2^{ll'} = (-1)^{I_2 + I_3 - I_{14} - (m-l)} \begin{pmatrix} m-l \\ m-l' \end{pmatrix}. \tag{3.15d}$$

In order to compute the action of B_i on the integral representation (3.11) we perform the change of integration variables

$$B_1: \quad t_i \to \eta + (1-\eta) t_i, \tag{3.16a}$$

$$B_2: \quad t_i \to \eta t_i, \tag{3.16b}$$

and transform the integral representation (3.11) back into its initial form.

Since the calculation is long, we shall first present the result and then explain its derivation in more detail.

For the KZ amplitudes (3.13) we find

$$B_i : \; W_\lambda(z,\zeta) \to \sum_\mu (B_i)_{\lambda\mu} W_\mu(z,\zeta), \tag{3.17}$$

where the exchange matrix $B_1 = B\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix}$ is upper triangular,

$$B\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix}_{\lambda\mu} = (-1)^{I_1+I_2-I_{34}-\mu} e^{i\pi(-\Delta_1-\Delta_2+\Delta_{34}+\lambda\alpha+\mu\gamma+\delta\lambda(\lambda-1))} \binom{\mu}{\lambda}_+, \tag{3.18a}$$

$$\binom{\mu}{\lambda}_+ = \frac{(\lambda+1)_+ \ldots (\lambda+(\mu-\lambda))_+}{(1)_+ \ldots (\mu-\lambda)_+}, \quad (\mu)_+ = \sum_{n=0}^{\mu-1} e^{2i\pi\delta n}, \tag{3.18b}$$

while $B_2 = B\begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix}$ is lower triangular and can be expressed in terms of B_1 as:

$$B\begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} = FB\begin{bmatrix} I_3 & I_2 \\ 0 & I_{14} \end{bmatrix} F. \tag{3.18c}$$

Here F is the antidiagonal matrix

$$F_{\lambda\mu} = \delta_m^{\lambda+\mu}, \qquad F^2 = 1; \tag{3.19}$$

in general, the operator $B\begin{bmatrix} I & J \\ L & R \end{bmatrix}$ changes the (neighbouring) factors of labels I, J between the states of *minimal isospin* L and R in a 4-point block.

We now turn to the derivation of the expression for B_1. Simple algebraic manipulations, using the formula

$$t_i - \eta = (1-\eta)t_i - \eta(1-t_i),$$

show that the exchange matrix b_1 (3.15b) compensates the change of the kernel (3.12a) under the permutation $\alpha \leftrightarrow \gamma$.

Applying the transformations (3.15a), (3.16a) to $g_{\lambda I}(\eta)$ (3.11b), we obtain an integral with λ contours in the interval $(0,\eta)$ and $m-\lambda$ contours in the interval $(0,1)$, which overlap with the first λ ones. We split the second set of contours into two parts and obtain expressions with μ $(\mu \geq \lambda)$ integrations from 0 to η and $m - \mu$ integrations from η to 1. This explains the upper triangular form of B_1. The coefficient is found as follows: the factor $\binom{\mu}{\lambda}_+$ comes from the symmetrization of the new $\mu - \lambda$ contours with respect to the old λ ones. Note that since the kernel (3.12a) is totally symmetric in t_i (because of the sum over all $m!$ permutations), every configuration enters with weight one. The factor $e^{i\pi\delta\lambda(\lambda-1)}$ comes from the reordering of the λ contours (their configuration is inverted when B_1 (3.15a) is applied). The remaining phase factors in (3.18a) are computed in a similar way.

The derivation of (3.18c) for B_2 follows the same pattern.

According to (3.17) the $m+1$ dimensional vector space $\mathcal{L} = \mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of 4-point blocks satisfying the KZ equation is closed under the action of the exchange algebra. The $\mathcal{A}_k(su_2)$ fusion rules (2.63) imply that for positive integer k and for $\lambda_3 + \lambda_4 \geq k+1$ (or $\lambda_1 + \lambda_2 \geq k+1$) there is a ("physical") subspace $\mathcal{P} = \mathcal{P}(\lambda_1, \ldots, \lambda_4)$ of $\mathcal{L}(\lambda_1, \ldots, \lambda_4)$ which is also closed under the action of B_1 and B_2. One can diagonalize either B_1 or B_2 in $\mathcal{P}$, thus obtaining the s-channel or the u-channel basis which corresponds to the OPE $V_{\lambda_1} V_{\lambda_2}$ or $V_{\lambda_2} V_{\lambda_3}$ respectively. It is, however, impossible to diagonalize B_1 in the whole space $\mathcal{L}$ if $\min(\lambda_1 + \lambda_2, \lambda_3 + \lambda_4) \geq k+2$, as is manifest from (3.18a) (the simplest example being provided for $k = 2 = \lambda_i$, $i = 1, \ldots, 4$). A similar restriction holds for B_2 if $\min(\lambda_1 + \lambda_4, \lambda_2 + \lambda_3) \geq k+2$.

As we shall demonstrate in Secs. 3C and 3D below there exists a natural extension of CVO to chiral fields with an internal quantum symmetry U_q. The correlation functions of these fields appear as U_q-extended conformal blocks that are invariant under the combined action $B_i \cdot \mathcal{B}_i$ of the KZ and U_q exchange operators at the price of involving the big space $\mathcal{L}$. This is why one is bound to use a regular basis in $\mathcal{L}$ of the type considered here.

Using (3.18) one can verify that B_i satisfies the (parameter-free) Yang-Baxter equation

$$B\begin{bmatrix} I_2 & I_3 \\ 0 & I_{14} \end{bmatrix} B\begin{bmatrix} I_1 & I_3 \\ I_2 & I_4 \end{bmatrix} B\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix} = B\begin{bmatrix} I_1 & I_2 \\ I_3 & I_4 \end{bmatrix} B\begin{bmatrix} I_1 & I_3 \\ 0 & I_{24} \end{bmatrix} B\begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix}$$

$$= (-1)^{I_1+I_2+I_3-I_4} e^{-i\pi(\Delta_1+\Delta_2+\Delta_3-\Delta_4)} F, \tag{3.20}$$

where F is defined in (3.19).

Although the explicit form of the fusion matrix (3.19) appearing in the right-hand side of (3.20) is basis dependent, its square, the monodromy matrix

$$B\begin{bmatrix} I_2 & I_1 \\ 0 & I_{34} \end{bmatrix} B\begin{bmatrix} I_3 & I_1 \\ I_2 & I_4 \end{bmatrix} M\begin{bmatrix} I_2 & I_3 \\ 0 & I_{14} \end{bmatrix} B\begin{bmatrix} I_1 & I_3 \\ I_2 & I_4 \end{bmatrix} B\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix}, \tag{3.21a}$$

where

$$M\begin{bmatrix} I_2 & I_3 \\ 0 & I_{14} \end{bmatrix} = B\begin{bmatrix} I_3 & I_2 \\ 0 & I_{14} \end{bmatrix} B\begin{bmatrix} I_2 & I_3 \\ 0 & I_{14} \end{bmatrix}, \tag{3.21b}$$

is a multiple of the unit matrix independent of any choices. The resulting identity is related to the fact that we are dealing with a (projective) representation of the braid group on the Riemann sphere.

3C U_q-coherent state operators and invariant n-point functions

In the preceding section we have written a basis W_μ in the $(m+1)$-dimensional space of conformal current algebra blocks. The question arises of whether one can introduce the notion of a *chiral field* (acting, possibly, in a bigger space than the physical CVO) so that W_μ appear as coefficients in the expansion of a chiral Green function G in certain internal symmetry invariants $\mathcal{J}_\mu$. If that is the cas then each of the representations λ of $\mathcal{A}_k(su_2)$ will appear with some finite multiplicity (the multiplicity of the vacuum, $\lambda = 0$, being 1). It is clear from the expansion formula

$$\phi_{\lambda\overline{\lambda}}(z,\zeta;\overline{z},\overline{\zeta}) = V_{\lambda\alpha}(z,\zeta)\overline{V}^{\alpha}_{\overline{\lambda}}(\overline{z},\overline{\zeta}), \tag{3.22}$$

(summation being understood over the repeated index α) that the chiral fields V_λ ($= V_{\lambda\alpha}$) are not determined uniquely by the *physical field* $\phi_{\lambda\overline{\lambda}}$. Indeed, a transformation

$$V_\lambda \to V_\lambda S \quad (\text{or } V_{\lambda\alpha} \to V_{\lambda\alpha'} S_\alpha^{\alpha'}), \quad \overline{V}_{\overline{\lambda}} \to S'^{-1} \overline{V}_{\overline{\lambda}}, \tag{2.23}$$

leaves $\phi_{\lambda\overline{\lambda}}$ invariant and can be viewed as a first kind gauge transformation for the chiral components of ϕ. Thus, the internal gauge symmetry of V_λ arises quite naturally. Furthermore, the monodromy invariance of ϕ,

$$\phi_{\lambda\overline{\lambda}}(e^{2\pi i} z, \zeta; e^{-2\pi i}\overline{z}, \overline{\zeta}) = \phi_{\lambda\overline{\lambda}}(z, \zeta; \overline{z}, \overline{\zeta}), \tag{3.23}$$

is recovered for

$$V_\lambda(e^{2\pi i} z, \zeta) = V_\lambda(z, \zeta) M, \quad \overline{V}_{\overline{\lambda}}(e^{-2\pi i}\overline{z}, \overline{\zeta}), = M^{-1} \overline{V}_{\overline{\lambda}}(\overline{z}, \overline{\zeta}). \tag{3.24}$$

Fields V_λ with a non-trivial monodromy obey (as we have seen) a braid group statistics. An internal symmetry algebra that commutes with the statistics operator for a braid group statistics cannot be a group algebra. A lucid analysis by Mack and Schomerus [6] shows that such a *quantum symmetry* can, in general, be described by a *weak quasi Hopf algebra* U_q (the notion of a quasi Hopf algebra with a weakened co-associativity condition was introduced earlier by Drienfeld [19] in analysing the Knizhnik-Zamolodchikov equations). Applying their general scheme to the critical Ising model, Mack and Schomerus have proved that in a *positive metric framework* for the chiral theory with an internal quantum symmetry, one does need a co-product $\Delta : U_q \to U_q \otimes U_q$ that is only almost co-associative and which, applied to the unit element in U_q, gives a projection in $U_q \otimes U_q$. Following a parallel development [3, 16], we shall demonstrate that if one allows for an indefinite metric in the U_q-extended state space of the (gauge dependent) chiral theory, one can describe the quantum symmetry by the more regular concept of a co-associative Hopf algebra or *quantum group.*

One can hardly avoid learning about quantum groups these days. We shall not, therefore, interrupt our exposition to introduce the notion of a *quantumm universal enveloping* (QUE) *algebra* (see, e.g. [20,21], and the Lecture Notes volume cited in [3]).

The chiral quantum symmetry of diagonal $\mathcal{A}_k(su_2)$ models is given, as we shall demonstrate below, by the simplest and best studied QUE algebra

$$U_q := U_q(sl_2) \quad \text{with } q = e^{i\frac{\pi}{k+2}}. \tag{3.25}$$

The internal (quantum) symmetry index of each field will be again substituted by a formal variable u. We thus obtain a U_q *coherent state operator* [7]

$$V_\lambda(z,\zeta;u) = \sum_{n=0}^{\lambda} \frac{u^n}{(n)_-!} V_\lambda^{2n-\lambda}(z,\zeta) \tag{3.26}$$

of weight λ $(= 0,1,2,\ldots)$ which satisfies the U_q covariance conditions $[H, V_\lambda(u)] = (2u\frac{\partial}{\partial u} - \lambda)V_\lambda(u)$ or $[H, V_\lambda^m] = mV_\lambda^m$, so that

$$\begin{aligned}
q^H V_\lambda(u) &= V_\lambda(q^2u)q^{H-\lambda}, && (3.27\text{a})\\
[E, V_\lambda(u)] &= D_- V_\lambda(u) q^{-H} && \\
\text{or } [E, V_\lambda^m] &= V_\lambda^{m+2} q^{-H}, && (3.27\text{b})\\
FV_\lambda(u) - q^{-\lambda}V_\lambda(q^2u)F &= -q^{\lambda+1}u^{u+2}D_+\left(\frac{V_\lambda(u)}{u^\lambda}\right) && \\
\text{or } FV_\lambda^{2n-\lambda} - q^{2n-\lambda}V_\lambda^{2n-\lambda}F &= [\lambda-n+1][n]V_\lambda^{2n-\lambda-2} && (3.27\text{c})
\end{aligned}$$

where $D_\pm$ are the q-derivatives:

$$D_\pm f(u) = \frac{f(q^{\pm 2}u) - f(u)}{(q^{\pm 2}-1)\,u}, \quad D_\pm u^n = (n)_\pm u^{n-1} \tag{3.28}$$

and the q-numbers $[n]$ and $(n)_\pm$ are given by

$$(n)_\pm = \frac{q^{\pm 2n}-1}{q^{\pm 2}-1} = q^{\pm(n-1)}[n]. \tag{3.29}$$

(We omit the spectator arguments (z,ζ) whenever just considering the u-transformation properties.) The U_q invariance of the vacuum is expressed by

$$X\,|0\rangle = \epsilon(X)\,|0\rangle, \tag{3.30}$$

$\epsilon : U_q \to \mathbb{C}$ being the co-unit (or trivial representation of U_q – see Appendix B). The action of V_λ on the vacuum can be written in terms of the U_q raising operator E as

$$V_\lambda(z,\zeta;u)\,|0\rangle \;=\; e_-(uE)V_\lambda^{-\lambda}(z,\zeta)\,|0\rangle \tag{3.31a}$$

$$\langle 0|\,V_\lambda(z,\zeta;u) \;=\; \langle 0|\,V_\lambda^{\lambda}(z,\zeta)e_-(u\gamma(E)) \tag{3.31b}$$

where the q-exponents $e_\pm(x)$ are given by

$$e_\pm(x) \;=\; \sum_{n=0}^{\infty}\frac{x^n}{(n)_\pm!},\quad (n)_\pm! = (n)_\pm(n-1)_\pm!,\quad (0)_\pm! = 1 \tag{3.32a}$$

$$e_+(x)e_-(-x) \;=\; 1, \tag{3.32b}$$

and γ is the antipode,

$$\gamma(E) = -Eq^H. \tag{3.33}$$

We shall demonstrate that one can define a monodromy operator acting on the U_q degrees of freedom in such a way that it would compensate the world sheet monodromy. In order to do this in a constructive manner, we shall need a basis of n-point U_q invariants. We shall write down such a basis for arbitrary n which appears as a simple q-deformation of the basis of $SU(2)$ invariants (given, for $n=4$, by (3.9)). We note that customarily one uses an alternative basis of *tree graph invariants* associated with Clebsch-Gordan expansions of tensor products of U_q representations [8]. Our choice (related, as we shall see, to the basis W_λ of solutions (3.13) (3.11) of the KZ equation) is both simpler in appearence and has the advantage of being well-behaved in the physically interesting case of q a root of $(-)1$.

Proposition 3.1 *For a given set of positive weights $\lambda_1,\ldots,\lambda_n$ (satisfying certain stability conditions implied by the requirement that the system (3.36) below has a solution with $\mu_{ij}\in\mathbb{Z}_+$) there exists a basis of U_q invariant monomials*

$$\mathcal{J}_{\{\mu\}}^{(\lambda_1\ldots\lambda_n)} = \begin{bmatrix}\lambda_1\ldots\lambda_n\\ u_1\ldots u_n\end{bmatrix}_{\{\mu\}} = \prod_{1\le i<j\le n} w_{ij} \tag{3.34}$$

where $w_{ij}\equiv w(u_i,u_j;\mu_{ij},\rho_{ij})$ are q-deformations of $(u_i-u_j)^{\mu_{ij}}$:

$$w_{ij} = \prod_{n=0}^{\mu_{ij}-1}\left(q^{-\frac{1}{2}-\rho_{ij}}u_i - q^{-\frac{1}{2}+\rho_{ij}+\mu_{ij}-2n}u_j\right), \tag{3.35}$$

$\mu_{ij}(=\mu_{ji})$ *are non-negative integers satisfying the conservation law*

$$\sum_j \mu_{ij} = \lambda_i, \qquad \mu_{ii} = 0 \tag{3.36}$$

and the selection rules

$$\mu_{ij}\mu_{kl} = 0 \quad \textit{if } i < k < j < l \quad (\textit{or } k < i < l < j). \tag{3.37}$$

If $\mu_{ij} > 0$, *then* $2\rho_{ij}$ *is determined as an integer:*

$$2\rho_{ij} = \sum_{i<k\leq j} \lambda_k + \mu_{ij} - 2 \sum_{i\leq k<l\leq j} \mu_{kl}. \tag{3.38}$$

The constraints (3.36) and (3.37) leave room for an $n-3$ *(integer) parameter family of invariants.*

Remarks:

(1) For $q = e^{i\frac{\pi}{p}}$ $(p = k+2 = 3, 4, \ldots)$ the exponents ρ_{ij} in (3.35) are determined up to an integer multiple of p. Eq. (3.38) gives the unique integer-valued solution of the E-invariance requirement for generic q.

(2) We shall prove that for $q \neq 1$ the selection rule (3.37) is *necessary* for U_q invariance. Thus in the q-deformed case the factorized *regular basis* (3.34) is essentially unique.

Proof: The QUE covariance law (3.27) for $V_\lambda(u)$ implies the following constraints on n-point U_q invariants (viewed as "vacuum expectation values" of products of V_λ)

$$H : \begin{bmatrix} \lambda_1, & \ldots, & \lambda_n \\ z^2 u_1, & \ldots, & z^2 u_n \end{bmatrix} = z^{\lambda_1+\lambda_2+\ldots+\lambda_n} \begin{bmatrix} \lambda_1, & \ldots, & \lambda_n \\ u_1, & \ldots, & u_n \end{bmatrix} \tag{3.39a}$$

(we assume this strong H-invariance condition, valid for any complex $z \neq 0$ rather than just for $z = q$, in order to exclude spurious invariants for q a root of 1);

$$E \ : \ \sum_{k=1}^{n} q^{\lambda_{k+1}+\ldots+\lambda_n} D_{k-} \left[\begin{array}{cccccc} \lambda_1, & \ldots, & \lambda_k, & \lambda_{k+1}, & \ldots, & \lambda_n \\ u_1, & \ldots, & u_k, & q^{-2}u_{k+1}, & \ldots, & q^{-2}u_n \end{array} \right] = 0, \tag{3.39b}$$

$$F \ : \ \sum_{k=1}^{n} q^{-\lambda_1-\ldots-\lambda_{k-1}} (qu_k)^{\lambda_k+2} \times$$

$$D_{k+} \left\{ u_k^{-\lambda_k} \left[\begin{array}{cccccc} \lambda_1, & \ldots, & \lambda_{k-1}, & \lambda_k, & \ldots, & \lambda_n \\ q^2 u_1, & \ldots, & q^2 u_{k-1}, & u_k, & \ldots, & u_n \end{array} \right] \right\} = 0. \tag{3.39c}$$

We shall first prove that $\mathcal{J}_{\{\mu\}}$ (3.34) satisfy (3.39) for μ_{ij}, ρ_{ij} obeying (3.36,37,38). Inserting $\mathcal{J}_{\{\mu\}}$ with arbitrary parameters ρ_{ij}, μ_{ij} in (3.39a) we obtain the homogeneity relation

$$\sum_{i<j} \mu_{ij} = \frac{1}{2} \sum_i \lambda_i \tag{3.40}$$

(which is a corollary of (3.36)).

The requirement for E-invariance implies (3.37) and (3.38). To demonstrate this we shall need some formulae for D_- derivatives and the properties of w_{ij}. For the reader's convenience we list them together with the corresponding formulae for D_+ derivatives necessary for checking F-invariance.

A basic tool for working with $D_\pm$ is the q-deformed Leibniz formula

$$D_\pm \{f(u)g(u)\} = \{D_\pm f(u)\} g(u) + f(q^{\pm 2}u)\{D_\pm g(u)\}. \tag{3.41}$$

The $D_\pm$ derivatives of $w(u,v;\mu;\rho)$ are given by

$$D_{u-} w(u,v;\mu;\rho) = q^{-\frac{\mu}{2}-\rho}[\mu] w(u,v;\mu-1;\rho+\frac{1}{2}), \tag{3.42a}$$

$$D_{v-} w(u,v;\mu;\rho) = -q^{1-\frac{\mu}{2}+\rho}[\mu] w(u,v;\mu-1;\rho-\frac{1}{2}), \tag{3.42b}$$

$$D_{u+} w(u,v;\mu;\rho) = q^{-1+\frac{\mu}{2}-\rho}[\mu] w(u,v;\mu-1;\rho-\frac{1}{2}), \tag{3.42c}$$

$$D_{v+} w(u,v;\mu;\rho) = -q^{\frac{\mu}{2}+\rho}[\mu] w(u,v;\mu-1;\rho+\frac{1}{2}). \tag{3.42d}$$

We shall also need the transformation law of w under dilation of one of the arguments,

$$w(q^{\pm 1}u,v;\mu;\rho) = q^{\pm\frac{\mu}{2}} w(u,v;\mu;\rho \mp \frac{1}{2}), \tag{3.43a}$$

$$w(u,q^{\pm 1}v;\mu;\rho) \;=\; q^{\pm\frac{\mu}{2}} w(u,v;\mu;\rho\pm\frac{1}{2}), \tag{3.43b}$$

and the reduction formulae

$$\begin{aligned} w(u,v;\mu;\rho) &= \left(q^{-\frac{\mu}{2}-\rho}u - q^{\frac{\mu}{2}+\rho}v\right) w(u,v;\mu-1;\rho-\frac{1}{2}) \\ &= \left(q^{-\frac{\mu}{2}-\rho-1}u - q^{\frac{\mu}{2}+\rho+1}v\right) w(u,v;\mu-1;\rho+\frac{1}{2}). \end{aligned} \tag{3.44}$$

Inserting (3.34) in (3.39b) and using (3.41,42,43) we obtain a polynomial in u_i which is written as a linear combination of products of factors $w_{ij}(u_i,u_j;\tilde{\mu}_{ij};\tilde{\rho}_{ij})$ (where $\tilde{\rho}_{ij}$, $\tilde{\mu}_{ij}$ differ by (half) an integer from μ_{ij}, ρ_{ij}). Eq. (3.44) allows us to extract and cancel certain common factors, thus reducing the degree of the polynomial in u_i.

We can think of a graphical representation of the invariant (3.34): marking the ordered set of points $(1,2,\dots,n)$ on a segment $(1,n)$, we associate an arc connecting i and j to every pair (i,j) for which the integer μ_{ij} defined by (3.35) is different from 0. Then Eq. (3.37) has an obvious graphical meaning: no two arcs are allowed to intersect.

To prove this (and at the same time to derive the relation (3.38)), it is convenient first to derive (3.38) for $j=i+1$ and then to cancel the factors $w_{i\,i+1}$ (which are never excluded by (3.37)). The next step is to establish (3.37) whenever one of the factors is $\mu_{i\,i+2}$ and then continue the procedure recursively.

In order to verify that (3.36) (on top of (3.37-38)) ensures F-invariance, we shall need the following result.

For each graph satisfying (3.37), the function $\mathcal{J}_{\{\mu\}}$ contains at least one triple of consecutive vertices $(l-1,l,l+1)$ such that $w_{l-1\,l}w_{l\,l+1}$ carries the entire u_l dependence. Moreover, Eq. (3.39c) implies that $\mu_{l-1\,l}$ and $\mu_{l\,l+1}$ satisfy (3.36),

$$\mu_{l-1\,l} + \mu_{l\,l+1} = \lambda_l, \tag{3.45}$$

while the remaining (u_l-independent) factor satisfies an equation of the type

(3.39c) for F-invariance of a $(n-1)$-point function

$$\begin{bmatrix} \lambda_1, & \dots, & \tilde{\lambda}_{l-1}, & \tilde{\lambda}_{l+1}, & \dots, & \lambda_n \\ u_1, & \dots, & \tilde{u}_{l-1}, & \tilde{u}_{l+1}, & \dots, & u_n \end{bmatrix}, \tag{3.46a}$$

where

$$\begin{aligned} \tilde{\lambda}_{l-1} &= \lambda_{l-1} - \mu_{l-1\,l}, \qquad \tilde{\lambda}_{l+1} = \lambda_{l+1} - \mu_{l\,l+1}, \\ \tilde{u}_{l\mp 1} &= u_{l\mp 1} q^{\pm\frac{1}{2}\mu_{l\pm 1\,l}}. \end{aligned} \tag{3.46b}$$

The crucial point is that for this reduced $(n-1)$-point invariant, we can again apply the above procedure and reduce it further. After $n-2$ steps we obtain an equation for a 2-point function, the solution of which is given by a single w_{ij}, and prove that (3.36) is necessary and sufficient for F-invariance. It follows also that the total number of factors w_{ij} in the product (3.34) can not exceed $2n-3$.

The monomials (3.34) for different μ_{ij} are clearly independent. On the other hand their number concides with the dimension of the space of n-point U_q invariants which is equal to the number of linearly independent (undeformed) $SU(2)$ invariants. Indeed, for $q \to 1$, every monomial (3.34) becomes an $SU(2)$ invariant (w_{ij} becomes $(\zeta_i - \zeta_j)^{\mu_{ij}}$). Conversely, the identity

$$\zeta_{ik}\zeta_{jl} = \zeta_{ij}\zeta_{kl} + \zeta_{il}\zeta_{jk} \quad (\zeta_{ij} = \zeta_i - \zeta_j) \tag{3.47}$$

allows us to write every $SU(2)$ invariant as a linear combination of such monomials.

Thus $\mathcal{J}_{\{\mu\}}$ (3.34) provides a basis in the space of all n-point U_q invariants.

We note that the proof of Proposition 3.1 is constructive and we shall write down as an illustration the values of μ_{ij} and ρ_{ij} for $n = 3, 4$.

For $n = 3$ an invariant exists if

$$\mu_{12} = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3), \quad \mu_{23} = \frac{1}{2}(\lambda_2 + \lambda_3 - \lambda_1), \quad \mu_{12} = \frac{1}{2}(\lambda_1 + \lambda_3 - \lambda_2), \tag{3.48a}$$

are non-negative integers; then

$$2\rho_{12} = \mu_{23}, \quad 2\rho_{23} = \mu_{13}, \quad 2\rho_{13} = \mu_{12}. \tag{3.48b}$$

The 2-point function is obtained as a special case for $\lambda_3 = 0 = \mu_{23} = \mu_{13}$ $(= \rho_{12})$. For $n = 4$ and λ_i satisfying (3.8) μ_{ij} are given by the expression (3.9b) for the undeformed invariants:

$$
\begin{aligned}
&0 \le \mu_{14} \equiv \mu \le m = \lambda_4 (= \min(\lambda_i,\, 1 \le i \le 4)), \qquad \mu_{12} = m_{12} + m - \mu, \\
&\mu_{34} = \lambda_{14} + \lambda_{23}, \quad \mu_{23} = m_{23} + \mu, \quad 2\mu_{13} = \lambda_{12} + \lambda_{34} \quad (\mu_{24} = 0) \\
&2m_{12} = \lambda_{14} + \lambda_{23}, \quad 2m_{23} = \lambda_{34} - \lambda_{12}, \quad \lambda_{ij} = \lambda_i - \lambda_j
\end{aligned}
\tag{3.49a}
$$

the corresponding ρ_{ij} being

$$
\begin{aligned}
&2\rho_{12} = \mu_{23}, \quad 2\rho_{23} = \frac{1}{2}(\lambda_{14} - \lambda_{23}) + m - \mu, \\
&2\rho_{34} = \mu, \quad 2\rho_{13} = -m_{12}, \quad 2\rho_{14} = \mu - \lambda_1.
\end{aligned}
\tag{3.49b}
$$

We note that the constraints (3.8b) ensure the stability condition $m_{ij} \ge 0$ and the inequalities

$$\rho_{i\,i+1} \ge 0, \qquad \rho_{13} \le 0, \qquad \rho_{14} \le 0. \tag{3.50}$$

They simplify the consideration of amplitudes with various exchanges of the arguments $1, 2, 3$, since μ_{13} remains non-negative under any permutation of λ_1, λ_2 and λ_3 provided that the inequalities (3.8b) are satisfied.

3D $\mathcal{R}$-matrix, exchange operators; monodromy-free chiral Green's functions and their 2-dimensional counterparts

The main characteristic of a quantum group is its *almost co-commutativity*. There are elements X of U_q (e.g. E and F for $U_q(sl_2)$) such that the coproduct $\Delta(X)$ differs from the permuted coproduct

$$\Delta'(X) = \sigma \cdot \Delta(X) \qquad (\sigma \cdot A \otimes B = B \otimes A), \tag{3.51}$$

but the breaking of co-commutativity is, in a sense, minimal. There is an invertible element, the "*universal $\mathcal{R}$-matrix*", of (a certain closure of) the tensor product of the QUE algebra with itself, that intertwines the two coproducts

$$\Delta'(X) = R\Delta(X)R^{-1}, \qquad \mathcal{R} \in U_q \otimes U_q. \tag{3.52}$$

It is just the difference of $\mathcal{R}$ from a multiple of the unit operator that distinguishes a QUE algebra from an undeformed universal enveloping algebra. The same $\mathcal{R}$ matrix, on the other hand, allows us to define an *exchange* (or *statistic*) *operator* in the tensor product of any two U_q modules, and as a consequence, in the space of U_q invariants. Indeed, if

$$\Delta(A) = \sum_i A_L^i \otimes A_R^i, \qquad R = \sum_n X_n \otimes Y_n, \tag{3.53}$$

(each factor in the tensor products being an element of U_q) and if $\underline{\lambda}$ and $\underline{\mu}$ are two representations of U_q, we define

$$\Delta^{\lambda\mu}(A) = \sum_i \underline{\lambda}\left(A_L^i\right) \otimes \underline{\mu}\left(A_R^i\right) \text{ etc.;}$$

then the exchange operator $\hat{R}^{\lambda\mu}$ is given by $R^{\lambda\mu}$ followed by the permutation:

$$\hat{\mathcal{R}}^{\lambda\mu} = PR^{\lambda\mu} = \sum_n \underline{\mu}(Y_n) \otimes \underline{\lambda}(X_n)P \tag{3.54a}$$

$$\hat{\mathcal{R}}^{\lambda\mu}: \qquad \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \to \mathcal{V}_\mu \otimes \mathcal{V}_\lambda \qquad (PV_1 \otimes V_2 = V_2 \otimes V_1). \tag{3.54b}$$

The intertwining property (3.52) implies the U_q covariance condition

$$\hat{\mathcal{R}}^{\lambda\mu}\Delta^{\lambda\mu}(A) = \Delta^{\mu\lambda}(A)\hat{R}^{\lambda\mu}, \tag{3.55}$$

which in turn allows us to go from $\hat{\mathcal{R}}^{\lambda\mu}$ to a (projected) operator $\mathcal{B}^{\lambda\mu}$ acting in the space of U_q invariants. It is important for the applications that Drinfeld [20] has given from the outset a construction of the universal $\mathcal{R}$-matrix which allows us to write down explicit formulae for the exchange operators. For U_q (3.25) $\mathcal{R}$ has the simple form

$$\mathcal{R} = \mathcal{Q}\mathcal{S}, \quad \mathcal{Q} = q^{-\frac{1}{2}H\otimes H}, \quad \mathcal{S} = e_+\left((q^{-1} - q)E \otimes F\right), \tag{3.56}$$

where the q exponent is defined in (3.32). In verifying the intertwining property (3.52) it is first useful to establish the relation

$$\mathcal{S}\Delta_q(X) = \Delta_{q^{-1}}(X)\mathcal{S} \text{ for } X = E, F, \tag{3.57}$$

where we have made explicit the dependence of the coproduct on the deformation parameter q:

$$\begin{aligned} \Delta_q(E) &= E \otimes q^{-H} + 1 \otimes E, \\ \Delta_q(F) &= F \otimes 1 + q^H \otimes F. \end{aligned} \tag{3.58}$$

In applying (3.56) it is important to note that for any pair of finite-dimensional representations $\underline{\lambda}$ and $\underline{\mu}$ of U_q, the exponent $\mathcal{S}$ reduces to a polynomial since both $\underline{\lambda}(E)$ and $\underline{\mu}(F)$ are nilpotent. We can further simplify our task by choosing a lowest weight vector in the second factor of the product of vectors on which $\mathcal{R}$ acts. Thus, the action of $\mathcal{B}$ on U_q invariants can be evaluated from the simple relation

$$\hat{\mathcal{R}} V_\lambda(u) V_\mu(0) \hat{\mathcal{R}}^{-1} = q^{-\frac{\lambda\mu}{2}} V_\mu(0) V_\lambda(q^\mu u). \tag{3.59}$$

We apply (3.59) to compute the action of an exchange operator $\mathcal{B}_i = \mathcal{B}_i^{\lambda_1 \ldots \lambda_4}$ on 4-point invariants. The result is

$$\mathcal{B}_i : \mathcal{J}_\mu^{(\lambda)} \to (\mathcal{B}_i)_\mu^\nu \mathcal{J}_\nu^{(\lambda)} \qquad (\lambda) = (\lambda_1, \ldots, \lambda_4) \tag{3.60}$$

(with a summation, as usual, over the repeated index ν from 0 to $m = \lambda_4$), where B_i are the following deformations of the permutation matrices b_i (3.15):

$$\begin{aligned} (\mathcal{B}_1)_\mu^\nu &= \mathcal{B}\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix}_\mu^\nu \\ &= (-1)^{I_1+I_2-I_{34}-\mu} q^{I_1(I_1+1)+I_2(I_2+1)-(I_{34}+\mu)(I_{34}+\nu+1)+I_{12}(\mu-\nu)} \begin{bmatrix} \mu \\ \nu \end{bmatrix}; \end{aligned} \tag{3.61}$$

here we are using the (q-)binomial coefficient (for the q-numbers $[n]$ (3.29))

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \frac{[\mu]!}{[\nu]![\mu-\nu]!}. \tag{3.62}$$

The second exchange operator $\mathcal{B}_2$ is obtained from $\mathcal{B}_1$ by the *duality transformation*

$$\mathcal{B}_2 = \mathcal{B}\begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} = \mathcal{F}\mathcal{B}\begin{bmatrix} I_3 & I_2 \\ 0 & I_{14} \end{bmatrix}\mathcal{F} \quad (\mathcal{F}^2 = 1), \tag{3.63}$$

where (in the regular basis of U_q invariants we are using) $\mathcal{F} = F$ (3.19). We also verify the Yang-Baxter equation ($\mathcal{B}_1\mathcal{B}_2\mathcal{B}_1 \sim \mathcal{F}$) obtained from its current algebra counterpart (3.29) by the substitution

$$e^{i\pi(\Delta_4-\Delta_1-\Delta_2-\Delta_3)} \to q^{I_1(I_1+1)+I_2(I_2+1)+I_3(I_3+1)-I_4(I_4+1)}. \tag{3.64}$$

Observing that $\mathcal{B}_1$ is a lower triangular matrix, we deduce that its eigenvalues coincide with its diagonal elements, obtained for $\mu = \nu$. We find, in particular, for $I_4 = 0(= \mu = \nu)$ the exchange operator for the 3-point function which appears as a phase factor:

$$\mathcal{B}\begin{bmatrix} I_1 & I_2 \\ 0 & I_3 \end{bmatrix} = (-1)^{I_1+I_2-I_3} q^{I_1(I_1+1)+I_2(I_2+1)-I_3(I_3+1)}. \tag{3.65}$$

Similarly, $\mathcal{B}_2$ is an upper triangular matrix whose eigenvalues are obtained from those of $\mathcal{B}_1$ by the substitution $I_1 \leftrightarrow I_3$.

We are now prepared to formulate the main result of this chapter.

Proposition 3.2 *The U_q extended chiral 4-point function*

$$G_4 = G\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ z_1\zeta_1 u_1 , & z_2\zeta_2 u_2 , & z_3\zeta_3 u_3 , & z_4\zeta_4 u_4 \end{pmatrix} = \sum_{\lambda=0}^{m} W_\lambda(z,\zeta)\mathcal{J}_\lambda(u) \tag{3.66}$$

is invariant under the combined action of the exchange operators (3.17) and (3.60) (provided that q is given by (3.25)). It is the unique (up to an overall normalization factor) Möbius, $SU(2)$ and U_q invariant solution of the KZ equation satisfying this property. The constant $N_{\lambda_1 \dots \lambda_4}$ appearing in (3.11b) as well as the (symmetric) structure constants $C_{\lambda\mu\nu}$ of the theory, are determined by the standard s- and u-channel factorization properties in terms of the normalization of 2-point functions.

Proof. The first part of the Proposition follows from the explicit form of the braed matrices (3.18) (3.61 - 63). The expansion (3.66) is invariant under the combined action of the KZ and U_q exchange operators since

$$^tB_i\mathcal{B}_i = \mathbb{1} \qquad \text{for } q^{I(I+1)} = e^{i\pi\Delta_{2I}}. \tag{3.67}$$

We now proceed to formulate the factorization property precisely and to outline the algorithm for computing the structure constants.

First of all, we note that for a generic q (i.e. for $|q| = 1$, q not a root of 1) Eq. (3.66) is equivalent to an s- or a u-channel expansion. To this end, we note that the s-channel U_q-blocks can be written in the form

$$\mathcal{S}_{I_{34}+\lambda}(u) = \sigma_{\lambda\mu}\mathcal{J}_\mu(u) \left(\equiv \sum_{\mu=0}^{m} \sigma_{\lambda\mu}\mathcal{J}_\mu(u)\right), \tag{3.68}$$

where σ is a lower triangular matrix,

$$\sigma_{\lambda\mu} = \sigma_{\lambda\mu}^{I_1\ldots I_4} = \frac{[2I_{34}+\lambda+\mu]![I_{34}-I_{12}+\lambda]!}{[2I_{34}+2\lambda]![I_{34}-I_{12}+\mu]!} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}, \tag{3.69}$$

$\lambda = 0, \ldots, m (= \lambda_4 = 2I_4)$. They form a $\mathcal{B}_1$ diagonal basis:

$$\begin{aligned} \mathcal{B}^s \begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix} &= \sigma^{I_2 I_1 I_3 I_4} \mathcal{B} \begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix} \left(\sigma^{I_1 I_2 I_3 I_4}\right)^{-1} \\ &= (-1)^{I_1+I_2-I_{34}} q^{I_1(I_1+1)+I_2(I_2+1)-I_{34}(I_{34}+1)} \times \\ &\qquad \operatorname{diag}\left\{(-1)^l q^{-l(2I_{34}+l+1)}, \quad l = 0, \ldots, \lambda_4\right\}. \end{aligned} \tag{3.70}$$

Remark. Although (3.70) is not a similarity transformation (for $I_{12} \neq 0$) the σ-factors in the various exchange operators, e.g.

$$\mathcal{B}^s \begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} = \sigma^{I_1 I_3 I_2 I_4} \mathcal{B} \begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} \left(\sigma^{I_1 I_2 I_3 I_4}\right)^{-1},$$

appear in such a way that the U_q counterpart of the Yang-Baxter equation (3.20),

$$\mathcal{B} \begin{bmatrix} I_2 & I_3 \\ 0 & I_{14} \end{bmatrix} \mathcal{B} \begin{bmatrix} I_1 & I_3 \\ I_2 & I_4 \end{bmatrix} \mathcal{B} \begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix} = \\ (-1)^{I_1+I_2+I_3-I_4} q^{I_2(I_2+1)+I_3(I_3+1)-I_4(I_4+1)} \mathcal{F} \tag{3.71}$$

is left invariant, provided that $\mathcal{F}$ also transforms according to

$$\mathcal{F} \to \mathcal{F}^s \begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} = \sigma^{I_3 I_2 I_1 I_4} \mathcal{F} \left(\sigma^{I_1 I_2 I_3 I_4}\right)^{-1}. \tag{3.72}$$

Similarly, we define the s-channel KZ blocks by

$$S_{I_{34}+\lambda}(z,\zeta) = W_\nu(z,\zeta)\sigma_{\nu\lambda}^{-1} = S^l_{I_{34}+\lambda}(z) s_{I_{34}+l}(\zeta), \tag{3.73a}$$

where

$$s_{I_{34}+l}(\zeta) = \sigma_{1ll'} J_{l'}(\zeta), \qquad \sigma_1 = \sigma_1^{I_1 \dots I_4} = \lim_{q\to 1} \sigma^{I_1 \dots I_4} \tag{3.73b}$$

(σ_1 is obtained from σ (3.69) by going back to the undeformed factorials). Thus, for generic q, we can write G_4 as an s-channel expansion:

$$G_4 = W_\lambda(z,\zeta)\mathcal{J}_\lambda(u) = S_{I_{34}+\lambda}(z,\zeta)\mathcal{S}_{I_{34}+\lambda}(u). \tag{3.74}$$

The s-channel factorization condition then reads

$$N_{2I_{34}+2\lambda} \lim_{\zeta_4\to\zeta_3} \zeta_{34}^{\lambda-\mu} \lim_{z_4\to z_3} z_{34}^{\Delta_3+\Delta_4-\Delta(I_{34}+\lambda)} S_{I_{34}+\lambda}(z,\zeta) = \\ W_3\begin{pmatrix} \lambda_1 & \lambda_2 & 2I_{34}+2\lambda \\ z_1\zeta_1 & z_2\zeta_2 & z_3\zeta_3 \end{pmatrix} C_{2I_{34}+2\lambda\,\lambda_3\lambda_4}, \tag{3.75}$$

where W_3 is the 3-point function (2.38), $(k+2)\Delta(I) = I(I+1)$, $N_\mu = C_{\mu\mu 0}$ is the normalization constant of the KZ 2-point function,

$$\langle 0|\, V_\mu(z_1,\zeta_1)V_\mu(z_2,\zeta_2)\,|0\rangle = N_\mu\, z_{12}^{-2\Delta_\mu}\, \zeta_{12}^\mu, \tag{3.76}$$

and $C_{\lambda\mu\nu}$ are the symmetric structure constants which define the normalization of the 3-point function. (Note that the corresponding limit for the amplitude $S^l_{I_{34}+\lambda}$ in (3.73a) is proportional to δ^l_λ.) A similar factorization formula should be valid for the u-channel expansion.

As already noted the transition matrix σ is ill defined for q given by (3.25). It is important, however, that for such q the s-channel blocks (3.73) do exist, provided the fusion rule (2.39) is respected, – i.e. for

$$I_1 + I_2 + I_{34} + \lambda \le k \qquad (q^{k+2} = -1). \tag{3.77}$$

The same observation is valid for u-channel blocks with low intermediate weights. These "physical amplitudes" should again satisfy the factorization condition (3.75) (with the same symmetric structure constants $C_{\lambda\mu\nu}$ in both channels).

In order to explore the factorization properties we shall need the normalization of the leading term of the functions $g_{\lambda\lambda}$ ($g_{m-\lambda\, m-\lambda}$) (3.11b) (3.12) for

$\eta \to 0$ ($\eta \to 1$). They are expressed in terms of the integrals (Dot 1])

$$\begin{aligned} f_n(a,b,\delta) &= \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \prod_{i=1}^n t_i^a (1-t_i)^b \prod_{i>j} (t_i - t_j)^{2\delta} \\ &= \prod_{k=1}^n \frac{\Gamma(k\delta)\Gamma(1+a+(k+1)\delta)\,\Gamma(1+b+(k-1)\delta)}{\Gamma(\delta)\Gamma(2+a+b+(n+k-2)\delta)}. \end{aligned} \tag{3.78}$$

Indeed, setting $f_0 \equiv 1$, we find

$$\begin{aligned} N^{-1} \lim_{\eta\to 0} \left\{ \eta^{\lambda(1-\delta(2I_{34}+\lambda+1))} g_{\lambda\lambda}(\eta) \right\} &= f_\lambda\left((I_{34}-I_{12}+1)\delta - 1, (I_{34}+I_{12}+1)\delta - 1, \delta\right) \times \\ & \qquad f_{m-\lambda}\left(2(I_{34}+\lambda+1)\delta - 1, (I_{14}+I_{23}+1)\delta - 1, \delta\right) \\ &=: A_s^{I_{34}+\lambda}(I_1, I_2, I_3, I_4), \end{aligned} \tag{3.79}$$

δ is given by (3.12b) ($(k+2)\delta = 1$). For the determination of all structure constants it is sufficient to consider the "elastic amplitude" with

$$I_2 = I_3 \geq I_1 = I_4. \tag{3.80}$$

(We shall then only write the first two arguments I_1, I_2 of A_s^I (3.79).) The lowest u-channel weight is 0 in this case, and we have

$$N^{-1} \lim_{\eta\to 1} (1-\eta)^{\lambda[1-\delta(\lambda+1)]} g_{mm}(\eta) = f_{2I_1}\left((2I_{21}+1)\delta - 1, 2\delta - 1, \delta\right) := A_u^0(I_1 I_2). \tag{3.81}$$

For the combination of normalization and structure constants which is independent of arbitrary choices, we find

$$\frac{C^2_{I_1 I_2 I}}{N_{I_1} N_{I_2} N_I} = \frac{A_s^I(I_1, I_2)}{A_u^0(I_1, I_2)} \frac{[I_1+I_2+I+1]![I_1+I_2-I]![I_1+I-I_2]![I_2+I-I_1]!}{[2I_1]![2I_2]![2I]!\left([2I_1+1][2I_2+1][2I+1]\right)^{\frac{1}{2}}}. \tag{3.82}$$

We shall determine the overall factor N in (3.11b) after evaluating the inner product of quantum group invariants which give the coupling between left and right U_q blocks and, in particular, the normalization of each of them.

The QUE algebras U_q and $U_{q^{-1}}$ have inverse monodromies. Since the world sheet monodromies of the analytic and the antianalytic sectors of a 2-dimensional CFT compensate each other, and we describe the internal "quantum symmetry" of the analytic sector by U_q, we should associate the algebra

$U_{q^{-1}}$ with the antianalytic sector. We are looking for a $U_{q^{-1}} \otimes U_q$ invariant sesqui-linear form that couples irreducible representations of $U_{q^{-1}}$ and U_q (and can be extended in a factorizable way to tensor products of such representations). For representations of weight $\lambda \leq k$, such an inner product only differs from 0 if the left and the right factors have the same weight. Then, we can write

$$\left(\overline{u}^{\overline{n}}, u^n\right) = (-1)^n q^{(\frac{\lambda}{2}-n)(\lambda-1)} \begin{bmatrix} \lambda \\ n \end{bmatrix}^{-1} \delta_{n+\overline{n},\lambda}. \tag{3.83}$$

The inner product of two (regular) 4-point invariants of type (3.34) (3.49) is

$$\begin{aligned} Z_{\mu\nu}(I_1, I_2, I_3, I_4) &:= \left(\mathcal{J}_\mu(\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4)\mathcal{J}_\nu(u_1, u_2, u_3, u_4)\right)_{\lambda_1\lambda_2\lambda_3\lambda_4} \\ &= (-1)^{\mu+\nu} \frac{[\mu]![\nu]![\mu - I_{12} + I_{34}]![\nu - I_{12} + I_{34}]!}{[2I_1]![2I_2]![2I_3]![2I_4]!} \sum_{\rho=0}^{\min(\mu,\nu,k(I))} T_\rho(\mu, \nu; I_i), \end{aligned} \tag{3.84a}$$

where $\mu, \nu = 0, \ldots, m$,

$$k(I) = k - I_1 - I_2 - I_3 + I_4, \tag{3.84b}$$

$$\begin{aligned} T_\rho(\mu, \nu; I_i) &= [I_1 + I_2 + I_{34} + \rho + 1]![2I_{34} + 2\rho + 1] \times \\ &\frac{[I_1 + I_2 - I_{34} - \rho]![2I_4 - \rho]![I_{12} + I_{34} + \rho]![2I_3 + \rho + 1]!}{[2I_{34} + \mu + \rho + 1]![2I_{34} + \nu + \rho + 1]![\mu - \rho]![\nu - \rho]![\rho]![I_{34} - I_{12} + \rho]!}. \end{aligned} \tag{3.84c}$$

One can prove that the matrix Z is positive semidefinite and intertwines the U_q exchange operators and their transposed:

$$\mathcal{B}\begin{bmatrix} I_1 & I_2 \\ 0 & I_{34} \end{bmatrix} Z(I_1, I_2, I_3, I_4) = Z(I_2, I_1, I_3, I_4)\ {}^t\mathcal{B}\begin{bmatrix} I_2 & I_1 \\ 0 & I_{34} \end{bmatrix}, \tag{3.85a}$$

$$\mathcal{B}\begin{bmatrix} I_2 & I_3 \\ I_1 & I_4 \end{bmatrix} Z(I_1, I_2, I_3, I_4) = Z(I_1, I_3, I_2, I_4)\ {}^t\mathcal{B}\begin{bmatrix} I_3 & I_2 \\ I_1 & I_4 \end{bmatrix}. \tag{3.85b}$$

This property, together with the established braid invariance of the 4-point functions, guarantees the monodromy invariance of the 2-dimensional Green's function

$$G^{(\lambda_i)}(z, \overline{z}; \zeta, \overline{\zeta}) = \left(\overline{G}_4(\overline{z}, \overline{\zeta}, \overline{u}), G_4(z, \zeta, u)\right)_{\lambda_1\ldots\lambda_4}, \tag{3.86}$$

which can be written alternatively as an s- (or a u-) channel expansion:

$$G^{(\lambda_i)}(z,\bar{z};\zeta,\bar{\zeta}) = \sum_{\nu=0}^{\min(m,k(I))} g_\nu^{(I_i)} \overline{S}_{I_{34}+\nu}(\bar{z},\bar{\zeta}) S_{I_{34}+\nu}(z,\zeta), \tag{3.87}$$

where (in view of (3.84))

$$\begin{aligned} g_\nu^{(I_i)} &= [I_1+I_2+I_{34}+\nu+1]![2I_3+\nu+1]![2I_{34}+\nu]![\nu]! \times \\ &\quad \frac{[I_1+I_2-I_{34}-\nu]![I_{12}+I_{34}+\nu]![2I_4-\nu]!}{[2I_1]![2I_2]![2I_3]![2I_4]![2I_{34}+2\nu]![2I_{34}+2\nu+1]!}. \end{aligned} \tag{3.88}$$

Putting all results together, we can write the square of the normalization constant $N_{\lambda_1\ldots\lambda_4}$ in (3.11b) (that enters the 2-dimensional Green's function) as

$$N^2_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{C^2_{\lambda_1\lambda_2\lambda_{34}} C^2_{\lambda_{34}\lambda_3\lambda_4}}{N^2_{\lambda_{34}} g_0^{(I_i)} \left(A_s^{I_{34}}(I_1\ldots I_4)\right)^2}. \tag{3.89}$$

The reader who has been lost in this rather bulky sketch of a still more complicated argument, should retain at least two points.

(1) Proposition 3.2 provides the solution of a highly nonlinear problem: how to construct correlation functions with (z-dependent) factorization properties reflecting the existence of OPE in all channels (or an offshell unitarity condition in the terminology of the old days). By solving a linear system of KZ equations and imposing linear braid relations, we have reduced this nonlinear problem to a (0-dimensional) numerical one, which has been completely solved (at least for the 4-point function).

(2) It is also quite remarkable that the simple $U_{q^{-1}} \otimes U_q$ invariant contraction (3.83) has allowed us to produce a well-behaved unitary 2-dimensional theory with no unphysical states, starting from a pair of indefinite metric chiral theories involving non-unitary indecomposable representations of both $\mathcal{A}_k(su_2)$ and U_q. In particular, all undesired representations of U_q appear to be in the kernel of the semidefinite sesquilinear form (3.84). Setting, for instance, $\lambda_1 = \lambda_4 = 0$, $\lambda_2 = \lambda_3 = \lambda$, we obtain, as a very

special case of (3.84), the formula for the "norm squared" of a 2-point function:

$$\|w(u_1,u_2;\lambda,0)\|^2 = [\lambda+1]. \tag{3.90}$$

Here, w is given by (3.35),

$$\begin{aligned} w(u_1,u_2;\lambda,0) &= \left[q^{-\frac{1}{2}}u_1(-)q\frac{1}{2}u_2\right]^{\lambda} \\ &:= \sum_{n=0}^{\lambda}\begin{bmatrix}\lambda\\ n\end{bmatrix}(-1)^{\lambda-n}q^{\frac{1}{2}\lambda-n}u_1^n u_2^{\lambda-n}. \end{aligned} \tag{3.91}$$

(We invite the reader to verify that Eq. (3.90) is a simple consequence of (3.83)). The quantum dimension of the irreductible U_q module $\mathcal{V}_\lambda$ vanished precisely for the first value, $\lambda = k+1$, forbidden by the constraint (2.39 a) for $\lambda_3 = 0$, $\lambda_1 = \lambda_2 = \lambda$. It is also easy to see that the effective U_q structure constants squared, obtained from (3.84) for $I_4 = 0 (= \mu = \nu)$,

$$Z_{00}(I_1,I_2,I_3,0) = [I_1+I_2+I_3+1]!\frac{[I_1+I_2-I_3]![I_2+I_3-I_1]![I_1+I_3-I_2]!}{[2I_1]![2I_2]![2I_3]!}, \tag{3.92}$$

vanish for weights violating the fusion rule (2.39).

4. *The case of a minimal conformal model.*

We deal in this section with diagonal (A-series) minimal conformal models for which the chiral observable algebra $\mathcal{A}_c$ is generated by the stress energy tensor T with central charge

$$c = c_{pp'} = 1 - 6\frac{(p-p')^2}{pp'}, \; p, p' \text{ coprimes}. \tag{4.1}$$

The discussion being parallel to that of the su_2 current algebra we shall only spell out the differences.

4A. *Chiral conformal 4-point blocks.*

The 2-dimensional Bose fields of a minimal conformal model are label by (integer) weights λ, λ' in the range

$$0 \le \lambda \le p-2, \quad 0 \le \lambda' \le p'-2. \tag{4.2}$$

If we introduce the rational spins [14] J, J'

$$2J = \lambda - \frac{p}{p'}\lambda', \quad 2J' = \lambda' - \frac{p'}{p}\lambda\left(= -\frac{p'}{p}2J\right) \tag{4.3}$$

then the conformal weights are given by

$$\Delta_{\lambda\lambda'} = J(J+1)\frac{p'}{p} - J = J'(J'+1)\frac{p}{p'} - J'. \tag{4.3}$$

The basis of 4-point conformal blocks [13] can be substituted (for $m := \lambda_4 \leq \lambda_i, m' = \lambda'_4 \leq \lambda'_i$) by

$$G^{\lambda\lambda'}_{\mu\rho}(z) = h(z_{13}, z_{14}, z_{24}, z_{34})\eta^{\Delta_{34}-\Delta_1-\Delta_2}(1-\eta)^{\Delta_{14}-\Delta_2-\Delta_3}g_{\mu\rho}(\eta) \tag{4.5}$$

where h is given by (3.3b) with $\Delta_i \equiv \Delta_{\lambda_i\lambda'_i}$ and

$$\begin{aligned} g_{\mu\rho}(\eta) = &\int_0^\eta dt_1 \ldots \int_0^{t_{\mu-1}} dt_\mu \int_\eta^1 dt_{\mu+1} \ldots \int_{t_{m-1}}^1 dt_m \times \\ &\times \int_0^\eta dt'_1 \ldots \int_{t'_{m'-1}}^1 dt'_{m'} P_m(t_i; \alpha, \beta, \gamma, \delta) P_{m'}(t'_i; \alpha', \beta', \gamma', \delta') \times \\ &\times \prod_{i=1}^m \prod_{j=1}^{m'} (t_i - t'_j)^{-2} \end{aligned} \tag{4.6}$$

with

$$P_n(t_i; a, b, c, d) = \prod_{i=1}^n t_i^{a-1}(1-t_i)^{b-1}|\mu - t_i|^{c-1} \prod_{i<j}^n |t_i - t_j|^{2d} \tag{4.7}$$

The parameters $\alpha, \ldots, \delta$, and $\alpha', \ldots, \delta'$, are obtained from (3.12b) by the substitutions

$$I_i \to J_i, K+2 \to \frac{p}{p'}, \text{ and } I'_i \to J'_i, \quad K+2 \to \frac{p'}{p}, \tag{4.8}$$

respectively. It follows that the 4-point conformal blocks depend on the rational spins (4.3) (rather than on λ_i and on λ'_j, separately). There is an argument in [14] that this is true for all n-point blocks.

4B. *The quantum group for a diagonal minimal model.*

It can be demonstrated that the quantum group for a diagonal minimal conformal model is a twisted product of U_q and $U_{q'}$ with

$$q = \exp(i\pi\frac{p'}{p}), q' = \exp(i\pi\frac{p}{p'}). \tag{4.9}$$

The commutation relations of $q^{\pm H}$, E, F and $q^{\pm H'}$, E', F' are given by (1), (and $[X, Y']=0$ for $X\in U_q$, $Y'\in U_{q'}$) while the modified coproducts read

$$\Delta(E)=E\otimes q^{-H}(-1)^{H'}+1\!\!1\otimes E,\quad \Delta(F)=F\otimes(-1)^{H'}+q^{H}\otimes F, \tag{4.10a}$$

$$\Delta(E')=E'\otimes q'^{-H'}(-1)^{H}+1\!\!1\otimes E',\quad \Delta(F')=F'\otimes(-1)^{H}+q'^{H'}\otimes F'. \tag{4.10b}$$

$(-1)^H$ and $(-1)^{H'}$ are sign factors, since $(-1)^{2H}=(-1)^{2H'}=1$, which commute with all generators. (These formulae are similar but not identical to the modified coproduct (5.13) used in ref. [22].) The universal R matrix for this coproduct is

$$R=\mathrm{i}^{H\otimes H'+H'\otimes H}$$

$$\times q^{-(1/2)H\otimes H}e_+((q^{-1}-q)E\otimes F(-1)^{H'})q'^{-(1/2)H'\otimes H'}e'_+((q'^{-1}-q')E'\otimes F'(-1)^{H}). \tag{4.11}$$

Technically, it is convenient to introduce (following (4.3)) fractional valued Cartan generators,

$$\tilde{H}=H-\frac{p}{p'}H',\quad \tilde{H}'=H'-\frac{p'}{p}H\quad (q^{\tilde{H}}=q^{H}(-1)^{H'},\text{etc.}),$$

$$\tilde{F}=F(-1)^{H'},\quad \tilde{F}'=F'(-1)^{H},\quad \tilde{E}=E,\quad \tilde{E}'=E'. \tag{4.12}$$

In this way we formally recover two independent quantum groups $\tilde{U}_q$ and $\tilde{U}_{q'}$ ($\tilde{U}_q$ is generated by $\tilde{E}$, $\tilde{F}$, $q^{\pm\tilde{H}}$ which satisfy (1.5)). The counterpart of (4.11) reads

$$R=q^{-(1/2)\tilde{H}\otimes\tilde{H}}e_+((q^{-1}-q)\tilde{E}\otimes\tilde{F})e'_+((q'^{-1}-q')\tilde{E}'\otimes\tilde{F}'),\quad q^{-(1/2)\tilde{H}\otimes\tilde{H}}=q'^{-(1/2)\tilde{H}'\otimes\tilde{H}'}. \tag{4.13}$$

As a consequence the n-point polynomial invariants also factorize into a product of two invariants (one for $\tilde{U}_q$ the other for $\tilde{U}_{q'}$):

$$J^{I,I'}_{\mu\mu'}(u_1,\ldots,u_n;u'_1,\ldots,u'_n)=N^{I,I'}_{\mu\mu'}J^{I,I'}_{\mu}(u_1,\ldots,u_n)J^{I,I'}_{\mu'}(u'_1,\ldots,u'_n). \tag{4.14}$$

The u-dependent part is similar to (3.34,35), where μ_{ij} satisfy (3.36, 37) and in the expression for ρ_{ij} (3.38) I_k is replaced by J_k (4.3):

$$\rho_{ij}=\sum_{k=i+1}^{j}J_k-\tfrac{1}{2}\mu_{ij}-\sum_{\substack{i\leq k\\ 0<l-k<j-i}}\mu_{kl}. \tag{4.15}$$

The u'-dependent part is obtained from u-dependent one by the substitution $q\to q'$, $I_i\to I'_{i'}$, $J_k\to J'_k$.

Applying the statistics operator PR to the four-point invariants we also find the corresponding matrix $\mathscr{B}$. We verify that for appropriate values of the coefficients $N^{I,I'}_{\mu\mu'}$ in (4.14) $\mathscr{B}$ can be made equal to the inverse transposed of the exchange matrix for the conformal blocks (4.5–7). This yields the braid invariance of the Green functions for the extended chiral minimal model:

$$G^{I,I'}(z,u,u')=\sum_{\mu=0}^{m}\sum_{\mu'=0}^{m'}G^{I,I'}_{\mu\mu'}(z)J^{I,I'}_{\mu\mu'}(u,u'). \tag{4.16}$$

As a corollary of the factorization of the $\widetilde{U}_q\times\widetilde{U}_{q'}$ invariants (4.14) their monodromy invariant contraction is a product of two expressions similar to (3.84), where all deformed factorials are calculated for q and q' given by (4.9) and the level k in the upper limit of the sum is replaced by $p-2$ and $p'-2$, respectively. Repeating the computation which yields (3.87) for the chiral functions (4.46) and their antianalytic counterparts we recover the two-dimensional monodromy invariant Green functions and reproduce the usual BPZ fusion rules

$$[I_1, I'_1] \times [I_2, I'_2] = \sum_{I=|I_{12}|}^{\min(I_1+I_2,\, p-2-I_1-I_2)} \; \sum_{I'=|I'_{12}|}^{\min(I'_1+I'_2,\, p'-2-I'_1-I'_2)} [I, I'] \,. \tag{4.17}$$

The conformal weights (4.4) of the fields in a minimal model satisfy the equation $\Delta_{\lambda\lambda'} = \Delta_{p-2-\lambda\; p'-2-\lambda'}$, so each weight appears twice for $\lambda \leqslant p-2$, $\lambda' \leqslant p'-2$. In the extended chiral model (and hence also in the resulting two-dimensional model) these two fields with equal dimensions are not equivalent, because their $U_q \times U_{q'}$ n two-point function is zero. So there are $(p-1)(p'-1)$ different fields in the model. However, we can choose a subset of $\frac{1}{2}(p-1)(p'-1)$ fields closed under the fusion rules (4.17), such that every conformal dimension appears exactly once. There always exist two different (among the following three) subsets with these properties:

for $p-p'$ odd (which includes the unitary models with $p'=p-1$) this is the quantum so(4) series

$$\{\phi_{I,I'} : I+I' \in \mathbb{Z}\} \quad (2I = \lambda,\; 2I' = \lambda'); \tag{4.18a}$$

for p odd it is the $U_q(so(3)) \times U_{q'}(su(2))$ series

$$\{\phi_{I,I'} : I \in \mathbb{Z}\}\,; \tag{4.18b}$$

for p' odd it is the $U_q(su(2)) \times U_{q'}(so(3))$ series

$$\{\phi_{I,I'} : I' \in \mathbb{Z}\}\,. \tag{4.18c}$$

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