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Addendum to " An Extension of the Theory of Fredholm Determinants "

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Addendum to

"**An extension of the theory of Fredholm determinants**"

by D. Ruelle

In deriving (2.12) we have used the inequality

$$\sum_{\vec{b}\in J_k^{(\ell)}} \| \mathfrak{M}_k^{(\ell)} \chi_{\vec{b}} \| \leq \text{const } (e^{P+\varepsilon})^{\ell} \tag{2.13}$$

which we shall now prove. Given $\beta > 0$, we let $\varphi_{\omega\beta} = |\varphi_\omega| + \beta\|\varphi_\omega\|$, and define $\mathfrak{M}_{k\beta}, \mathfrak{M}_{k\beta}^{(\ell)}$ with φ_ω replaced by $\varphi_{\omega\beta}$. We first check that

$$\|\mathfrak{M}_k^{(\ell)} \chi_{\vec{b}}\| \leq C(\beta) \,\|\mathfrak{M}_{k\beta}^{(\ell)} \chi_{\vec{b}}\|_o \tag{2.14}$$

where $C(\beta)$ does not depend on ℓ. We may indeed write

$$(\mathfrak{M}_k^{(\ell)} \chi_b)_{\vec{j}}\,(x) \;=\; \int \mu(d\omega_1)\ldots\mu(d\omega_\ell)$$

$$\varepsilon\varphi_{\omega_\ell}(x)\ldots\varphi_{\omega_1}(\psi_{\omega_2}\ldots\,\psi_{\omega_\ell}x)$$

where the integral is restricted to those $(\omega_1,\ldots,\omega_\ell)$ for which there is a permutation π such that

$$\pi(v(j_o,\bar{\omega}),\ldots,v(j_k,\bar{\omega})) = \vec{b}\ ,$$

and $\varepsilon = \text{sign } \pi$. Therefore

$$\| \mathfrak{M}_k^{\ell} \chi_{\vec{b}}\|_o \leq \| \mathfrak{M}_{k\beta}^{\ell}\, \chi_{\vec{b}}\|_o \,.$$

If $x,y \in X_{j_o} \cap \ldots \cap X_{j_k}$, we have also

$$|(\mathcal{M}_k^{(\ell)}\chi_{\vec b})_{\vec j}(x) - (\mathcal{M}_k^{(\ell)}\chi_{\vec b})_{\vec j}(y)| \le \int \mu(d\omega_1) \dots \mu(d\omega_\ell)$$

$$|\varphi_{\omega_\ell}(x) \dots \varphi_{\omega_1}(\psi_{\omega_2} \dots \psi_{\omega_\ell} x) - \varphi_{\omega_\ell}(y) \dots \varphi_{\omega_1}(\psi_{\omega_2} \dots \psi_{\omega_\ell} y)|$$

$$\le \sum_{i=1}^{\ell} \int \mu(d\omega_1) \dots \mu(d\omega_k)\, \varphi_{\omega_\ell \beta}(x) \dots \varphi_{\omega_{i+1}\beta}(\psi_{\omega_{i+2}} \dots \psi_{\omega_\ell} x)$$

$$|\varphi_{\omega_i}(\psi_{\omega_{i+1}} \dots \psi_{\omega_\ell} x) - \varphi_{\omega_i}(\psi_{\omega_{i+1}} \dots \psi_{\omega_\ell} y)|$$

$$\varphi_{\omega_{i-1}\beta}(\psi_{\omega_i} \dots \psi_{\omega_\ell} y) \dots \varphi_{\omega_1 \beta}(\psi_{\omega_2} \dots \psi_{\omega_\ell} y) \ .$$

We may assume that $\|\varphi_\omega\|$ is bounded uniformly with respect to ω (this can be achieved by a change of the measure μ). We may then write

$$|\varphi_{\omega_i}(\psi_{\omega_{i+1}} \dots \psi_{\omega_\ell} x) - \varphi_{\omega_i}(\psi_{\omega_{i+1}} \dots \psi_{\omega_\ell} y)|$$

$$\le \|\varphi_{\omega_i}\| \, (\theta^{\ell - i} d(x,y))^\alpha \le \text{const } \varphi_{\omega_i \beta}(\psi_{\omega_{i+1}} \dots \psi_{\omega_\ell} x)\ \theta^{\alpha(\ell - i)} d(x,y)^\alpha$$

and similarly

$$\varphi_{\omega_r \beta}(\psi_{\omega_{r+1}} \dots \psi_{\omega_\ell} y)$$

$$\le \varphi_{\omega_r \beta}(\psi_{\omega_{r+1}} \dots \psi_{\omega_\ell} x)\ (1 + \text{const } \theta^{\alpha(\ell - r)}) \ . \qquad (2.15)$$

Therefore

$$\frac{|(\mathcal{M}_k^{(\ell)}\chi_{\vec b})_{\vec j}(x) - (\mathcal{M}_k^{(\ell)}\chi_{\vec b})_{\vec j}(y)|}{d(x,y)^\alpha} \le \text{const } \|\mathcal{M}_{k\beta}^{(\ell)} \chi_{\vec b}\|_o$$

and (2.14) follows.

From (2.15) we also obtain

$$(\mathfrak{M}_{k\beta}^{(\ell)} \chi_{\vec{b}})_{\vec{j}}(y) \leq C'(\beta)\, (\mathfrak{M}_{k\beta}^{(\ell)} \chi_{\vec{b}})_{\vec{j}}(x)$$

where $C'(\beta)$ does not depend on ℓ . Therefore

$$\sum_{\vec{b}\in J_k^{(\ell)}} \| \mathfrak{M}_{k\beta}^{(\ell)} \chi_{\vec{b}} \|_o \leq C'(\beta) \sum_{\vec{j}} \sup_x |(\mathfrak{M}_{k\beta}^{(\ell)} 1)_{\vec{j}}(x)|$$

$$\leq C''(\beta) \| \mathfrak{M}_{k\beta}^{\ell} 1 \|_o$$

and finally

$$\sum_{\vec{b}\in J_k^{(\ell)}} \| \mathfrak{M}_k^{(\ell)} \chi_{\vec{b}} \| \leq C(\beta)C''(\beta) \| \mathfrak{M}_{k\beta}^{\ell} 1 \|_o$$

$$\leq C'''(\beta)\, (e^{P(\beta)+\varepsilon/2})^{\ell}$$

where $e^{P(\beta)}$ is the spectral radius of $\mathfrak{M}_{k\beta}$. Note that $\mathfrak{M}_{k\beta}$ is close in norm to $|\mathfrak{M}_k|$ for β small :

$$\| \mathfrak{M}_{k\beta} - |\mathfrak{M}_k| \| \leq \beta \int \mu(d\omega)\, \| \varphi_\omega \| \; .$$

Using the upper semicontinuity of the spectral radius we may thus choose β such that

$$\sum_{\vec{b}\in J_k^{(\ell)}} \| \mathfrak{M}_k^{(\ell)} \chi_{\vec{b}} \| \leq C'''(\beta)\, (e^{P+\varepsilon})^{\ell}$$

i.e., (2.13) holds as announced.

A similar argument may be used to obtain the inequality

$$\sum_{b \in J^{(\ell)}} \| \mathfrak{M}^{(\ell)}((.-x(b))^n \chi_b) \| \leq \mathrm{const}(e^{P+\varepsilon})^{\ell} (\theta'^{\ell})^{|n|}$$

which is needed in the proof of (3.11).