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SOME RESULTS ABOUT HOLONOMIC $\mathcal{E}$ - MODULES

By Jan-Erik BJÖRK

Introduction.

If X is a complex analytic manifold and $T^*(X)$ its holomorphic cotangent bundle (which is a new complex analytic manifold whose dimension is $2\dim(X)$) , then we find the sheaf $\mathcal{E}_X$ of micro-local differential operators of finite order. A coherent sheaf of left $\mathcal{E}_X$ - modules has a support given by a ξ - conic complex analytic subset of $T^*(X)$ where ξ denote covector variables. The module is holonomic if its support is of dimension $n = \dim(X)$. Since $\mathrm{supp}(\mathcal{M})$ are involutive for any coherent $\mathcal{E}_X$ - module an equivalent condition for $\mathcal{M}$ to be holonomic is that $\mathrm{supp}(\mathcal{M})$ is Lagrangian which by definition means that the fundamental 1 - form ξdx has a vanishing pull back to the dense open subset of regular points of $\mathrm{supp}(\mathcal{M})$.

If $\mathcal{D}_X$ is the sheaf of differential operators with holomorphic coefficients and if $\mathcal{N}$ is a coherent $\mathcal{D}_X$ - module then we get the extended $\mathcal{E}_X$ - module $\mathcal{E}_X \underset{\pi^{-1}(\mathcal{D}_X)}{\otimes} \pi^{-1}(\mathcal{N})$ where $\pi : T^*(X) \to X$ is the projection. The characteristic variety $SS(\mathcal{N})$ equals $\mathrm{supp}(\mathcal{E}_X \otimes \pi^{-1}(\mathcal{N}))$. In particular $\mathcal{N}$ is a holonomic $\mathcal{D}_X$ - module if and only if its extended $\mathcal{E}_X$ - module is holonomic.

Suppose now that $\mathcal{M}$ is a holonomic $\mathcal{E}_X$ - module and take a point $(x^o, \xi^o) = p \in \mathrm{supp}(\mathcal{M})$. The stalk $\mathcal{E}_X(p)$ contains $\mathcal{D}_X(x^o)$ as a subring and hence the $\mathcal{E}_X(p)$ - module $\mathcal{M}(p)$ has an underlying $\mathcal{D}_X(x^o)$ - module structure. We may ask if the $\mathcal{D}_X(x^o)$ - module $\mathcal{M}(p)$ is holonomic. This is not true in general. However we can prove

Main Theorem If $\mathrm{supp}(\mathcal{M})$ has a generic position at p then $\mathcal{M}(p)$ is a holonomic $\mathcal{D}_X(x^o)$ - module and if $\mathcal{N}$ is a holonomic sheaf of $\mathcal{D}_X$- modules which extends $\mathcal{M}(p)$, i.e. the stalk $\mathcal{N}(x^o) \cong \mathcal{M}(p)$ as a left $\mathcal{D}_X(x^o)$ - module then $\mathcal{M} \cong \mathcal{E}_X \otimes \pi^{-1}(\mathcal{N})$ in some open and conic neighborhood of p .

Remark. To say that supp($\mathcal{M}$) has a generic position at p means that the complex line $\mathbb{C}^* p$ is an isolated fiber in the ξ - conic set supp$(\mathcal{M}) \cap \pi^{-1}(x^o)$. The conclusion in the Main Theorem is this : First $\mathcal{M}(p)$ is a holonomic $\mathcal{D}_X(x^o)$ - module and since $\mathcal{D}_X$ is a coherent sheaf of rings it follows that there exists a unique holonomic sheaf $\mathcal{N}$ of $\mathcal{D}_X$ - modules defined in some small open neighborhood of x^o such that its stalk $\mathcal{N}(x^o)$ equals the holonomic $\mathcal{D}_X(x^o)$ - module $\mathcal{M}(p)$. The main theorem asserts now that the extended $\mathcal{E}_X$ - module $\mathcal{E}_X \otimes \pi^{-1}(\mathcal{N})$ equals (or rather : Is isomorphic to) the given holonomic $\mathcal{E}_X$ - module $\mathcal{M}$ in some small conic neighborhood of p .

In [3] the main theorem was proved for regular holonomic modules. Here we prove it in the general case, i.e. we need not the assumption that $\mathcal{M}$ "has regular singularieties" in the sense of [6]. Besides the proof is rather different from that in [3]. Here we are going to use methods from Malgrange's work [7] where the Main Theorem was proved when dim(X) = 1 and the proof below has been inspired by various remarks and suggestions which also are due to Malgrange. In fact, the Main Theorem was more or less implicit in Malgrange's work [7].

Remarks. We are going to use various basic results about the sheaf $\mathcal{E}_X$. The reader may consult [8] or [6] or [1 : Chapter 4] for relevant background. In § 2 we also discuss some consequences of the Main Theorem.

1. Proof of the Main Theorem.

Working locally we can assume that the base manifold X is $\mathbb{C}^{n+1}$ where $(x,t) = (x_1 \dots x_n, t)$ are the coordinates and then $T^*(X)$ is the 2n+2 dimensional (x,t,ξ,τ) - space where $\xi = (\xi_1 \dots \xi_n)$ and hence $\tau dt + \xi_1 dx_1 + \dots + \xi_n dx_n$ is the fundamental 1 - form.

Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-module defined in some open and conic neighborhood of $p = (o,o,o,dt)$. Assume that $\operatorname{supp}(\mathcal{M})$ has a generic position at p which means that if $\Lambda = \operatorname{supp}(\mathcal{M})$ and if Ω is a small conic open neighborhood of p then $\pi^{-1}(o,o) \cap \Lambda \cap \Omega$ is reduced to ${}^{x}p = (o,o,o,\tau dt)$ with $\tau \neq 0$.

Before we enter the actual proof of the Main Theorem we need some preliminary results of a geometric nature.

1.1. <u>The hypersurface</u> $\pi(\Lambda \cap \Omega)$. Since $\Lambda = \operatorname{supp}(\mathcal{M})$ has a generic position at p and at the same time is a conic Lagrangian it follows that for a suiable choice of a conic neighborhood Ω of p there exists a polydise Δ centered at the origin in the (x,t)-space such that the following holds :

1.2. <u>Lemma</u>. $\pi(\Lambda \cap \Omega) = S$ is a complex analytic hypersurface in and $\Lambda \cap \Omega = \Omega \cap \overline{T^*_{S_{reg}}}$ where S_{reg} is the regular part of S and $\overline{T^*_{S_{reg}}}$ is the conic Lagrangian defined in the whole of $\pi^{-1}(\Delta)$.

For a detailed proof we refer to [3 : page 906]. It only requires the Weierstrass preparation theorem and the fact that the conic Lagrangian Λ has a generic position at p and at the same time it is a complex analytic set of pure dimension $n+1$.

<u>Remark</u>. The position of the hypersurface S is not arbitrary since $T^*_{S_{reg}}$ has a generic position at p. In fact, we can choose S so that the equality in Lemma 1.2. holds and at the same time $S = \varphi^{-1}(0)$ where $\varphi(x,t) \in \mathcal{O}(\Delta)$ is in the Weierstrass form with respect to t. In fact, from [3 : page 907] one finds that if $(x,t) \in S$ is close to the origin then $|t| \ll |x|$ so the projection $(x,t) \to x$ is proper on S and so on.

Finally, S can be chosen so that $\pi^{-1}(o,o) \cap \overline{T^*_{S_{reg}}}$ is reduced to the complex line ip. With $S = \varphi^{-1}(0)$ as above this implies that if $(x,t) \in S_{reg}$

is close to the origin then the corresponding point $(x,t,d_x\varphi,\partial\varphi/\partial t) \in T^*_{S_{reg}}$ after a normalisation gives $(x,t,d_x\varphi/\partial\varphi/\partial t,dt)$ and this point gets close to (o,o,o,dt). In other words, one has $|\partial\varphi/\partial x_v| \ll |\partial\varphi/\partial t|$ for all $1 \leq v \leq n$ as $(x,t) \in S_{reg} \approx (o,o)$. See again [3] for this.

1.3. The study of $\mathcal{M}(p)$.

We work at stalks for the moment and put $\mathcal{E}_p = \mathcal{E}_X(o,o,o,dt)$ and $\mathcal{D}_o = \mathcal{D}_X(o,o)$ to simplify the notations. Let $\mathcal{M}$ be a holonomic sheaf of $\mathcal{E}_X$-modules defined in a small conic neighborhood of p and assume that $\text{supp}(\mathcal{M})$ is in a generic position at p. Then we can prove

1.4. Lemma. $\mathcal{M}(p)$ is a holonomic $\mathcal{D}_o$-module and the equality $\mathcal{M}(p) = \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p)$ holds.

The proof requires several steps. First we can find some good filtration Γ of the holonomic sheaf $\mathcal{M}$ which is defined in a whole conic neighborhood Ω. Now Γ_o/Γ_{-1} is a coherent sheaf of modules over the sheaf of rings $\mathcal{E}_X(0)/\mathcal{E}_X(-1) \cong \mathcal{O}_{T^*(X)}(0)$. = The sheaf of holomorphic functions in $T^*(X)$ which are (ξ,τ)-homogenous of order zero.

We have $\text{supp}(\Gamma_o/\Gamma_{-1}) = \text{supp}(\mathcal{M})$ and since $\text{supp}(\mathcal{M})$ has a generic position at p the geometric results above in particular show that the projection $(x,t,\xi,\tau) \to (x,t,\tau)$ is proper with finite fibers over $\Lambda \cap \Omega$ – where $\Lambda = \text{supp}(\mathcal{M})$ and Ω is a suitable conic neighborhood of $p = (o,o,o,dt)$.

This implies that Γ_o/Γ_{-1} is coherent as a sheaf of modules over the subring $\pi^{-1}(\mathcal{O}_X)$ of $\mathcal{O}_{T^*(X)}(0)$ and passing to the stalk at p we get.

1.5. Lemma. $\Gamma_p(p)/\Gamma_{-1}(p)$ is a finitely generated module over the local ring ring $\{x,t\} = \mathcal{O}_X(o,o)$.

Next, we recall that $\pi(\Lambda \cap \Omega) = S = \varphi^{-1}(0)$ where $\varphi(x,t)$ is in the

Weierstrass form, i.e. $\varphi(x,t) = t^e + \varphi_1(x)t^{e-1} + \dots + \varphi_e(x)$ can be assumed. Now $\varphi = 0$ on $\mathrm{supp}(\Gamma_o/\Gamma_{-1})$ so the Nullstellen Satz implies that some power of φ annhilates Γ_o/Γ_{-1} - at leadt if we stay in a small conic neighborhood of p . In particular : $\exists$ N with $\varphi^N\ (\Gamma_o(p)/\Gamma_{-1}(p)) = 0$ and then Lemma 1.5. even gives that $\Gamma_o(p)/\Gamma_{-1}(p)$ is a finitely generated module over the local ring $\mathbb{C}\{x\}$.

1.6. The ring $\mathcal{G}(p)$.

Put $\mathcal{G} = \mathcal{E}_X(0) \cap \mathcal{E}_X(x,D_t)$, i.e. it is the sheaf of micro-local differential operators of order ≤ 0 which only depend on x and D_t . We find the stalk $\mathcal{G}(p)$ and now we can prove

1.7. Lemma. $\Gamma_o(p)$ is a finitely generated $\mathcal{G}(p)$ - module.

Proof. Since $\Gamma_o(p)/\Gamma_{-1}(p)$ is a finitely generated $\mathbb{C}\{x\}$ - module this follows after divisions in the ring $\mathcal{E}_X(p)$. To be precise, we can choose a finite set $u_1 \dots u_s$ in $\Gamma_o(p)$ whose images in $\Gamma_o(p)/\Gamma_{-1}(p)$ generate this $\mathbb{C}\{x\}$ - module. If we then use that $\Gamma_{-m}(p) = D_t^{-m}\,\Gamma_o(p)$ for all $m \geq 1$ one can prove that $\Gamma_o(p) = \mathcal{G}(p)u_1 + \dots + \mathcal{G}(p)u_s$ using a division with bounds in the ring $\mathcal{E}_X(p)$. See [8] and also [1 : Chapter 4] for such divisions with bounds in the stalk of $\mathcal{E}_X$.

So far we have not removed "micro-local terms" since negative powers of D_t do not belong to $\mathcal{D}_X(o,o) = \mathcal{D}_o$. However, at this stage we can use methods from [7] and obtain

1.8. Lemma. $\Gamma_o(p)$ is a finitely generated $\mathbb{C}\{x,t\}$ - module.

Proof. We have $\Gamma_o(p) = \mathcal{G}(p)u_1 + \dots + \mathcal{G}(p)u_s$. This sum is in general not direct, i.e. $\Gamma_o(p)$ need not be a free $\mathcal{G}(p)$ - module. However, using a finite set $u_1 \dots u_s$ of generators from Lemma 1.7. we can express the action by the element t on the left $\mathcal{E}_p(0)$ - module $\Gamma_o(p)$ and find that $tu_j = P_{1,j}(x,D_t)u_1 + \dots + P_{sj}(x,D_t)u_s$

where $\mathcal{P} = (P_{\nu j}(x,D_t)$ is some $\mathcal{C}(p)$ - valued matrix.

Recall that $\mathcal{C}(p)$ = the stalk of the sheaf $\mathcal{E}_X(0) \cap \mathcal{E}_X(x,D_t)$ and it is filtered when we use $\mathcal{C}(-m) = \mathcal{E}_X(-m) \cap \mathcal{E}_X(x,D_t)$ for $m \geq 1$. In particular we find $\mathcal{C}(p)/\mathcal{C}(-1)(p) \cong \{x\}$. Consider the corresponding principal symbols of the elements $P_{\nu j}(x,D_t)$ which form the matrix $\sigma_o(\mathcal{P})(x)$ whose entries belong to $\mathbb{C}\{x\}$. Now one has

<u>Sublemma 1</u>. The scalar matrix $\sigma_o(\mathcal{P})(o)$ is nilpotent.

<u>Proof of Sublemma 1</u>. We have the Weierstrass polyonmial $\varphi(x,t)$ which gives $\varphi^N \Gamma_o(p) \subset \Gamma_{-1}(p)$ for some $N \geq 1$. With $\varphi = t^e + \varphi_1(x)t^{e-1} + \ldots + \varphi_e(x)$ it follows that $t^{eN}\underline{u} \in \mathcal{C}(-1)(p)\underline{u} + m\,\Gamma_o(p)$ where m is the maximal ideal in the local ring $\{x\}$, i.e. it uses only that $\varphi_\nu(0) = 0$ for all ν.

Passing to the matrix $\mathcal{P}$ which expresses the action by t on the left $\mathcal{E}_p(0)$ - module $\Gamma_o(p)$ we find that $\mathcal{P}^{eN}\underline{u} \subset \mathcal{C}(-1)(p)\underline{u} + m\Gamma_o(p)$ and this implies that $\sigma_o(\mathcal{P})^{eN}$ has all its entries in m and hence the scalar matrix which arises if $x = 0$ is nilpotent.

<u>Proof continued</u>. Let us now introduce the free $\mathcal{C}(p)$ - module $F = \mathcal{C}(p)\varepsilon_1 \oplus \ldots \oplus \mathcal{C}(p)\varepsilon_s$ and <u>define</u> $t\underline{\varepsilon} = \mathcal{P}\underline{\varepsilon}$ and in general, if $g(x,t) = \sum_{\nu \geq o} g_\nu(x)t^\nu \in \mathbb{C}\{x,t\}$ we put $g(x,t)\underline{\varepsilon} = \sum_{\nu \geq o} g_\nu(x)\mathcal{P}^\nu \underline{\varepsilon}$. We claim that this gives an $\mathbb{C}\{x,t\}$ - module structure on F. To prove it is suffices to show that whenever $g(x,t) \in \{x,t\}$ then $\sum g_\nu(x)\mathcal{P}^\nu$ converges in the ring of $\mathcal{C}(p)$ - valued matrices. This convergence is an easy consequence of Sublemma 1 and the existence of certain norms from [8]. To be precise, using the norms from [8] (see also |Bj : page 143|) we can define a Banach algebra norm over some subring of $\mathcal{C}(p)$ such that all the entries of $\mathcal{P}$ have a finite $\| \;\|$ - norm and in addition $\|\mathcal{P}^{Ne}\| \ll 1$ can be assumed thanks to Sublemma 1.

<u>Summing up</u>, F has a natural $\mathbb{C}\{x,t\}$ - module structure. By the mapping which sends $\varepsilon_j \to u_j$ for $1 \leq j \leq s$ we see that $F \to \mathcal{M}(p)$ is $\mathbb{C}\{x,t\}$ - linear and surjective.

So if we can prove that F is a finitely generated $\mathbb{C}\{x,t\}$-module then Lemma 1.8. follows.

To prove this we consider $F/(x_1F+\dots+x_nF)$ which first becomes a free module of rank s over the ring $\mathcal{G}(p)\cap\mathcal{E}(t,D_t)$ = The ring of germs of micro-local differential operators which only depend on D_t and have order ≤ 0. Observe that t operates as above, i.e. $t\underline{\varepsilon} = P\underline{\varepsilon}$ where $\sigma_o(P)(o)$ is nilpotent. Using Malgrange's result from [7]
we conclude that $F/(x_1F+\dots+x_nF)$ is a free $\{t\}$-module of rank s and if $H_1 \dots H_s$ are chosen in F so that their images give a free basis we get $F = \mathbb{C}\{t\}H_1\oplus\dots\oplus\mathbb{C}\{t\}H_s + mF$ where $mF = x_1F+\dots+x_nF$. Repeating the recursion formulas from [7] we can then prove that $F = \mathbb{C}\{x,t\}\varepsilon_1\oplus\dots\oplus\mathbb{C}\{x,t\}\varepsilon_s$.

1.9. Proof of Lemma 1.4.

Armed with Lemma 1.8. we can prove Lemma 1.4. rather easily. First we show that $\mathcal{M}(p)$ is a finitely generated $\mathcal{B}_o$-module. To see this we study the elements $D_{x_v}/D_t = R_v \in \mathcal{E}_X(0)$ when $1\leq v\leq n$. Each R_v preserves Γ_o. Now Lemma 1.8. means that $\Gamma_o(p)$ is a noetherian $\mathbb{C}\{x,t\}$-module and this implies that we can find $\mathcal{E}_X(0)$-valued sections $S_v(x,t,D_x,D_t)$ of the forme $S_v = R_v^m + r_1(x,t)R_v^{m-1}+\dots+r_m(x,t)$ with some $m \gg 0$ such that $S_v u_j = 0$ for all v and all $1\leq j\leq s$.

Multiplying each S_v by D_t^m to the left we find differential operators $Q_v = D_v^m + Q_{1,v}(x,t,D_t)D_v^{m-1}+\dots+Q_m(x,t,D_t)$ - where $D_v = D_{x_v}$ are used - and here $Q_v u_j = 0$ for all pairs v and j.

Let us then take an element R in $\mathcal{E}_p$. By succesive divisions in the ring $\mathcal{E}_p$ we can write $R = R_1Q_1+\dots+R_nQ_n + P$ where the remainder term P is given as a finite sum of the forme $\Sigma P_\alpha(x,t,D_t)D_x^\alpha$. Here Σ extends over a finite set of multi-indices, i.e. $0<\alpha_v<m$ hold for all $1\leq v\leq n$ with the integer m as above.

Also, $P_\alpha(x,t,D_t)$ are micro-local differential operators which only depend on x,t and D_t.

Now $\mathcal{M}(p) = \mathcal{E}_p u_1 + \dots + \mathcal{E}_p u_s$ and using the divisions above we get $\mathcal{M}(p) = \sum_\alpha \sum_j \mathcal{E}_p(x,t,D_t)(D_x^\alpha u_j)$. Finally, consider some $R(x,t,D_t) \in \mathcal{E}_p(x,t,D_t)$ and expand it with respect to D_t. One part is a finite sum where the D_t-powers are $> -mn$ and the remainder belongs to $\mathcal{E}_p(-mn) \cap \mathcal{E}_p(x,t,D_t)$. If we call it R' we have $R'(x,t,D_t)D_x^\alpha \in \mathcal{E}_p(0)$ for all α as above.
Now $\mathcal{E}_p(0)u_j \in \mathbb{C}\{x,t\}H_1 + \dots + \mathbb{C}\{x,t\}H_s \subset \mathcal{D}_0 H_1 + \dots + \mathcal{D}_0 H_s$ where $H_1 \dots H_s$ were found from Lemma 1.8.

We conclude that the $\mathcal{D}_0$-module $\mathcal{M}(p)$ is generated by the finite set $\{H_1 \dots H_s\}$ and $\{D_t^{-v} D_x^\alpha u_j : 1 \le j \le s : 0 \le v < mn$ and $|\alpha| < mn\}$.

<u>The holonomicity of</u> $\mathcal{M}(p)$. Since $\mathcal{M}(p)$ is a finitely generated $\mathcal{D}_0$-module it is holonomic if each cyclic $\mathcal{D}_0$-submodule is holonomic. Let us take some $u \in \Gamma_0(p) \in \mathcal{M}(p)$ and prove that $\mathcal{D}_0 u$ is $\mathcal{D}_0$-holonomic. Exactly as above we find an n-tuple of $\mathcal{D}_0$-elements $Q_1 \dots Q_n$ where

$$Q_v = D_v^m + Q_{1,v}(x,t,D_t)D_v^{m-1} + \dots + Q_{v,m}(x,t,D_t) - \text{here } Q_{v,j}(x,t,D_t) \in \mathcal{D}_0(j),$$

ie. their orders with respect to D_t are $\le j$, and $Q_1 u = \dots = Q_n u = 0$.

In addition we have the Weierstrass polynomial $\varphi(x,t)$ which gave $\varphi^N \Gamma_0(p) \subset \Gamma_{-1}(p)$ and hence $(\varphi^N D_t)\Gamma_0(p) \subset \Gamma_0(p)$ and since $\Gamma_0(p)$ is a finitely generated $\mathbb{C}\{x,t\}$-module it follows that

$$\mathbb{C}\{x,t\}u + (\varphi^N D_t)\mathbb{C}\{x,t\}u + (\varphi^N D_t)^2 \mathbb{C}\{x,t\}u + \dots$$

is stationary. This gives some differential operator of the form

$$R(x,t,D_t) = (\varphi^N D_t)^w + r_1(x,t)(\varphi^N D_t)^{w-1} + \dots + r_w(x,t)$$

such that $Ru = 0$.

Now it is easy to chech that the cyclic $\mathcal{D}_o$ - module defined by $\mathcal{D}_o/$

$$\mathcal{D}_o / [\mathcal{D}_o Q_1 + \dots + \mathcal{D}_o Q_n + \mathcal{D}_o R]$$

is holonomic $\Rightarrow$ The quotient module $\mathcal{D}_o u$ is holonomic too.

<u>The mapping</u> $\mathcal{M}(p) \to \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p)$. We have proved that the $\mathcal{D}_o$ - module $\mathcal{M}(p)$ is holonomic. Put $M = \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p)$ which becomes a finitely generated $\mathcal{E}_p$ - module, i.e. this uses only that $\mathcal{M}(p)$ is a finitely generated $\mathcal{D}_o$ - module. By $u \to 1 \otimes u$ (1 = the identity in $\mathcal{E}_p$) we get a left $\mathcal{D}_o$ - linear mapping from the $\mathcal{D}_o$ - module $\mathcal{M}(p)$ into M .

<u>Claim</u>. The mapping $\mathcal{M}(p) \to M$ is surjective.

To prove this we use that any holonomic $\mathcal{D}_o$ - module is cyclic so we can find $u \in \mathcal{M}(p)$ such that $\mathcal{M}(p) = \mathcal{D}_o u = \mathcal{D}_o / L$ where $L = \{Q \in \mathcal{D}_o : Qu = 0\}$. Now $\mathcal{E}_p$ is a flat $\mathcal{D}_o$ - module $\Rightarrow M = \mathcal{E}_p / \mathcal{E}_o L$ and the surjectivity amounts to prove that $\mathcal{E}_p = \mathcal{D}_o + \mathcal{E}_p L$. To prove it we first find an n - tuple $Q_1 \dots Q_n$ in the left ideal L where $Q_v = D_v^m + Q_{v,1}(x,t,D_t)D_v^{m-1} + \dots + Q_{v,m}(x,t,D_t)$ with ord $(Q_{v,j}) \leq j$ for all v and j . To be precise, they are found as in the beginning of § 1.9.

By divisions in $\mathcal{E}_p \Rightarrow \mathcal{E}_p = \mathcal{E}_p Q_1 + \dots + \mathcal{E}_p Q_n + \Sigma\, D_x^{\alpha} \mathcal{E}_p(x,t,D_t)$ where $|\alpha| < mn$ in Σ . Since $\mathcal{E}_p Q_j \in \mathcal{E}_p L$ for all j the inclusion $\mathcal{E}_p \subset \mathcal{D}_o + \mathcal{E}_p L$ follows if we can prove that $\mathcal{E}_p(x,t,D_t) \in \mathcal{D}_o + L$. To see this we take some $R(x,t,D_t)$ and expand it with respect to D_t . A finite sum occurs with non-negative powers of D_t and it belongs already to $\mathcal{D}_o$. So we can assume that $R = \Sigma\, r_v(x,t)D_t^{-v}$ with $v \geq 1$ inside Σ .

Now we use that $\mathcal{M}(p)$ is a left $\mathcal{E}_p$ - module and that D_t is invertible as an element in the ring $\mathcal{E}_p$. Since $\mathcal{M}(p) = \mathcal{D}_o u$ we find for each $v \geq 1$ some $Q_v \in \mathcal{D}_o$ with $D_t^{-v} u = Q_v u$. Observe here that $D_t^{-v} u \in \Gamma_o(p)$ (or at last to some $\Gamma_m(p)$) when the $\mathcal{M}(p)$ - element $u \in \Gamma_m(p)$) hold for all v . It means that <u>we can find a fixed integer</u> m so that all $Q_v \in \mathcal{D}_o(1)$, i.e. a bound exists on these germs of differential operators.

Now another "division with bounds" is available - (see for example |Bj : page 140-141| for the technique) and it shows that $\Sigma\, r_v(x,t)Q_v$ can be arranged so that it converges in the ring $\mathcal{D}_o$. Call this $\mathcal{D}_o$ - element Q .

To get $R \in \mathcal{E}_p L + \mathcal{D}_o$ it only remains to see that $R-Q \in \mathcal{E}_p L$. First it is obvious that $Ru = Qu$ by the preceeding construction. To be precise, our choice of Q first implies that $(R-Q)u \in \Gamma_{-v}(p)$ for all $v \geq 1$ and then $\bigcap_{v \geq 1} \Gamma_{-v}(p) = 0$ is used.

It remains only to prove

<u>Sublemma</u> $\mathcal{E}_p L = \{R \in \mathcal{E}_p : Ru = 0$ in the given left $\mathcal{E}_p$ - module $\mathcal{E}_p u\} = \mathcal{L}$.

<u>Proof.</u> Using the flatness of $\mathcal{E}_p$ over $\mathcal{D}_o$ one easily gets that $\mathcal{L} \subset \mathcal{E}_p L + \mathcal{E}_p(-m)$ for all $m \geq 1$. Now we use that the left ideal $\mathcal{E}_p L$ in the ring $\mathcal{E}_p$ is closed, i.e. $\mathcal{E}_p L = \bigcap_{m \geq 1} [\mathcal{E}_p L + \mathcal{E}_p(-m)]$.

<u>Summing up</u>, we have proved that $\mathcal{M}(p)$ is a holonomic $\mathcal{D}_o$ - module and the canonical mapping $\mathcal{M}(p) \to \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p)$ is surjective. It remains to prove that it is injective. We postpone the proof of the injectiveness until § 1.14 below.

1.10. <u>The holonomic</u> $\mathcal{D}_X$ - <u>module</u>.

Since $\mathcal{D}_X$ is a coherent sheaf of rings there exists a unique" germ of a holonomic sheaf $\mathcal{N}$" of left $\mathcal{D}_X$ - modules which is defined in some polydise Δ centered at the origin and the stalk $\mathcal{N}(o,o)$ is the holonomic $\underline{\mathcal{D}_o}$ - module $\mathcal{M}(p)$.

We can arrange this polydisc Δ so that $\pi(\Lambda \cap \Omega) = T^X_{S_{reg}}$ with $S \subset \Delta$ as in Lemma 1.2. Now we can prove

1.11. <u>Lemma</u> $\pi(\overset{\circ}{SS}(\mathcal{N})) \subset S$.

Here $\overset{\circ}{SS}(\mathcal{N})$ is the part of the characteristic variety of $\mathcal{N}$ which is outside the zero-section.

Proof of Lemma 1.11. Recall that $S = \varphi^{-1}(0)$. We have $\mathcal{N} = \mathcal{D}_X u$ where $\mathcal{M}(p) = \mathcal{D}_o u$ was found as above and the proof of Lemma 1.9. has shown that there exist differential operators $Q_1 \dots Q_n$ such that $Q_v u = 0$ for all v . Shrinking Δ if necessary we can take $Q_v \in \Gamma(\Delta, \mathcal{D}_X)$ here. In addition we find some $R \in \Gamma(\Delta, \mathcal{D}_X)$ where $\sigma(R)(x,t,\xi,\tau)$ is of the form $(\varphi^N \tau)^m$ with some $m \geq 1$, i.e. it follows from the proof of the holonomicity of $\mathcal{M}(p)$.

If $(x,t) \in \Delta - \varphi^{-1}(0)$ and if $(x,t,\xi,\tau) \in SS(\mathcal{N}) \subset \sigma(R)^{-1}(0)$ we see that $\tau = 0$. Next, $\sigma(Q_v)(x,t,\xi,\tau) = 0$ on $SS(\mathcal{N})$ and if $\sigma(Q_v)(x,t,\xi,\sigma) = 0$ we see tha that $\xi_v = 0$. This hold for all $v \Rightarrow \xi = 0$ on $\pi^{-1}(\Delta - S) \cap SS(\mathcal{N})$ which gives Lemma 1.11.

1.12. The module $\tilde{\mathcal{N}}$. Put $\tilde{\mathcal{N}} = \mathcal{E}_X \otimes_{\pi^{-1}(\mathcal{D}_X)} \pi^{-1}(\mathcal{N})$ which now is a holonomic $\mathcal{E}_X$-module defined in $\pi^{-1}(\Delta)$ and supp $(\tilde{\mathcal{N}}) = SS(\mathcal{N})$. The stalk $\tilde{\mathcal{N}}(p) = \mathcal{E}_p \otimes \mathcal{N}(o,o) = \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p) = M$. We have seen that $\mathcal{M}(p) \to \mathcal{E}_p \otimes_{\mathcal{D}_o} \mathcal{M}(p)$ is surjective. <u>This canonical mapping is even left</u> $\mathcal{E}_p$ <u>- linear</u> - it was not stated explicitly before but can be proved by similar methods as in the proof of Lemma 1.9.

Hence the left $\mathcal{E}_p$ - linear and surjective mapping $\mathcal{M}(p) \to \tilde{\mathcal{N}}(p)$ exists. By coherence $\Rightarrow \mathcal{M} | \tilde{\Omega} \to \mathcal{N} | \tilde{\Omega}$ is surjective in a small conic neighborhood $\tilde{\Omega}$ of p . Shrinking Δ and Ω we can take $\tilde{\Omega} = \Omega$ and the surjectivity implies that $\text{supp}(\tilde{\mathcal{N}}) \cap \Omega \subset \text{supp}(\mathcal{M}) \cap \Omega$. In particular $\text{supp}(\tilde{\mathcal{N}})$ has a generic position at p . At the same time we already know that the conic Lagrangian $\text{supp}(\tilde{\mathcal{N}}) = SS(\mathcal{N})$ satisfies $\pi(SS(\mathcal{N})) \subset S$ and then a geometric result gives

1.13. Lemma $SS(\tilde{\mathcal{N}}) \subset \Lambda \cap \Omega = \Omega \cap \overline{T^*_{S_{reg}}}$.

See again [3] for this general result, to be precise we are using [3. Lemma 5.1.2 page 928].

1.14. Proof that $\mathcal{M}(p) \to M$ is injective. Using the holonomic $\mathcal{D}_X$ - module $\mathcal{N}$ and its extended holonomic $\mathcal{E}_X$ - module $\tilde{\mathcal{N}}$ it amounts to prove that the mapping from

$\mathcal{N}(o,o) \to \tilde{\mathcal{N}}(p)$ is injective. So take $u_o \in \mathcal{N}(o,o)$ and assume that $1 \otimes u_o = 0$ in the stalk $\tilde{\mathcal{N}}(p)$. Using the inclusion in Lemma 1.13. and the generic position at $p \Rightarrow \exists$ some small polydisc Δ_o centered at the origin in $\mathbb{C}^{n+1}$ such that $1 \otimes \pi^{-1}(u_o)$ is zero in $\tilde{\mathcal{N}}$ on the whole of $\pi^{-1}(\Delta_o) \cap \mathring{SS}(\mathcal{N})$. It means that $\mathring{SS}(\mathcal{D}_X u_o) \cap \pi^{-1}(\Delta_o)$ is empty and hence $\mathcal{D}_X u_o$ is a connection in side Δ_o. In particular, its stalk at the origin contains an element which is annihilated by D_t. This gives a contradiction since D_t is a bijective operator on $\mathcal{N}(o,o) =$ left $\mathcal{E}_p$ - module $\mathcal{M}(p)$. In other words, use that D_t^{-1} exists in the ring $\mathcal{E}_p$.

2. Some consequences of the Main Theorem.

We have proved that if $X = \mathbb{C}^{n+1}$ and $p = (o,o,o,dt)$ and if $\mathcal{M}$ is a holonomic $\mathcal{E}_X$ - module whose support has a generic position at p then $\mathcal{M} = \mathcal{E}_X \otimes \pi^{-1}(\mathcal{N})$ holds in some conic neighborhood Ω of p where $\mathcal{N}$ is a holonomic $\mathcal{D}_X$ - module defined in some polydisc Δ. Here we also have $\mathring{SS}(\mathcal{N}) \cap \Omega = \Lambda \cap \Omega$ where $\Lambda = \operatorname{supp}(\mathcal{M})$. The holonomic $\mathcal{D}_X$ - module $\mathcal{N}$ is quite special. For example, its characteristic variety is contained in $\overline{T^*_{S_{reg}}} \cup T^*_\Delta(\Delta)$ = a conic Lagrangian of $\pi^{-1}(\Delta)$.

Remark. Observe that the zero-section can appear, i.e. the support of the $\mathcal{D}_X$ - module $\mathcal{N}$ in the Main Theorem can be the whole polydisc. Already a 1 - dimensional case illustrates this phenomenon, i.e. consider the case when $X = \mathbb{C}^1$ and here $\mathcal{E}(t,D_t)/\mathcal{E}(t,D_t)(t-\alpha D_t^{-1})$ is holonomic and we assume that the complex scalar α is not an integer. The holonomic $\mathcal{D}_o$ - module $\mathcal{M}(p)$ cannot be supported by the origin for then any cyclic generator of it is annihilated by some power of t which is impossible.

We can use the Main Theorem to get a new proof of the existence of regular holonomic modules, i.e. if $\mathcal{M}$ is a holonomic $\mathcal{E}_X$ - module and if $\mathcal{M}^\infty = \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}$ is its extended $\mathcal{E}_X^\infty$ - module one can use quantized contact transformations and assume that $\operatorname{supp}(\mathcal{M})$ has a generic position at a preassigned point of its support and use the Main Theorem to find the unique $\mathcal{E}_X$ - submodule $\mathcal{M}_{reg}$ of $\mathcal{M}^{\infty}$ which is regular holonomic and satisfies $\mathcal{M}^{\infty}_{reg} = \mathcal{M}^{\infty}$. See [3] and also [2] for this.

In addition we can extend results from [4] to the irregular case.

Finally we mention that the Main Theorem suggests the study of holonomic $\mathcal{D}_X$ - modules $\mathcal{N}$ whose characteristic varieties are reduced to $\overline{T^*_{S_{reg}}}$ and a (possibly empty) union with the zero-section where the conormal variety $\overline{T^*_{S_{reg}}}$ has a generic position at (o,o,o,dt). This leads to a delicate study. For example, it is not true that any such hypersurface works and one may ask to what extent the $\mathcal{D}_X$ - module $\mathcal{N}$ is "determined by monodromy of its solution complex and so on. In fact, we may ask if the Main Theorem eventually leads to a structure theory for holonomic $\mathcal{E}_X$ - module with irregular singularities. This is a very ambitious program. See Malgrange's work for the 1 - dimensional classification of holonomic $\mathcal{E}$ - modules which need not be regular.

In the regular case one has the <u>Riemann Hilbert Correspondence</u> and it suggests that we try to determine all perverse sheaves which are related to special holonomic $\mathcal{D}_X$ - modules arising from the Main Theorem. For example, let S be a hypersurface in X so that $\overline{T^*_{S_{reg}}} = \mathrm{supp}(\mathcal{M})$ for some holonomic $\mathcal{E}_X$ - module. Here we take $X = \mathbb{C}^{n+1}$ and assume as above that $\mathrm{supp}(\mathcal{M})$ has a generic position at (o,o,o,dt). Then we try to "classify" all these holonomic $\mathcal{E}_X$ - modules which in addition have R.S. (i.e. "regular singularities in the sense of [6]). Already an answer when $n = 1$, i.e. $\dim(X) = 2$ would be interesting.

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